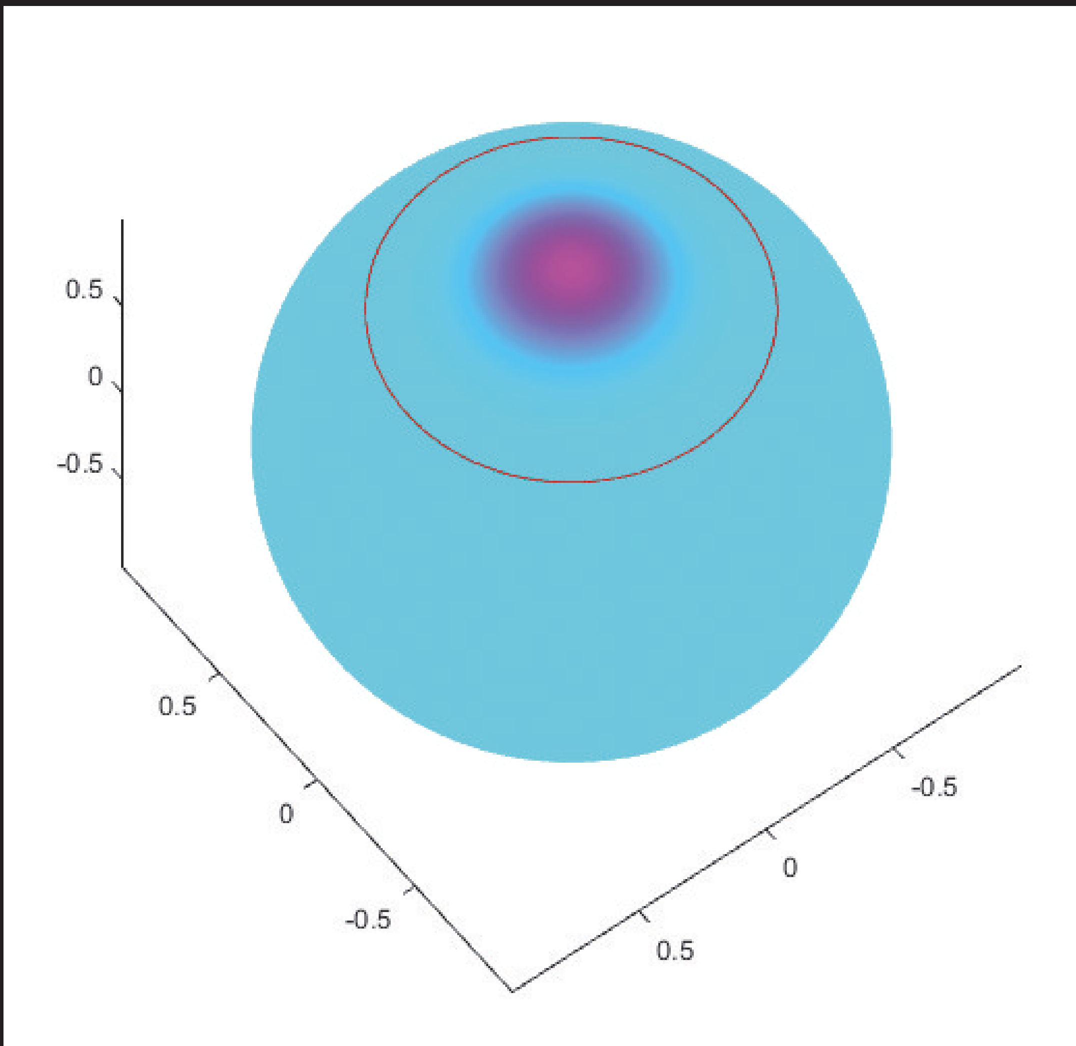
Katrin Seibert



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Zusammenfassung

Die spin-gewichteten Kugelflächenfunktionen, definiert von Newman und Penrose (1966), bilden eine Orthonormalbasis des $L^2(\Omega)$ auf der Einheitssphäre Ω und haben ein breit gefächertes Anwendungsgebiet.
Wir präsentieren eine einheitliche mathematische Theorie der spin-gewichteten Kugelflächenfunktionen inklusive Auflistung ihrer bereits bekannten Eigenschaften auf mathematische Weise, verknüpft mit der Notation der Theorie der Kugelflächenfunktionen. Des Weiteren verwenden wir die spin-gewichteten Kugelflächenfunktionen zur Konstruktion tensorieller Slepian-Funktionen auf der Sphäre.
Slepian-Funktionen sind räumlich konzentriert und spektral begrenzt. Für skalare und vektorielle Messdaten auf der Sphäre werden sie in diversen Bereichen wie Geodäsie, Kosmologie und der biomedizinischen Bildgebung verwendet. Ihre Konzentriertheit in einer ausgewählten Region auf der Sphäre ermöglicht die lokale Inversion, wenn nur regionale Messdaten gegeben sind oder ermöglichen es, regionale Information zu extrahieren.
Wir konzentrieren uns auf die Analyse von Tensorfeldern mit Hilfe von Slepian-Funktionen, wie sie z.B. von der Satellitenmission GOCE stammen. Unsere Konstruktion mit spin-gewichteten Kugelflächen-funktionen erzielt für tensorwertige Felder diverse numerische Vorteile und erlaubt eine effizientere Konstruktion der tensoriellen Slepian-Funktionen für sphärische Kappen, z.B. eine lokale Basis auf der sphärischen Kappe für die Polarisation der kosmischen Hintergrundsstrahlung (CMB).

Abstract

The spin-weighted spherical harmonics of Newman and Penrose (1966) form an orthonormal basis of $L^2(\Omega)$ on the unit sphere Ω and have a huge field of applications.
We present a unified mathematical theory, which implies the collection of already known properties of the spin-weighted spherical harmonics, recapitulated in a mathematical way, and connected to the notation of the spherical harmonics. In addition, we use spin-weighted spherical harmonics to construct tensor Slepian functions on the sphere.
Slepian functions are spatially concentrated and spectrally limited. For scalar and vectorial data on the sphere, they are utilized in a variety of disciplines, including geodesy, cosmology, and biomedical imaging. Their concentration within a chosen region of the sphere allows for local inversions when only regional data are available, or enable the extraction of regional information.
We focus on the analysis of tensorial fields, as collected e.g. in the GOCE mission, by means of Slepian functions. For tensorial data, Slepian functions have already been constructed by Eshagh (2009) in the basis of the tensor spherical harmonics of Martinec (2003).
By using spin-weighted spherical harmonics, our theory offers several numerical advantages. Furthermore, we present a method for an efficient construction of tensor Slepian functions for spherical caps. In this context, we are able to construct a localized basis on the spherical cap for the cosmic microwave background (CMB) polarization.

Contents

Chapter 1

Introduction

This thesis deals with a unified mathematical theory of the spin-weighted spherical harmonics of Newman and Penrose [63]. They are used to construct the tensor Slepian functions on the sphere and to recapitulate the scalar and vector Slepian functions [43, 67, 78, 82]. Furthermore, we use the spin-weighted spherical harmonics to construct a commuting operator for the spherical cap, which enables us to find an efficient method for calculating the Slepian functions on spherical caps. Moreover, we use this method on the example of the CMB (cosmic microwave background) polarization.

That functions cannot have a temporal (or spatial) and spectral finite support at the same time is well known. However, we can find signals that are timelimited and optimally concentrated in the spectral domain. This leads us to a concentration problem, which was solved by Slepian, Landau, and Pollak in the early 1960s for Euclidean domains like the real line [52, 84, 85]. Therefore, the timelimited and optimally concentrated functions are called the Slepian functions.
In geosciences, we have data measured on planetary surfaces or parts of it. So, the use of the method of Slepian, Landau, and Pollak is also useful for spherical problems. On the unit sphere, Albertello, Sansò, and Sneeuw [2] and Simons, Dahlen, and Wieczorek [78, 82] deal with the scalar Slepian functions.
Furthermore, the method of Slepian functions has been extended to the vectorial case by Plattner and Simons [67, 83] and by Jahn and Bokor [43].

Note that there also exists an alternative method to the method of Slepian functions to obtain a spatially localized basis. This method is based on the construction of an operator, which is composed of a projection operator and a multiplication operator. The operator is closely connected to the theory of orthogonal polynomials. Here, an eigenvalue problem for that operator is to solve. This method was developed by Erb on the real line [19] and by Erb and Mathias on the sphere [20]. However, in this thesis, we deal with the method of Slepian functions since they are more common in various disciplines.

For data on the sphere in scalar and vectorial form, the Slepian functions have proven to be a viable and versatile tool. They have been applied in a variety of fields including geodesy, planetary magnetism, cosmology, and biomedical imaging (see [43, 67] and the references therein). Furthermore, Khalid, Kennedy, and McEwen [48] investigated scalar Slepian functions on the ball and Michel, Orzlowski, and Schneider [59] considered vector Slepian functions on the ball.

The Slepian functions are orthonormal on the entire sphere as well as orthogonal on the

concentration domain [82]. Further, we know that they are optimally concentrated within a region of interest. Thus, they are relevant and advantageous for local problems.
The scalar Slepian functions are a basis transformation of the spherical harmonics and the vector Slepian functions are a basis transformation of the vector spherical harmonics of Hill [40]. So, with the method of the Slepian functions, we transform a global basis into a localized one.

For the construction of the Slepian functions, we have to solve a concentration problem, which leads us to an eigenvalue problem that can be rewritten as an integral equation. The kernel matrix of the eigenvalue problem is supposed to be ill-conditioned. So, the solution of the problem is unstable. For special regions, it is possible to formulate a commuting operator to the kernel function of the integral equation. This leads to an alternative problem and thus to an alternative eigenvalue problem with a matrix, which commutes with the kernel matrix. This commuting matrix has the same eigenvectors like the kernel matrix. Furthermore, the commuting matrix is a tridiagonal matrix and thus we solve the numerically stable alternative eigenvalue problem.
The commuting operator for scalar Slepian functions can be formulated for the spherical cap [37, 78, 82] and for the spherical double cap and the belt [80]. For arbitrarily shaped regions, the ill-conditioned eigenvalue problem must be solved (see [81] for further details). The commuting operator for the vector Slepian functions for the spherical cap is given by [43].

Satellite gravity gradiometry data (SGG data), like the data recorded by the satellite mission GOCE, is given in the form of tensors. Moreover, the CMB polarization is also given in form of tensors and is for practical reasons only given on parts of the sphere, mainly on spherical caps. So, an extension of the method of Slepian functions to tensors is also useful. As a realistic example, we will look at the CMB polarization by tensor Slepian functions for synthetic data.

The analysis of CMB anisotropies is used for studies of the early universe and for classical cosmology [45, 46, 53, 62]. The CMB anisotropies are measured in terms of temperature and polarization anisotropies on parts of the sphere, mainly on the hemisphere. The temperature anisotropies are scalar fields that can be described by scalar Slepian functions [1]. The polarization anisotropies are tensor fields and can be separated into an electric and a magnetic component. They are given in series of electric and magnetic tensor spherical harmonics, which we connect to the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27].

However, tensor Slepian functions on the sphere have been rarely investigated so far. Eshagh [22] has previously developed the tensor Slepian functions for the basis of the tensor spherical harmonics of Martinec [56]. We use a different ansatz for the construction. There is a huge spectrum of tensor spherical harmonics from which we choose the tensor basis (see [90]). We decided to use as basis the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27], which we transform to a spin-weighted basis system. This enables us to decouple the concentration problem to spin-weighted eigenvalue problems. So, the kernel matrices have a blockdiagonal structure and we analyze every block for itself. These blocks can be reduced to scalar problems, depending on the spin-weight. Furthermore, we also have the tool to investigate the CMB polarization by tensor Slepian functions.

The spin-weighted eigenvalue problem has the advantage that we can formulate a commuting operator for the spherical cap for it. By solving the spin-weighted eigenvalue problem,

we do not only get the tensor Slepian functions, but also the scalar and the vector Slepian functions. So, we have a unified formulation of the Slepian problem depending on a spin weight, which enables us to calculate all types of Slepian functions. We need spin weight zero for the scalar problem, spin weight zero and ± 1 for the vector problem, and spin weight zero, ± 1, and ± 2 for the tensor problem. The spin-weighted Slepian functions are scalar fields. To get the vectorial or the tensorial ones, we have to multiply the corresponding spin-weighted Slepian functions with spherical unit vectors or with the tensor product of spherical unit vectors.
So, the scalar, vector, and tensor spherical harmonics that we use are connected to the spin-weighted spherical harmonics.

All those advantages for the construction of the Slepian functions lead us to analyze the spin-weighted spherical harmonics in more detail. The spin-weighted spherical harmonics, as defined by Newman and Penrose [63], have a huge field of applications. Mainly, they are used in quantum mechanics [18], but also for the theory of gravitation, in early universe and classical cosmology, and in geophysics [12, 15, 63, 91, 97]. An important fact is that the spin-weighted spherical harmonics of spin weight zero are the well-known spherical harmonics. So, the spin-weighted spherical harmonics are also called the generalized spherical harmonics [12].

We see that the spin-weighted spherical harmonics are a viable and versatile tool. They form an orthonormal basis on the $\mathrm{L}^2(\Omega)$. However, a major problem occurs that is conditioned by this huge field of applications. To date, it exists no complete theory of the spin-weighted spherical harmonics and hence, there is no unified notation given. Therefore, we treat the spin-weighted spherical harmonics to a large extent in this thesis. Here, we show a unified mathematical theory of the spin-weighted spherical harmonics. Furthermore, we recapitulate the definition and properties of the spin-weighted spherical harmonics in a mathematical way and formulate and prove multiple new properties. We choose this mathematical notation such that it is connected for spin weight zero to the theory of the spherical harmonics.

For this purpose, we recapitulate and connect different definitions of the spin-weighted spherical harmonics. They can be defined as function of spin weight [91], defined by the spin raising and lowering operators $\eth$ and $\overline{\eth}$ as iteration or directly connected to the spherical harmonics [34, 63], and defined by the Wigner D-function [53, 96, 97]. We do not only recapitulate recursion relations for the spin-weighted spherical harmonics [93], but also prove new ones.
Furthermore, we formulate the Parseval identity, a Christoffel-Darboux formula, a finite series expansion, and a Sturm-Liouville differential equation. This differential equation leads to a differential operator, a spin-weighted version of the Beltrami operator [49]. Based on this operator, we show with the completeness of the spin-weighted spherical harmonics that they are the unique eigenvectors of the spin-weighted Beltrami operator. Moreover, we formulate Green's second surface identity for this operator. Still based on the spin-weighted Beltrami operator, we formulate the set Harm_n^N of the $(*, N)$-harmonic functions of spin weight N and degree n, which is spanned by the spin-weighted spherical harmonics. This notation is connected to the definition of the set Harm_n of the harmonic and homogeneous polynomials of degree n, which is spanned by the spherical harmonics [33, 28, 58].
Besides, we formulate the spin-weighted spherical harmonics in terms of the Jacobi polynomials [18, 77, 93].

As an additional feature, we introduce the spin-weighted Legendre polynomials and the spin-weighted associated Legendre functions. Here, we show properties like some recursion relations, the orthonormality with respect to a weighted scalar product, a finite series expansion, and a Christoffel-Darboux formula.

The outline of this thesis is as follows:

We start with the fundamental tools that are needed in this thesis. Here, we first denote the basics and notations. Then, we introduce the tensor nomenclature we need in Chapter 6 for calculations with tensor fields. Next, we collect spherical nomenclature that is important for this work, because all our functions will act on the unit sphere or on parts of it. Therefore, we necessarily have to introduce a system of functions on the unit sphere that form an orthonormal basis therein. These functions are the scalar spherical harmonics. Furthermore, in Chapter 5 and in Chapter 6, we want to look at the vector and the tensor Slepian functions. So, in the fundamentals, we introduce the vector spherical harmonics of Hill [40], the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27], and their properties.

Chapter 3 deals with the theory of the spin-weighted spherical harmonics. Here, we start with the definition of the spin weight and a function of spin weight. Therefore, we have to define the functions o^i and $\hat{o}^i$, $i = 1, 2$, which deliver us the spin weight of a function.
Following, we define the spin-weighted spherical harmonics with the help of the spin raising and lowering operators $\eth$ and $\overline{\eth}$. Here, we use the definition of Newman and Penrose [63], which connects the spin-weighted spherical harmonics to the spherical harmonics by an iterative use of the operators $\eth$ and $\overline{\eth}$. Furthermore, we reformulate this definition into a recursion formulation which connects the spin-weighted spherical harmonics of spin weight N to those of spin weight $N \pm 1$.
Next, we show different properties of the spin-weighted spherical harmonics. For example, we prove known and new recursion relations, formulate a Christoffel-Darboux formula, and recapitulate a Sturm-Liouville differential equation, which leads us to the spin-weighted Beltrami operator.
In the next section, we deal with the addition theorem for the spin-weighted spherical harmonics. However, we need to introduce the Wigner D-function at first, through which we represent the spin-weighted spherical harmonics. This property of the spin-weighted spherical harmonics is of upmost importance. The Wigner D-function is already very well investigated for example by [18, 93]. With help of this representation, we write the spin-weighted spherical harmonics in a finite series expansion and thus in a finite series expansion with the functions o^i and $\hat{o}^i$, $i = 1, 2$. From the finite series expansion, we can prove that the spin-weighted spherical harmonics are bounded. Furthermore, we can show conditions for different derivatives of the spin-weighted spherical harmonics. With these properties, we define a set of functions that fulfills these conditions. Then, we formulate Green's second surface identity for the spin-weighted Beltrami operator. Furthermore, the definition of the spin-weighted spherical harmonics by the Wigner D-function enables us to express them in terms of Jacobi polynomials. In addition, the formulation by the Wigner D-function enables us to prove the orthonormality of the spin-weighted spherical harmonics on $\mathrm{L}^2(\Omega)$.
Additionally, we pay attention on the spin raising and lowering operator and its properties. Here, we connect, for example, these operators to the spin-weighted Beltrami operator.
In the following section, we describe the kernel of the spin raising and lowering operators. Also, we show the completeness of the spin-weighted spherical harmonics and conclude from this the Parseval identity. Then, we define the set of the $(*, N)$-harmonic functions of spin

weight N and degree n. The spin-weighted spherical harmonics span this set. With all these properties, we prove the uniqueness of the spin-weighted spherical harmonics as the eigenfunctions of the spin-weighted Beltrami operator.
Further, we introduce in the next section the spin-weighted Legendre polynomials. These are really polynomials that are orthonormal on $[-1, 1]$ with respect to a weighted L^2-scalar product. Furthermore, we show recursion relations, a Christoffel-Darboux formula, and a finite series expansion for the spin-weighted Legendre polynomials. Moreover, we also define the spin-weighted associated Legendre functions and look at analogous properties of them.
In the section about additional properties of the spin-weighted spherical harmonics, we mainly collect recursion relations for the spin-weighted spherical harmonics, which are given in [93] for the Wigner D-function.
In the last section of this chapter, we show the relation between the spin-weighted spherical harmonics and the scalar, vector, and tensor spherical harmonics that we introduced in the previous chapter.

Some of the properties of the spin-weighted spherical harmonics from Chapter 3 like the recursion relations, the Christoffel-Darboux formula, the orthonormality, the completeness, the spin-weighted Beltrami operator from the Sturm-Liouville differential equation, and the addition theorem, we need to prove that the commuting operator for the spin-weighted eigenvalue problem from the Slepian problem commutes with the kernel matrix of our problem. Additionally, the relation between the two kinds of spherical harmonics from Section 3.9 is important for the formulation of the Slepian eigenvalue problem. Other properties from Chapter 3 are given for reasons of mathematical correctness of the theory of the spin-weighted spherical harmonics.

After dealing with the spin-weighted spherical harmonics, we can now move on to the Slepian functions. In Chapter 4, we start with the scalar Slepian functions and explain the method of the Slepian functions on the sphere. This was previously investigated by [78, 82]. We start with the derivation of the scalar Slepian functions. Here, we start with a concentration problem. This is why we want to find bandlimited functions that are optimally concentrated within a region of interest. This region is a part of the unit sphere. Then, we transform this problem to an eigenvalue problem and further, to an integral equation. The solutions of the integral equation are the Slepian functions. We sort them by decreasing eigenvalue such that the first functions are the best concentrated ones in the region of interest and the last ones are the least concentrated functions. Furthermore, we choose the eigenvectors of the eigenvalue problem to be orthonormal by the Gram-Schmidt algorithm and thus we can show various properties for the Slepian functions in the next section. For example, the Slepian functions are orthonormal on the unit sphere and orthogonal on the region of interest.
Next, we introduce the Shannon number as a good estimate for significant eigenfunctions. This means that we get the number of functions that are well concentrated within the region of interest. The remaining functions can be omitted.

In Chapter 5, we do the same for vector fields [43, 67]. Here, the derivation of the vector Slepian functions is separated into the normal part and the tangential part of the vector fields. In analogy to the scalar case, we also show the properties of the vector Slepian functions and determine the Shannon number for the vector case.

In Chapter 6, we develop the method of the Slepian functions for tensor fields. The strategy is the same as before but it is a little bit more complicated. Here, the derivation is separated into four parts, the normal part, the left normal/right tangential part, the left tangential/right normal part, and the tangential part. In analogy to before, we gain the properties for the tensor Slepian functions like the orthonormality on the unit sphere and the orthogonality on the region of interest. Again, we can calculate the Shannon number.

In the previous chapters, we looked at the optimal concentration of bandlimited functions. In the same way, we can examine the spectral concentration of spacelimited functions. We do these considerations in Chapter 7. We do this for the scalar, the vector, and the tensor case and connect it to the results of the previous chapters.

All the cases, the scalar, the vector, and the tensor case, can be reduced to a general spin-weighted case, which depends on a fixed spin weight N. We look at the spin-weighted Slepian eigenvalue problem and integral equation in Chapter 8 for the spherical cap as region of interest. Here, we start with considerations about the spin-weighted kernel matrix. This kernel matrix is real, symmetric, and hermitian. Furthermore, numerical experiments have shown that this matrix is supposed to be ill-conditioned. So, we want to solve an alternative problem instead.
Therefore, we define the commuting operator to the kernel function from the integral equation for the spherical cap. We denote what it means that the operator and the kernel function commute and prove it. This commuting operator has the same eigenfunctions as the kernel function. Thus, we can solve the eigenvalue equation for the commuting operator instead of the integral equation.
For practical reasons, we reformulate the eigenvalue equation for the commuting operator and get an eigenvalue problem with a matrix, which commutes with the kernel matrix. This commuting matrix is tridiagonal and has the same eigenvector like the kernel matrix. So, the use of an alternative problem with a commuting operator is a valuable tool for the computation of Slepian functions.
Next, we determine the Shannon number for the spherical cap for the scalar, the vector, and the tensor Slepian functions.
In the last section of this chapter, we implement the tensor Slepian functions. Here, we plot the norm of them for different concentration degrees within a spherical cap.

Chapter 9 deals with an application of the method of tensor Slepian functions. Here, we investigate the CMB polarization on the spherical cap [45, 46, 53, 62]. Therefore, we first define the CMB polarization, which can be separated into an electric and a magnetic component [79]. These components can be written in the electric and magnetic tensor spherical harmonics, which we connect to the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27].
Next, we construct the CMB Slepian functions with the method described in Chapter 6 and thus for the spherical cap as described in Chapter 8.
We use the results to compare the method of the CMB Slepian functions to the basis of the electric and magnetic tensor spherical harmonics for contrived polarizations given on a spherical cap.

In Chapter 10, we summarize our results and provide suggestions for future researches.

Appendix A recapitulates different Gauß quadratures like the Gauß-Tschebyscheff quadrature and the Gauß-Legendre quadrature. We need these quadrature methods for the integration over the spherical cap for example, to calculate the coefficients of the Slepian functions for the CMB polarization. Furthermore, we mention an idea for a Gauß quadrature with spin-weighted Legendre polynomials.

In Appendix B, we collect point grids on the sphere, which we use for the implementation mainly of the CMB polarization test cases, but also for the implementation of the tensor Slepian functions from Chapter 8.5. Here, we introduce the integration grid that we use for the integration over the spherical cap from Appendix A, the HEALPix grid, which is an important point grid in cosmology, the Reuter grid, and finally, the Driscoll-Healy grid as plotting grid.

In addition, in Appendix C, we collect a list of used symbols in this thesis, which facilitates the readability of this thesis.

Chapter 2

Fundamentals

This chapter addresses the fundamentals that we need in this thesis. Being divided in six sections, we address the basics and notations first. Followed by the needed nomenclature for tensors and for spherical calculations. Thirdly, we look at different types of spherical harmonics, where we start with the scalar spherical harmonics. Next, the vector spherical harmonics of Hill [40] and finally, the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27] are being introduced.

2.1 Basics and Notations

Here are the basic notations we use in this thesis.

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ denote the sets of positive integers, non-negative integers, integers, real numbers, positive real numbers, and complex numbers. $\mathbb{R}^3$ and $\mathbb{C}^3$ denote the three-dimensional Euclidean space and the three-dimensional complex space.

Definition 2.1.1. *With capital letters like F, G, we denote scalar functions and everything that belongs to the scalar case. With small letters like f, g, we label vector functions and with bold letters like $\boldsymbol{f}, \boldsymbol{g}$, we label tensor functions (mostly of second rank) and everything that belongs to that case.*

Definition 2.1.2. *The set $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ denotes the canonical orthonormal basis of $\mathbb{R}^3$ given by*

$$\varepsilon^1 := \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \varepsilon^2 := \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \varepsilon^3 := \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Next, we define function spaces and its associated norms.

Definition 2.1.3. *Let $D \subset \mathbb{R}^n$, compact, and $W \subset \mathbb{R}$ be given. Then, $\mathrm{C}^{(k)}(D, W)$ is the space of all functions $F : D \to W$, which are at least differentiable to order $k \in \mathbb{N}_0$, where the k-th derivative is continuous. If $W = \mathbb{R}$, then we denote $\mathrm{C}^{(k)}(D, \mathbb{R}) := \mathrm{C}^{(k)}(D)$ and if $k = 0$, then we write $\mathrm{C}^{(0)}(D, \mathbb{R}) := \mathrm{C}(D, W)$.*

Analogously, we define $\mathrm{c}^{(k)}(D, w)$ for all vector functions $f : D \to w$, $w \subset \mathbb{R}^3$, and $\mathbf{c}^{(k)}(D, \boldsymbol{w})$ for all second rank tensor functions $\boldsymbol{f} : D \to \boldsymbol{w}$, $\boldsymbol{w} \subset \mathbb{R}^{3\times 3}$.

Definition 2.1.4. *The norm for $F \in \mathrm{C}^{(p)}(D)$, $0 \le p \le \infty$, $D \subset \mathbb{R}^n$ compact, is given by*

$$\|F\|_{\mathrm{C}(D)} := \sup_{x\in D} |\boldsymbol{F}(x)|,$$

where $|y|$ denotes the Euclidean norm for $y \in \mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 2.1.5. *We call a vector field $f \in \mathrm{c}^{(p)}(D)$ and a tensor field $\boldsymbol{f} \in \boldsymbol{c}^{(p)}(D)$, $0 \leq p \leq \infty$, $D \subset \mathbb{R}^n$ compact, if its component functions with respect to the orthonormal basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ are in $\mathrm{C}^{(p)}(D)$, where*

$$\|f\|_{\mathrm{c}(D)} := \sup_{x \in D} |f(x)|$$

and

$$\|\boldsymbol{f}\|_{\mathbf{c}(D)} := \sup_{x \in D} |\boldsymbol{f}(x)|.$$

Finally, we collect three definitions and one lemma we need in this thesis.

Definition 2.1.6. *We define the inner product of two vectors $x, y \in \mathbb{C}^n$ by*

$$\langle x, y \rangle := x \cdot \overline{y} := \sum_{j=1}^{n} x_j \overline{y_j}.$$

Definition 2.1.7. *The Kronecker symbol is defined by*

$$\delta_{n,n'} := \begin{cases} 1, & n = n' \\ 0, & n \neq n' \end{cases}.$$

Definition 2.1.8. *With $\lfloor \cdot \rfloor$, we denote the Gaussian rounding function given for $l \in \mathbb{R}$ by*

$$\lfloor l \rfloor := \{k \in \mathbb{Z} \mid k \leq l\}.$$

Lemma 2.1.9 (Addition Theorems for the Cosine)**.** *From [4], we borrow the following well-known properties for the cosine*

$$\cos\alpha \cos\beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta)),$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta.$$

2.2 Tensor Nomenclature

In this section, we look at the required properties for the calculation with tensors [12, 28, 33]. Here, we follow mainly [33].

Definition 2.2.1. *A tensor T of rank $q \in \mathbb{N}$ is a functional $T : (\mathbb{R}^3)^q \to \mathbb{R}$ such that it is linear in all q arguments. This means that for all $\lambda_i \in \mathbb{R}$ and for all $x^{(i)} \in \mathbb{R}$ and $y \in \mathbb{R}^3$*

$$T\left(\lambda_1 x^{(1)}, \ldots, \lambda_q x^{(q)}\right) = T\left(x^{(1)}, \ldots, x^{(q)}\right) \prod_{i=1}^{q} \lambda_i,$$
$$T\left(x^{(1)}, \ldots, x^{(j)} + y, \ldots, x^{(q)}\right) = T\left(x^{(1)}, \ldots, x^{(q)}\right) + T\left(x^{(1)}, \ldots, x^{(j-1)}, y, x^{(j+1)}, \ldots, x^{(q)}\right).$$

The scalar multiplication and the addition of tensors is given as usually by

$$(\lambda_1 T_1 + \lambda_2 T_2)(x) = \lambda_1 T_1(x) + \lambda_2 T_2(x),$$

where $x \in (\mathbb{R}^3)^q$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and T_1 and T_2 are tensors of rank q.

Remark 2.2.2. *Let T be a tensor of rank $q \in \mathbb{N}$. The components of T with respect to $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ are defined by*

$$\boldsymbol{t}_{i_1,\dots,i_q} := T(\varepsilon^{i_1}, \dots, \varepsilon^{i_q}),$$

where $(i_1, \dots, i_q) \in \{1, 2, 3\}^q$.

Definition 2.2.3. *From now on, we denote a tensor of rank $q \in \mathbb{N}$ by $\boldsymbol{t} \in (\mathbb{R}^3)^q$, which can be written in the form*

$$\boldsymbol{t} = \sum_{i_1,\dots,i_q=1}^{3} \boldsymbol{t}_{i_1,\dots,i_q} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_q}.$$

Note that a scalar is a tensor of rank zero, a vector is a tensor of rank 1, and a matrix is a tensor of rank 2. If there is no declaration of the rank of a tensor, we mean in this work a second rank tensor.

Now, we define our operators for tensor calculation.

Definition 2.2.4. *Let $\boldsymbol{s}$ be a tensor of rank $p \in \mathbb{N}$ and $\boldsymbol{t}$ be a tensor of rank $q \in \mathbb{N}$. The tensor product delivers a tensor of rank $p + q$ and is given by*

$$\boldsymbol{s} \otimes \boldsymbol{t} := \sum_{i_1,\dots,i_p=1}^{3} \sum_{j_1,\dots,j_q=1}^{3} \boldsymbol{s}_{i_1,\dots,i_p} \boldsymbol{t}_{j_1,\dots,j_q} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_p} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_q}.$$

Note that in general $\boldsymbol{s} \otimes \boldsymbol{t} \neq \boldsymbol{t} \otimes \boldsymbol{s}$.

Definition 2.2.5. *Let $\boldsymbol{s}$ be a tensor of rank p and $\boldsymbol{t}$ be a tensor of rank q with $0 < q \leq p$, $p, q \in \mathbb{N}$. The double dot product delivers a tensor of rank $p - q$ and is given by*

$$(\boldsymbol{s} : \boldsymbol{t})_{i_1,\dots,i_{p-q}} := \sum_{j_1,\dots,j_q=1}^{3} \boldsymbol{s}_{i_1,\dots,i_{p-q},j_1,\dots,j_q} \boldsymbol{t}_{j_1,\dots,j_q}.$$

Note that the double dot product of two tensors of rank 1 delivers the product "$\cdot$" of two vectors. This means that for two vectors $x, y \in \mathbb{C}^3$

$$x : y = x \cdot y = \sum_{i=1}^{3} x_i y_i.$$

Special cases of the previous introduced tensor operations are pointed out in the following.

Remark 2.2.6. *The tensor product of two column vectors $x, y \in \mathbb{C}^3$ is given by*

$$x \otimes y := xy^{\mathrm{T}} = \sum_{i,j=1}^{3} x_i y_j \varepsilon^i \otimes \varepsilon^j = (x_i y_j)_{i,j=1,2,3}.$$

The double dot product of two second rank tensors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^{3\times 3}$ is defined by

$$(\boldsymbol{a} : \boldsymbol{b}) := \operatorname{tr}\left(\boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}\right) = \sum_{i,j=1}^{3} \boldsymbol{a}_{i,j} \boldsymbol{b}_{i,j}$$

and the double dot product of a tensor $\boldsymbol{s} \in (\mathbb{C}^3)^4$ of rank 4 and a tensor $\boldsymbol{t} \in \mathbb{C}^{3\times 3}$ of rank 2 *by*

$$(\boldsymbol{s} : \boldsymbol{t})_{i,j=1,2,3} = \sum_{k,l=1}^{3} \boldsymbol{s}_{i,j,k,l} \boldsymbol{t}_{k,l}.$$

Now, we show the properties for tensor calculations that we need in this thesis.

Lemma 2.2.7. *Let a, b, c and $d \in \mathbb{C}^3$ be vectors. Then, we get*

$$(a \otimes b) : (c \otimes d) = (a \cdot c)(b \cdot d).$$

Proof. We get the proposition directly by

$$\begin{aligned}(a \otimes b) : (c \otimes d) &= \sum_{i,j=1}^{3} (a \otimes b)_{i,j} (c \otimes d)_{i,j} = \sum_{i,j=1}^{3} a_i b_j c_i d_j \\ &= \left(\sum_{i=1}^{3} a_i c_i \right) \left(\sum_{j=1}^{3} b_j d_j \right) = (a \cdot c)(b \cdot d).\end{aligned}$$

□

Lemma 2.2.8. *Let a, b, c, and $d \in \mathbb{C}^3$ be vectors. Then, we obtain the following properties*

$$\begin{aligned}(a \otimes b)^{\mathrm{T}} &= b \otimes a, \\ a(b \cdot c) &= (a \otimes b)c, \\ (a \cdot b)(c \cdot d) &= a^{\mathrm{T}}(b \otimes c)d.\end{aligned}$$

Proof. The proof is straight forward.

- We get the first property directly by

$$(a \otimes b)^{\mathrm{T}} = (a_i b_j)_{i,j=1,2,3}^{\mathrm{T}} = (a_i b_j)_{j,i=1,2,3} = (b_j a_i)_{j,i=1,2,3} = b \otimes a.$$

- For the second property, we get on the one hand

$$a(b \cdot c) = a \left(\sum_{j=1}^{3} b_j c_j \right) = \left(\sum_{j=1}^{3} a_i b_j c_j \right)_{i=1,2,3}$$

  and on the other hand, we see

$$(a \otimes b)c = (a_i b_j)_{i,j=1,2,3} c = \left(\sum_{j=1}^{3} a_i b_j c_j \right)_{i=1,2,3}.$$

  So, both sides are equal.

- The left-hand side of the third equation is given by

$$(a \cdot b)(c \cdot d) = \left(\sum_{i=1}^{3} a_i b_i \right) \left(\sum_{j=1}^{3} c_j d_j \right) = \sum_{i,j=1}^{3} a_i b_i c_j d_j$$

and the right-hand side by

$$a^{\mathrm{T}}(b \otimes c)d = a^{\mathrm{T}}(b_i c_j)_{i,j=1,2,3} d = \left(\sum_{i=1}^{3} a_i b_i c_j \right)_{j=1,2,3} d = \sum_{i,j=1}^{3} a_i b_i c_j d_j.$$

So, both sides are equal. Consequently, we obtain the third property.

□

Lemma 2.2.9. *Let* $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, *and* $\boldsymbol{d} \in \mathbb{C}^{3\times 3}$ *be tensors of rank* 2. *Then, the following properties are fulfilled*

$$\boldsymbol{a}(\boldsymbol{b} : \boldsymbol{c}) = (\boldsymbol{a} \otimes \boldsymbol{b}) : \boldsymbol{c},$$
$$(\boldsymbol{a} : \boldsymbol{b})(\boldsymbol{c} : \boldsymbol{d}) = \boldsymbol{a} : [(\boldsymbol{b} \otimes \boldsymbol{c}) : \boldsymbol{d}].$$

Proof. Again, the proof is straight forward.

- The left-hand side can be written by

$$\boldsymbol{a}(\boldsymbol{b} : \boldsymbol{c}) = \boldsymbol{a} \left(\sum_{k,l=1}^{3} \boldsymbol{b}_{k,l} \boldsymbol{c}_{k,l} \right) = \left(\sum_{k,l=1}^{3} \boldsymbol{a}_{i,j} \boldsymbol{b}_{k,l} \boldsymbol{c}_{k,l} \right)_{i,j=1,2,3}$$

  and the right-hand side by

$$(\boldsymbol{a} \otimes \boldsymbol{b}) : \boldsymbol{c} = (\boldsymbol{a}_{i,j} \boldsymbol{b}_{k,l})_{i,j,k,l=1,2,3} : \boldsymbol{c} = \left(\sum_{k,l=1}^{3} \boldsymbol{a}_{i,j} \boldsymbol{b}_{k,l} \boldsymbol{c}_{k,l} \right)_{i,j=1,2,3}.$$

  So, both sides are equal and we obtain the first property.

- For the second property, we get on the one hand

$$(\boldsymbol{a} : \boldsymbol{b})(\boldsymbol{c} : \boldsymbol{d}) = \left(\sum_{i,j=1}^{3} \boldsymbol{a}_{i,j} \boldsymbol{b}_{i,j} \right) \left(\sum_{k,l=1}^{3} \boldsymbol{c}_{k,l} \boldsymbol{d}_{k,l} \right) = \sum_{i,j,k,l=1}^{3} \boldsymbol{a}_{i,j} \boldsymbol{b}_{i,j} \boldsymbol{c}_{k,l} \boldsymbol{d}_{k,l}$$

  and on the other hand, we see with the proof of the first property

$$\begin{aligned} \boldsymbol{a} : [(\boldsymbol{b} \otimes \boldsymbol{c}) : \boldsymbol{d}] &= \boldsymbol{a} : \left(\sum_{k,l=1}^{3} \boldsymbol{b}_{i,j} \boldsymbol{c}_{k,l} \boldsymbol{d}_{k,l} \right)_{i,j=1,2,3} \\ &= \sum_{i,j,k,l=1}^{3} \boldsymbol{a}_{i,j} \boldsymbol{b}_{i,j} \boldsymbol{c}_{k,l} \boldsymbol{d}_{k,l}. \end{aligned}$$

  Altogether, this leads to the proposition.

□

2.3 Spherical Nomenclature

Next, we look at the notations for spherical calculations [28, 33, 58]. We start with contents borrowed mainly from [58]. We begin with basic definitions.

Definition 2.3.1. *The unit sphere Ω of the three-dimensional Euclidean space $\mathbb{R}^3$ is represented by*

$$\Omega = \left\{ x \in \mathbb{R}^3 \;\middle|\; |x| = 1 \right\}.$$

Definition 2.3.2. *We denote $\varepsilon^r, \varepsilon^\varphi, \varepsilon^t$ as a local orthonormal basis on the unit sphere Ω given by*

$$\xi = \varepsilon^r = \begin{pmatrix} \sqrt{1-t^2}\ \cos\varphi \\ \sqrt{1-t^2}\ \sin\varphi \\ t \end{pmatrix}, \qquad \varepsilon^\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \qquad \varepsilon^t = \begin{pmatrix} -t\ \cos\varphi \\ -t\ \sin\varphi \\ \sqrt{1-t^2} \end{pmatrix},$$

where $t \in [-1,1]$ is the polar distance and $\varphi \in [0,2\pi)$ is the longitude. Note that ε^r is radially outward, ε^φ eastward and ε^t northward. Moreover, for $t=-1$, we obtain the South pole and for $t=1$ the North pole.

Definition 2.3.3. *We define the tensors [33]*

$$\mathbf{i}_{\tan}(\xi) := \varepsilon^\varphi \otimes \varepsilon^\varphi + \varepsilon^t \otimes \varepsilon^t$$

and

$$\mathbf{j}_{\tan}(\xi) := \varepsilon^t \otimes \varepsilon^\varphi - \varepsilon^\varphi \otimes \varepsilon^t$$

for $\xi = \xi(t,\varphi) \in \Omega$.

Note that we use in the following the notation for partial differentiation

$$\partial_x := \frac{\partial}{\partial x}.$$

Then, we can define the following well-known differential operators.

Definition 2.3.4. *The gradient is defined by*

$$\nabla_x := (\partial_{x_i})_{i=1,2,3} = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

and the Laplace operator by

$$\Delta_x := \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$$

for $x = (x_1,x_2,x_3)^{\mathrm{T}} \in D \subset \mathbb{R}^3$.

In the following, we introduce the spherical differential operators.

Definition 2.3.5. *The surface gradient*

$$\nabla_\xi^* := \varepsilon^\varphi \ \frac{1}{\sqrt{1-t^2}}\ \partial_\varphi + \varepsilon^t\ \sqrt{1-t^2}\ \partial_t$$

and the surface curl gradient

$$L_\xi^* := -\varepsilon^\varphi\ \sqrt{1-t^2}\ \partial_t + \varepsilon^t\ \frac{1}{\sqrt{1-t^2}}\ \partial_\varphi$$

for $\xi = \xi(t,\varphi) \in \Omega$ are differential operators on the sphere such that $\nabla_{r\xi} = \left(\xi\partial_r + \frac{1}{r}\ \nabla_\xi^*\right)$ for $r \in \mathbb{R}^+$, $\xi \in \Omega$ and $L_\xi^* = \xi \wedge \nabla_\xi^*$. $\xi = \xi(t,\varphi)$ is the polar coordinate representation of $\xi \in \Omega$.

Moreover,

$$\Delta^*_\xi := \partial_t\left(\left(1-t^2\right)\partial_t\right) + \frac{1}{1-t^2}\,\partial^2_\varphi$$

is the Beltrami operator such that $\Delta^* = \nabla^* \cdot \nabla^* = L^* \cdot L^*$.

Lemma 2.3.6. *With the previous definition, it is simple to show that*

$$\begin{aligned}\xi F(\xi) \cdot \nabla^*_\xi F(\xi) &= 0,\\ \xi F(\xi) \cdot L^*_\xi F(\xi) &= 0,\\ \nabla^*_\xi F(\xi) \cdot L^*_\xi F(\xi) &= 0\end{aligned}$$

for $\xi \in \Omega$ *and* $F \in \mathrm{C}^{(2)}(\Omega)$.

Lemma 2.3.7. *The Laplace operator can be decomposed into spherical components given for* $x = r\xi \in \mathbb{R}^3$, $r = |x| \in \mathbb{R}$, *and* $\xi \in \Omega$ *by*

$$\Delta_x = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta^*_\xi.$$

This enables us to write down the well-known Green's second surface identity [58]. Note that all integrals in this thesis are Lebesgue-integrals.

Theorem 2.3.8. *Green's second surface identity is given by*

$$\int_\Gamma \left(F(\xi)\Delta^*_\xi G(\xi) - G(\xi)\Delta^*_\xi F(\xi)\right)\, \mathrm{d}\omega(\xi) = \int_{\partial\Gamma}\left(F(\xi)\frac{\partial}{\partial\nu(\xi)}G(\xi) - G(\xi)\frac{\partial}{\partial\nu(\xi)}F(\xi)\right)\, \mathrm{d}\sigma(\xi),$$

where $F, G \in \mathrm{C}^{(2)}\left(\overline{\Gamma}\right)$, $\Gamma \subset \Omega$ *with a sufficiently smooth boundary and* ν *is the outward unit normal vector field to* $\partial\Gamma$.

Then, Green's second surface identity on the unit sphere is given by

$$\int_\Omega \left(F(\xi)\Delta^*_\xi G(\xi) - G(\xi)\Delta^*_\xi F(\xi)\right)\, \mathrm{d}\omega(\xi) = 0$$

for $F, G \in \mathrm{C}^{(2)}(\Omega)$.

Next, we deal with spherical function spaces.

Definition 2.3.9. *For* $1 \leq p < \infty$ *and a (Lebesgue) measurable set* $D \subset \mathbb{R}^n$, *we denote with* $\mathscr{L}^p(D, \mathbb{R}^m)$ *the space of all (Lebesgue) measurable functions* $F : D \to \mathbb{R}^m$ *with*

$$\int_D |F(x)|^p \, \mathrm{d}x < \infty.$$

Let $\mathscr{N}^p(D, \mathbb{R}^m)$ *be the space of all (Lebesgue) measurable functions* $F : D \to \mathbb{R}^m$ *with*

$$\int_D |F(x)|^p \, \mathrm{d}x = 0.$$

Then, we define $(\mathrm{L}^p(D, \mathbb{R}^m), \|\cdot\|_p)$ *by*

$$\mathrm{L}^p(D, \mathbb{R}^m) := \mathscr{L}^p(D, \mathbb{R}^m) / \mathscr{N}^p(D, \mathbb{R}^m),$$

where

$$\|F\|_p := \|F\|_{\mathrm{L}^p(D,\mathbb{R}^m)} := \left(\int_D |F(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Theorem 2.3.10. *The function space* $(\mathrm{L}^p(D,\mathbb{R}^m), \|\cdot\|_p)$ *is a Banach space.*

Note that functions, which are almost everywhere equal, are considered to be identical. So, all functions F, where $\|F\|_{\mathrm{L}^p(D)} = 0$, are identified with the zero function.

In this thesis, we only use $p = 2$ and $m = 1, 3, 3 \times 3$. The case $m = 1$ delivers scalar, $m = 3$ vectorial, and $m = 3 \times 3$ tensorial function spaces. So, we write $\mathrm{L}^2(D,\mathbb{R}) = \mathrm{L}^2(D)$, $\mathrm{L}^2(D,\mathbb{R}^3) = \mathrm{l}^2(D)$, and $\mathrm{L}^2(D,\mathbb{R}^{3\times 3}) = \mathbf{l}^2(D)$.

Theorem 2.3.11. $(\mathrm{L}^2(D), \|.\|_2)$ *is a Hilbert space. Furthermore, we denote with* $(\mathrm{l}^2(D), \|.\|_2)$ *the Hilbert space of square-integrable vector fields and with* $\left(\mathbf{l}^2(D), \|.\|_2\right)$ *the Hilbert space of square-integrable tensor fields [28, 33, 74].*

Theorem 2.3.12. *For all* $F \in \mathrm{C}(\Omega)$ *and for all* $p \in [1,\infty)$, *we get*

$$\|F\|_{\mathrm{L}^p(\Omega)} \leq (4\pi)^{\frac{1}{p}} \|F\|_{\mathrm{C}(\Omega)}.$$

Proof. The proposition follows directly by

$$\int_\Omega |F(\xi)|^p \, \mathrm{d}\omega(\xi) \leq \int_\Omega \max_{\eta\in\Omega} |F(\eta)|^p \, \mathrm{d}\omega(\xi) = \max_{\eta\in\Omega} |F(\eta)|^p \int_\Omega \mathrm{d}\omega(\xi) = \|F\|^p_{\mathrm{C}(\Omega)} 4\pi.$$

□

Theorem 2.3.13. *We get*

$$\overline{\mathrm{C}(\Omega)}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}} = \mathrm{L}^2(\Omega).$$

Definition 2.3.14. *We define the inner product on* $\mathrm{L}^2(\Omega)$ *for scalar functions* $F, G \in \mathrm{L}^2(\Omega)$ *by*

$$\langle F, G\rangle_{\mathrm{L}^2(\Omega)} := \int_\Omega F(\xi)\overline{G(\xi)} \, \mathrm{d}\omega(\xi).$$

The induced norm on $\mathrm{L}^2(\Omega)$ for $F \in \mathrm{L}^2(\Omega)$ coincides with the $\|\cdot\|_{\mathrm{L}^2(\Omega)}$-norm from above given by

$$\|F\|_{\mathrm{L}^2(\Omega)} := \sqrt{\langle F, F\rangle_{\mathrm{L}^2(\Omega)}}.$$

Analogously, we define the inner products for vector and tensor fields. We borrow this from [28, 33, 74].

Definition 2.3.15. *We define the inner product on* $\mathrm{l}^2(\Omega)$ *for vector fields* $f, g \in \mathrm{l}^2(\Omega)$ *by*

$$\langle f, g\rangle_{\mathrm{l}^2(\Omega)} := \int_\Omega f(\xi) \cdot \overline{g(\xi)} \, \mathrm{d}\omega(\xi).$$

Again, the induced norm on $\mathrm{l}^2(\Omega)$ for $f \in \mathrm{l}^2(\Omega)$ coincides with the $\|\cdot\|_{\mathrm{l}^2(\Omega)}$-norm from above given by

$$\|f\|_{\mathrm{l}^2(\Omega)} := \sqrt{\langle f, f\rangle_{\mathrm{l}^2(\Omega)}}.$$

Definition 2.3.16. *We define the inner product on* $\mathbf{l}^2(\Omega)$ *for tensor fields* $\boldsymbol{f}, \boldsymbol{g} \in \mathbf{l}^2(\Omega)$ *by*

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle_{\mathbf{l}^2(\Omega)} := \int_\Omega \boldsymbol{f}(\xi) : \overline{\boldsymbol{g}(\xi)} \, \mathrm{d}\omega(\xi).$$

Again, the induced norm on $\mathbf{l}^2(\Omega)$ for $\boldsymbol{f} \in \mathbf{l}^2(\Omega)$ coincides with the $\|\cdot\|_{\mathrm{L}^2(\Omega)}$-norm from above given by

$$\|\boldsymbol{f}\|_{\mathbf{l}^2(\Omega)} := \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle_{\mathbf{l}^2(\Omega)}}.$$

Theorem 2.3.17. *In analogy to the scalar case, we obtain for a vector field $f : \Omega \to \mathbb{R}^3$, $f \in \mathrm{c}(\Omega)$,*

$$\|f\|_{\mathrm{l}^2(\Omega)} \leq \sqrt{4\pi}\|f\|_{\mathrm{c}(\Omega)}$$

and

$$\overline{\mathrm{c}(\Omega)}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}} = \mathrm{l}^2(\Omega).$$

For a tensor field $\boldsymbol{f} : \Omega \to \mathbb{R}^{3\times 3}$, $\boldsymbol{f} \in \mathbf{c}(\Omega)$, we get

$$\|\boldsymbol{f}\|_{\mathbf{l}^2(\Omega)} \leq \sqrt{4\pi}\|\boldsymbol{f}\|_{\mathbf{c}(\Omega)}$$

and

$$\overline{\mathbf{c}(\Omega)}^{\|\cdot\|_{\mathbf{l}^2(\Omega)}} = \mathbf{l}^2(\Omega).$$

In this thesis, we use primarily the spaces of $\left(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)}\right)$, $\left(\mathrm{l}^2(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{l}^2(\Omega)}\right)$, and $\left(\mathbf{l}^2(\Omega), \langle\cdot,\cdot\rangle_{\mathbf{l}^2(\Omega)}\right)$.

2.4 Scalar Spherical Harmonics

As previously stated in the introduction, the scalar spherical harmonics are a fundamental tool in geosciences, in mathematics, and in physics. They have a huge field of applications and are already well-examined. For example, to construct a basis on the unit sphere, the spherical harmonics must be clearly known. Moreover, they are based on the Legendre polynomials, which are introduced as well. So, this chapter deals with the Legendre polynomials, the associated Legendre functions, the spherical harmonics, and their properties [28, 33, 58]. For example, we show recursion relations and the orthonormality not only for the Legendre polynomials and the associated Legendre functions, but also for the spherical harmonics. Furthermore, we introduce the function space for the spherical harmonics. Here, we mainly follow [58].

First, we start with the Legendre polynomials.

Definition 2.4.1. *A unique set of polynomials $\{P_n\}_{n\in\mathbb{N}_0}$ exists such that for all $n \in \mathbb{N}_0$*

1. *P_n is a polynomial of degree n on the domain $[-1, 1]$,*
2. *$\int_{-1}^{1} P_n(t)P_m(t)\ \mathrm{d}t = 0$ for all $n, m \in \mathbb{N}_0$, $n \neq m$,*
3. *$P_n(1) = 1$.*

These polynomials are called the Legendre polynomials.

Theorem 2.4.2. *The Legendre polynomials satisfy the Rodriguez formula*

$$P_n(t) = \frac{1}{2^n n!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left(t^2 - 1\right)^n,$$

where $t \in [-1, 1]$ and $n \in \mathbb{N}_0$.

Remark 2.4.3. *The Legendre polynomials fulfill for $t \in [-1, 1]$ the recursion relation*

$$P_n(t) - \frac{2n-1}{n}\, tP_{n-1}(t) + \frac{n-1}{n}\, P_{n-2}(t) = 0$$

for $n \geq 2$ with

$$P_0 \equiv 1, \quad P_1(t) = t.$$

Furthermore, with Theorem 2.4.2, the Legendre polynomials obviously satisfy for $t \in [-1, 1]$ the formula

$$P_n(-t) = (-1)^n P_n(t).$$

The next properties, like the recursion relations, the orthonormality, the differential equation, and the finite series version for the Legendre polynomials, we borrow from [33].

Theorem 2.4.4. *Further recursion relations for the Legendre polynomials for $t \in [-1, 1]$ are given by*

$$\begin{aligned}\left(t^2 - 1\right) P_n'(t) &= ntP_n(t) - nP_{n-1}(t),\\ &= -(n+1)tP_n(t) + (n+1)P_{n+1}(t)\end{aligned}$$

for $n \geq 0$ with $P_{-1} := 1$. From the previous remark, we get the recursion relation

$$(2n+1)tP_n(t) = nP_{n-1}(t) + (n+1)P_{n+1}(t).$$

Lemma 2.4.5. *The Legendre polynomials are orthogonal with respect to $\mathrm{L}^2([-1, 1])$. This means that for $n, m \in \mathbb{N}_0$*

$$\int_{-1}^{1} P_n(t)P_m(t)\ \mathrm{d}t = \frac{2}{2n+1}\ \delta_{n,m}.$$

Lemma 2.4.6. *The Legendre polynomials fulfill for $t \in [-1, 1]$ and $n \in \mathbb{N}_0$ the following differential equation*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(1 - t^2\right)\frac{\mathrm{d}}{\mathrm{d}t} + n(n+1)\right) P_n(t) = 0.$$

Lemma 2.4.7. *The Legendre polynomials can also be written for $t \in [-1, 1]$ and $n \in \mathbb{N}_0$ by*

$$P_n(t) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s \frac{(2n-2s)!}{2^n (n-2s)!(n-s)!s!}\ t^{n-2s}.$$

This follows from Theorem 2.4.2, the Leibniz rule and the binomial formula.

Lemma 2.4.8. *For the Legendre polynomials, we obtain the recursion relation for $t \in [-1, 1]$*

$$P_{n+1}'(t) - P_{n-1}'(t) = (2n+1)P_n(t)$$

for all $n \in \mathbb{N}_0$ with $P_{-1}(t) := 1$.

Proof. Let $n \in \mathbb{N}_0$ and $t \in [-1, 1]$. We know from the recursion relations of Theorem 2.4.4

$$\begin{aligned}\left(t^2 - 1\right) P_n'(t) &= ntP_n(t) - nP_{n-1}(t),\\ &= -(n+1)tP_n(t) + (n+1)P_{n+1}(t),\\ (2n+1)tP_n(t) &= nP_{n-1}(t) + (n+1)P_{n+1}(t).\end{aligned}$$

Then, with these relations, we obtain

$$
\begin{aligned}
P'_{n+1}(t) - P'_{n-1}(t) &= \frac{1}{t^2-1}\left[(n+1)tP_{n+1}(t) - (n+1)P_n(t) + ntP_{n-1}(t) - nP_n(t)\right] \\
&= \frac{1}{t^2-1}\left[t\left(nP_{n-1}(t) + (n+1)P_{n+1}(t)\right) - (2n+1)P_n(t)\right] \\
&= \frac{1}{t^2-1}\left[(2n+1)t^2P_n(t) - (2n+1)P_n(t)\right] \\
&= \frac{1}{t^2-1}\ (2n+1)P_n(t)(t^2-1) \\
&= (2n+1)P_n(t).
\end{aligned}
$$

□

Corollary 2.4.9. *We conclude from the previous lemma that*

$$\int_b^1 P_n(t)\ \mathrm{d}t = \frac{1}{2n+1}\left[P_{n-1}(b) - P_{n+1}(b)\right]$$

for all $n \in \mathbb{N}_0$ *with* $P_{-1}(t) := 1$ *and* $b \in [-1,1]$.

Proof. Let $n \in \mathbb{N}_0$ and $t \in [-1,1]$. From the previous lemma, we know that

$$P'_{n+1}(t) - P'_{n-1}(t) = (2n+1)P_n(t).$$

Then, with the Definition 2.4.1, we get

$$
\begin{aligned}
(2n+1)\int_b^1 P_n(t)\ \mathrm{d}t &= \int_b^1 \left[P'_{n+1}(t) - P'_{n-1}(t)\right]\ \mathrm{d}t \\
&= \underbrace{P_{n+1}(1)}_{=1} - P_{n+1}(b) - \underbrace{P_{n-1}(1)}_{=1} + P_{n-1}(b) \\
&= P_{n-1}(b) - P_{n+1}(b).
\end{aligned}
$$

□

Next, we deal with the associated Legendre functions and collect their properties from [33].

Definition 2.4.10. *The functions defined by*

$$P_{n,j}(t) := \left(1-t^2\right)^{\frac{j}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j P_n(t)$$

for $t \in [-1,1]$, $n \in \mathbb{N}_0$ *and* $j = 0,\dots,n$, *are the associated Legendre functions.*

Remark 2.4.11. *From Definition 2.4.10, we get directly with Definition 2.4.1 and with Remark 2.4.3 that for* $t \in [-1,1]$, $n \in \mathbb{N}_0$, *and* $j = 0,\dots,n$

$$
\begin{aligned}
P_{n,j}(1) &= \delta_{j,0}, \\
P_{n,j}(-t) &= (-1)^{n+j}P_{n,j}(t).
\end{aligned}
$$

Theorem 2.4.12. *The associated Legendre functions* $P_{n,j}$ *satisfy the following recursion* *relations for* $t \in [-1,1]$, $n \in \mathbb{N}_0$, *and* $j = 0, \ldots, n$

$$\begin{aligned}\left(t^2-1\right)\frac{\mathrm{d}}{\mathrm{d}t}P_{n,j}(t) &= ntP_{n,j}(t) - (n+j)P_{n-1,j}(t),\\ &= -(n+1)tP_{n,j}(t) + (n+1-j)P_{n+1,j}(t),\\ (2n+1)tP_{n,j}(t) &= (n+j)P_{n-1,j}(t) + (n+1-j)P_{n+1,j}(t),\end{aligned}$$

where $P_{n,j} := 0$ *for* $n < j$.

Lemma 2.4.13. *For the associated Legendre functions, we get for* $n, n' \in \mathbb{N}_0$ *and* $j = 0, \ldots, n$,

$$\int_{-1}^{1} P_{n,j}(t)P_{n',j}(t)\mathrm{d}t = \frac{2}{2n+1}\frac{(n+j)!}{(n-j)!}\,\delta_{n,n'}.$$

Lemma 2.4.14. *The associated Legendre functions fulfill the following differential equation for* $t \in [-1,1]$, $n \in \mathbb{N}_0$, *and* $j = 0, \ldots, n$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(1-t^2\right)\frac{\mathrm{d}}{\mathrm{d}t} + n(n+1) - \frac{j^2}{1-t^2}\right)P_{n,j}(t) = 0.$$

Lemma 2.4.15. *We can write the associated Legendre functions for* $t \in [-1,1]$, $n \in \mathbb{N}_0$, *and* $j = 0, \ldots, n$, *by [33]*

$$P_{n,j}(t) = \left(1-t^2\right)^{\frac{j}{2}} \sum_{k=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} (-1)^k \,\frac{(2n-2k)!}{2^n k!(n-k)!(n-j-2k)!}\, t^{n-j-2k}.$$

Now, we go on with the spherical harmonics, where we follow again mainly [58]. We start with the function spaces for the spherical harmonics.

Definition 2.4.16. *Let* $D \subset \mathbb{R}^3$ *be open and connected. A twice continuously differentiable function* $F \in \mathrm{C}^{(2)}(D)$ *is called harmonic, if for all* $x \in D$ *the equation*

$$\Delta_x F(x) = 0$$

is fulfilled.

Definition 2.4.17. *A, in general, complex-valued polynomial* P *on* $\mathbb{R}^m$, $m \in \mathbb{N}$, *is called homogeneous of degree* $n \in \mathbb{N}_0$, *if there exist constants* C_α, *which do not all vanish, such that*

$$P(x) = \sum_{|\alpha|=n} C_\alpha x^\alpha$$

for all $x \in \mathbb{R}^m$, *where* $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, $|\alpha| := \sum_{i=1}^m \alpha_i$, *and* $x^\alpha := \prod_{i=1}^m x_i^{\alpha_i}$.

$\mathrm{Hom}_n(\mathbb{R}^m)$ *denotes the set of the homogeneous polynomials of degree* n.

Definition 2.4.18. $\mathrm{Harm}_n\left(\mathbb{R}^3\right)$, $n \in \mathbb{N}_0$, *denotes the space of all homogeneous harmonic polynomials. We also define the spaces*

$$\mathrm{Harm}_{0\ldots n}\left(\mathbb{R}^3\right) := \bigoplus_{i=0}^{n} \mathrm{Harm}_i\left(\mathbb{R}^3\right)$$

and

$$\mathrm{Harm}_{0\ldots\infty}\left(\mathbb{R}^3\right) := \bigcup_{i=0}^{\infty} \mathrm{Harm}_{0\ldots i}\left(\mathbb{R}^3\right).$$

For $D \subset \mathbb{R}^3$, we denote

$$\mathrm{Harm}_n(D) := \left\{ P|_D \mid P \in \mathrm{Harm}_n\left(\mathbb{R}^3\right) \right\}$$

and

$$\mathrm{Harm}_{0\ldots\infty}(D) := \left\{ P|_D \mid P \in \mathrm{Harm}_{0\ldots\infty}\left(\mathbb{R}^3\right) \right\}.$$

From these definitions we obtain the space of the spherical harmonics. In the following, we look at them [58].

Definition 2.4.19. *The elements of* $\mathrm{Harm}_n(\Omega)$, $n \in \mathbb{N}_0$, *are called the spherical harmonics.*

Theorem 2.4.20. *For the dimension of* $\mathrm{Harm}_n(\Omega)$, *we get*

$$\dim \mathrm{Harm}_n(\Omega) = 2n + 1.$$

Now, we can introduce the spherical harmonics.

Theorem 2.4.21. *For* $Y_n \in \mathrm{Harm}_n(\Omega)$, $Y_m \in \mathrm{Harm}_m(\Omega)$, $n, m \in \mathbb{N}_0$, $m \neq n$, *we receive*

$$\langle Y_n, Y_m \rangle_{\mathrm{L}^2(\Omega)} = 0.$$

Definition 2.4.22. *The set* $\{Y_{n,j}\}_{j=-n,\ldots,n}$, $n \in \mathbb{N}_0$, *is a complete* $\mathrm{L}^2(\Omega)$*-orthonormal set in the function space* $\left(\mathrm{Harm}_n(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)}\right)$, *this means that*

1. $\langle Y_{n,j}, Y_{n,j'} \rangle_{\mathrm{L}^2(\Omega)} = \delta_{j,j'}$.
2. *If* $\langle F, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = 0$ *for all* $j = -n, \ldots, n$ *and* $F \in \mathrm{Harm}_n(\Omega)$, *then* $F = 0$.

Remark 2.4.23. *Because of the previous definition, all* $Y_n \in \mathrm{Harm}_n(\Omega)$, $n \in \mathbb{N}_0$, *can be written by*

$$Y_n = \sum_{j=-n}^{n} \langle Y_n, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}.$$

Definition 2.4.24. *The index n is called the degree and the index j is called the order of* $Y_{n,j}$.

Remark 2.4.25. *The set of the spherical harmonics* $\{Y_{n,j}\}_{n\in\mathbb{N}_0,\ j=-n,\ldots,n}$ *is an orthonormal set in the function space* $\left(\mathrm{Harm}_{0\ldots\infty}(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)}\right)$, *this means that*

$$\langle Y_{n,j}, Y_{n',j'} \rangle_{\mathrm{L}^2(\Omega)} = \delta_{n,n'}\delta_{j,j'}.$$

Remark 2.4.26. *It is obvious that for* $0 \leq p \leq q \leq \infty$ *[33]*

$$\mathrm{Harm}_{p\ldots q}(\Omega) = \mathrm{span}\left\{Y_{n,j}\right\}_{n=p,\ldots,q, j=-n,\ldots,n}.$$

The following theorems are very important for the use of the spherical harmonics.

Theorem 2.4.27. *The spherical harmonics are the eigenfunctions of the Beltrami operator, this means that for* $\xi \in \Omega$

$$\Delta^*_\xi Y_{n,j}(\xi) = -n(n+1) Y_{n,j}(\xi).$$

Theorem 2.4.28 (Spherical Addition Theorem)**.** *For $n \in \mathbb{N}_0$ and $\xi, \eta \in \Omega$, we obtain*

$$\sum_{j=-n}^{n} Y_{n,j}(\xi)\overline{Y_{n,j}(\eta)} = \frac{2n+1}{4\pi} \; P_n(\xi \cdot \eta).$$

Lemma 2.4.29. *For $t \in [-1,1]$ and $h \in (-1,1)$, we get*

$$\sum_{n=0}^{\infty} h^n P_n(t) = \frac{1}{\sqrt{1+h^2-2ht}}$$

and

$$\sum_{n=0}^{\infty} (2n+1) h^n P_n(t) = \frac{1-h^2}{(1+h^2-2ht)^{\frac{3}{2}}}.$$

Theorem 2.4.30 (Poisson integral formula)**.** *If F is continuous on Ω, then*

$$\lim_{h\to 1-} \sup_{\xi\in\Omega} \left| \frac{1}{4\pi} \int_{\Omega} \frac{(1-h^2)\, F(\eta)}{(1+h^2-2h(\xi\cdot\eta))^{\frac{3}{2}}} \; \mathrm{d}\omega(\eta) - F(\xi) \right| = 0.$$

Theorem 2.4.31. *Let $F \in \mathrm{C}(\Omega)$. Then,*

$$\sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} \, Y_{n,j}(\xi)$$

converges uniformly for all $\xi \in \Omega$ and for a fixed $h \in (0,1)$ and

$$\lim_{h\to 1-} \sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}(\xi) = F(\xi)$$

uniformly for all $\xi \in \Omega$.

Proof. Let $\xi \in \Omega$. We get the uniform convergence by

$$\begin{aligned}
&\sum_{n=N}^{\infty} h^n \sum_{j=-n}^{n} \left| \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} \, Y_{n,j}(\xi) \right| \\
&\leq \sum_{n=N}^{\infty} h^n \sum_{j=-n}^{n} \max_{m\in\mathbb{N}_0, k=-m,\ldots,m} \left| \langle F, Y_{m,k}\rangle_{\mathrm{L}^2(\Omega)} \right| \; \sqrt{\frac{2n+1}{4\pi}} \\
&= \underbrace{\max_{m\in\mathbb{N}_0, k=-m,\ldots,m} \left| \langle F, Y_{m,k}\rangle_{\mathrm{L}^2(\Omega)} \right|}_{<\infty, \text{ because } F\in\mathrm{C}(\Omega)\subset\mathrm{L}^2(\Omega)} \sum_{n=N}^{\infty} h^n (2n+1) \sqrt{\frac{2n+1}{4\pi}} \\
&\to 0 \text{ uniformly } \;(|h|<1)\,.
\end{aligned}$$

Furthermore, with Theorem 2.4.28, the spherical addition theorem, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} \, Y_{n,j}(\xi) &= \sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \int_{\Omega} F(\eta)\overline{Y_{n,j}(\eta)} \; \mathrm{d}\omega(\eta) Y_{n,j}(\xi) \\
&= \sum_{n=0}^{\infty} h^n \, \frac{2n+1}{4\pi} \int_{\Omega} F(\eta) P_n(\xi\cdot\eta) \; \mathrm{d}\omega(\eta).
\end{aligned}$$

Due to the dominated convergence theorem, we can interchange the summation and the integration. In addition, we use Lemma 2.4.29. Then,

$$\begin{aligned}\sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}(\xi) &= \frac{1}{4\pi}\int_\Omega \sum_{n=0}^{\infty} h^n(2n+1)P_n(\xi\cdot\eta)F(\eta)\,\mathrm{d}\omega(\eta)\\ &= \frac{1}{4\pi}\int_\Omega \frac{(1-h^2)\,F(\eta)}{(1+h^2-2h(\xi\cdot\eta))^{\frac{3}{2}}}\,\mathrm{d}\omega(\eta).\end{aligned}$$

With Theorem 2.4.30, the Poisson integral formula, we get

$$\sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}(\xi) \overset{h\to 1-}{\longrightarrow} F(\xi) \text{ uniformly for all } \xi\in\Omega.$$

□

In the following, we show the completeness of the spherical harmonics. First, we have to prepare this by some corollaries and a theorem.

Corollary 2.4.32. *The set $\{Y_{n,j}\}_{n\in\mathbb{N}_0, j=-n,\ldots,n}$ is closed in $\mathrm{C}(\Omega)$, this means that for all $F\in\mathrm{C}(\Omega)$ and for all $\varepsilon>0$, there exists $\sum_{n=0}^{L}\sum_{j=-n}^{n} D_{n,j}Y_{n,j}$ such that*

$$\left\| F - \sum_{n=0}^{L}\sum_{j=-n}^{n} D_{n,j}Y_{n,j}\right\|_{\mathrm{C}(\Omega)} \le \varepsilon.$$

Proof. Let $F\in\mathrm{C}(\Omega)$ and $\varepsilon>0$. Then, according to Theorem 2.4.31, a real number $h=h(\varepsilon)<1$ exists such that

$$\left\| F - \sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}\right\|_{\mathrm{C}(\Omega)} \le \frac{\varepsilon}{2}.$$

Additionally, because of the uniform convergence of the series for $L\to\infty$ with fixed $h=h(\varepsilon)$, there also exists $L=L(\varepsilon)\in\mathbb{N}_0$ such that

$$\left\| \sum_{n=0}^{\infty} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j} - \sum_{n=0}^{L} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}\right\|_{\mathrm{C}(\Omega)} \le \frac{\varepsilon}{2}.$$

So, we deduce

$$\left\| F - \sum_{n=0}^{L} h^n \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}\right\|_{\mathrm{C}(\Omega)} \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

□

Theorem 2.4.33. *The set $\{Y_{n,j}\}_{n\in\mathbb{N}_0, j=-n,\ldots,n}$ is closed in $\mathrm{C}(\Omega)$ with respect to $\|\cdot\|_{\mathrm{L}^2(\Omega)}$.*

Proof. From Theorem 2.3.12, we know that for all $G\in\mathrm{C}(\Omega)$ and for $p=2$

$$\|G\|_{\mathrm{L}^2(\Omega)} \le \sqrt{4\pi}\|G\|_{\mathrm{C}(\Omega)}.$$

With the previous corollary, we get the following. □

Corollary 2.4.34. *The set $\{Y_{n,j}\}_{n\in\mathbb{N}_0, j=-n,\ldots,n}$ is closed in $\left(\mathrm{L}^2(\Omega), \|\cdot\|_{\mathrm{L}^2(\Omega)}\right)$.*

Proof. From Theorem 2.3.13, we know that

$$\overline{\mathrm{C}(\Omega)}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}} = \mathrm{L}^2(\Omega).$$

Let $F \in \mathrm{L}^2(\Omega)$ and $\varepsilon > 0$. Then, $G \in \mathrm{C}(\Omega)$ exists such that

$$\|G - F\|_{\mathrm{L}^2(\Omega)} \leq \frac{\varepsilon}{2}.$$

We conclude from the previous theorem that for G a linear combination, given by $\sum_{n=0}^{L} \sum_{j=-n}^{n} D_{n,j} Y_{n,j}$, exists such that

$$\left\| G - \sum_{n=0}^{L} \sum_{j=-n}^{n} D_{n,j} Y_{n,j} \right\|_{\mathrm{L}^2(\Omega)} \leq \frac{\varepsilon}{2}.$$

All in all, we deduce

$$\left\| F - \sum_{n=0}^{L} \sum_{j=-n}^{n} d_{n,j} Y_{n,j} \right\|_{\mathrm{L}^2(\Omega)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

So, we can obtain the following important theorem.

Theorem 2.4.35. *The set* $\{Y_{n,j}\}_{n\in\mathbb{N}_0,\ j=-n,\ldots,n}$ *is complete in* $\left(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)}\right)$. *Consequently, for* $F, G \in \mathrm{L}^2(\Omega)$, *we get*

$$\lim_{L\to\infty} \left\| F - \sum_{n=0}^{L} \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j} \right\|_{\mathrm{L}^2(\Omega)} = 0.$$

This means that every $F \in \mathrm{L}^2(\Omega)$ *can be written uniquely in the* $\mathrm{L}^2(\Omega)$*-sense in terms of a Fourier series*

$$F = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} Y_{n,j}.$$

Moreover, the Parseval identity

$$\langle F, G\rangle_{\mathrm{L}^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} \overline{\langle G, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)}}$$

holds true.

Now, we can conclude the uniqueness of the spherical harmonics as the eigenfunctions of the Beltrami operator.

Theorem 2.4.36. *The spherical harmonics* $Y_{n,j}$ *are the only eigenfunctions of the Beltrami operator* Δ^* *to the eigenvalues* $-n(n+1)$.

Proof. This proof follows [58]. From Theorem 2.4.27, we know already that

$$\Delta^*_\xi Y_{n,j}(\xi) = -n(n+1) Y_{n,j}(\xi).$$

(1) Determination of the eigenvalues
We assume that there exists a $\lambda \neq -n(n+1)$ for all $n \in \mathbb{N}_0$ and a $K \in \mathrm{C}^{(2)}(\Omega)$ such that

$\Delta^* K = \lambda K$. We apply Green's second surface identity on the unit sphere, Theorem 2.3.8, so that for all $n \in \mathbb{N}_0$ and $j = -n, \ldots, n$

$$\begin{aligned} &\int_\Omega \Bigg(K(\xi)\, \underbrace{\overline{\Delta^*_\xi Y_{n,j}(\xi)}}_{= -n(n+1)\overline{Y_{n,j}(\xi)}} - \overline{Y_{n,j}(\xi)}\, \underbrace{\Delta^*_\xi K(\xi)}_{=\lambda K(\xi)} \Bigg) \, \mathrm{d}\omega(\xi) = 0 \\ \Leftrightarrow\ &\underbrace{(-n(n+1) - \lambda)}_{\neq 0} \int_\Omega K(\xi)\, \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) = 0 \\ &\Leftrightarrow \langle K, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = 0. \end{aligned}$$

From Theorem 2.4.35, we know that $\{Y_{n,j}\}_{n,j}$ is an orthonormal basis. So, we obtain $K = 0$ in $\mathrm{L}^2(\Omega)$ and therefore, λ is not an eigenvalue.

(2) Determination of the eigenfunctions
We assume that there exists a $\lambda = -k(k+1)$ for $k \in \mathbb{N}_0$ and a $K \in \mathrm{C}^{(2)}(\Omega)$, $K \neq 0$, such that $\Delta^* K = \lambda K$. In analogy to the previous considerations, we get for all $n \in \mathbb{N}_0$ and $j = -n, \ldots, n$ the equation

$$(-n(n+1) - \lambda) \langle K, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = 0,$$

where $-n(n+1) - \lambda \neq 0$ for $n \neq k$. Then, for all $n \in \mathbb{N}_0$, $n \neq k$, and $j = -n, \ldots, n$

$$\langle K, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = 0.$$

Again, we refer to Theorem 2.4.35 and use the fact that $\{Y_{n,j}\}_{n,j}$ is an orthonormal basis. So, we obtain

$$\begin{aligned} K &= \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \langle K, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} Y_{n,j} \\ &= \sum_{j=-k}^{k} \langle K, Y_{k,j} \rangle_{\mathrm{L}^2(\Omega)} Y_{k,j} \in \mathrm{Harm}_k(\Omega). \end{aligned}$$

□

That $\{Y_{n,j}\}_{n \in \mathbb{N}_0,\ j=-n,\ldots,n}$ is a complete $\mathrm{L}^2(\Omega)$-orthonormal set, does not lead to a unique basis. Therefore, the fully normalized spherical harmonics are introduced. The derivation can be done as shown in the proof of Theorem 3.4.9 for the spin-weighted case. That Theorem 3.4.9 holds true for the fully normalized spherical harmonics, as we will see in the proof of the corresponding theorem.

Definition 2.4.37. *We denote the (scalar) fully normalized spherical harmonics by*

$$Y_{n,j}(\xi(t,\varphi)) := X_{n,j}(t) e^{ij\varphi} := \begin{cases} (-1)^j \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-j)!}{(n+j)!}}\, P_{n,j}(t) e^{ij\varphi}, & j \geq 0 \\ (-1)^j\, \overline{Y_{n,-j}(\xi(t,\varphi))}, & j < 0 \end{cases}$$

and the fully normalized associated Legendre functions by

$$X_{n,j}(t) := \begin{cases} (-1)^j \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-j)!}{(n+j)!}}\, P_{n,j}(t), & j \geq 0 \\ (-1)^j X_{n,-j}(t), & j < 0 \end{cases},$$

where $\xi \in \Omega$ *and* $n \in \mathbb{N}_0$, $j = -n, \ldots, n$.

So, the set of fully normalized spherical harmonics constructs a basis of $\left(\mathrm{L}^2(\Omega), \langle \cdot, \cdot \rangle_{\mathrm{L}^2(\Omega)}\right)$, in which all $F \in \mathrm{L}^2(\Omega)$ can be represented.

There are algorithms for the fast and stable computation of Fourier series of functions $F \in \mathrm{L}^2(\Omega)$ and hence, for the spherical harmonics [68]. These algorithms are based on the fast Fourier transform for equispaced and for nonequispaced data [69]. This method is split into fast Fourier transforms for the exponentials and discrete Legendre function transforms for the fully normalized associated Legendre functions. This is because of the structure we gain from Definition 2.4.37 for the fully normalized spherical harmonics [16, 39, 50, 61, 86].

Theorem 2.4.38. *From Theorem 2.4.12, we conclude that the fully normalized spherical harmonics* $Y_{n,j}$ *fulfill the following recursion relations for* $\xi = \xi(t, \varphi) \in \Omega$, $n \in \mathbb{N}_0$, *and* $j = -n, \ldots, n$, *[33]*

$$\begin{aligned} \left(t^2 - 1\right) \partial_t Y_{n,j}(\xi) &= ntY_{n,j}(\xi) - (2n+1)c_{n,j}Y_{n-1,j}(\xi), \\ &= -(n+1)tY_{n,j}(\xi) + (2n+1)c_{n+1,j}Y_{n+1,j}(\xi), \\ tY_{n,j}(\xi) &= c_{n,j}Y_{n-1,j}(\xi) + c_{n+1,j}Y_{n+1,j}(\xi), \end{aligned}$$

where

$$c_{n,j} := \sqrt{\frac{(n-j)(n+j)}{(2n-1)(2n+1)}}$$

and $Y_{n,j} := 0$ *for* $n < |j|$.

Note that $c_{|j|,j} = 0$ for all $j \in \mathbb{Z}$ such that the terms with the spherical harmonics $Y_{n-1,j}$ for $n - 1 < j$ vanishes.

2.5 Vector Spherical Hamonics

In this section, we deal with the vector spherical harmonics of Hill [40]. Here, we summarize from [24, 26, 28, 33] and use mainly [33].

We start with the definition of the vector spherical harmonics that are separated into a normal and a tangential vector field for physically issues [33]. Furthermore, we handle the vector spherical harmonics consistent to the scalar spherical harmonics.

Definition 2.5.1. *The vector spherical harmonics of Hill [40] are defined by*

$$\begin{aligned} y_{n,j}^{(1)}(\xi) &:= \xi Y_{n,j}(\xi), \\ y_{n,j}^{(2)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} \nabla_\xi^* Y_{n,j}(\xi), \\ y_{n,j}^{(3)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} L_\xi^* Y_{n,j}(\xi) \end{aligned}$$

for $\xi \in \Omega$, $n \in \mathbb{N}_0$, $n \geq 0_i$, $j = -n, \ldots, n$, *and* $i = 1, 2, 3$ *with*

$$0_i := \begin{cases} 0, & i = 1 \\ 1, & i = 2, 3 \end{cases}.$$

Note that $y^{(1)}_{n,j}$ is normal and $y^{(2)}_{n,j}$, $y^{(3)}_{n,j}$ are tangential for the unit sphere Ω.

The index i is called the type of the vector spherical harmonics.

Remark 2.5.2. *With the notation from [33], we can write the vector spherical harmonics also by*

$$y^{(i)}_{n,j} = \left(\mu^{(i)}_n\right)^{-\frac{1}{2}} o^{(i)} Y_{n,j}$$

for $i = 1, 2, 3$, $n \in \mathbb{N}_0$, $n \geq 0_i$, and $j = -n, \ldots, n$, where

$$\mu^{(i)}_n := \begin{cases} 1, & i = 1 \\ n(n+1), & i = 2, 3 \end{cases}$$

and $o^{(i)} : \mathrm{C}^{(0_i)}(\Omega) \to \mathrm{c}(\Omega)$ with

$$o^{(1)}_\xi F(\xi) := \xi F(\xi),$$
$$o^{(2)}_\xi F(\xi) := \nabla^*_\xi F(\xi),$$
$$o^{(3)}_\xi F(\xi) := L^*_\xi F(\xi)$$

for $\xi \in \Omega$ and $F : \Omega \to \mathrm{C}^{(1)}(\Omega)$.

Corollary 2.5.3. *We get that*

$$y^{(i)}_{n,j}(\xi) \cdot \overline{y^{(i')}_{n,j}(\xi)} = 0$$

for $i \neq i'$ and $\xi \in \Omega$, because of Lemma 2.3.6.

In analogy to the scalar case, we define the function space for the vector spherical harmonics.

Definition 2.5.4. *The set $\mathrm{harm}^{(i)}_n(\Omega)$, $n \in \mathbb{N}_0$, $n \geq 0_i$, denotes the span of the vector spherical harmonics of degree n and type i. This means that for $i = 1, 2, 3$ and $0_i \leq p \leq q \leq \infty$, we get [24]*

$$\mathrm{harm}^{(i)}_{p\ldots q}(\Omega) := \mathrm{span}\left\{y^{(i)}_{n,j}\right\}_{n=p,\ldots,q, j=-n,\ldots,n}.$$

With

$$\mathrm{harm}_n := \mathrm{harm}^{(1)}_n, \qquad n = 0,$$
$$\mathrm{harm}_n := \bigoplus_{i=1}^{3} \mathrm{harm}^{(i)}_n, \qquad n \geq 1,$$

we define the space of the vector spherical harmonics of degree n.

Furthermore, we denote [74]

$$\mathrm{l}^2_{(i)}(\Omega) := \overline{\mathrm{harm}^{(i)}_{0_i\ldots\infty}(\Omega)}^{\|\cdot\|_{\mathrm{l}^2(\Omega)}}.$$

Lemma 2.5.5. $\left\{y^{(i)}_{n,j}\right\}_{i=1,2,3, n\in\mathbb{N}_0, n\geq 0_i,\ j=-n,\ldots,n}$ *is an orthonormal set in $\left(\mathrm{harm}_{0\ldots\infty}(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{l}^2(\Omega)}\right)$, this means that*

$$\left\langle y^{(i)}_{n,j}, y^{(i')}_{n',j'}\right\rangle_{\mathrm{l}^2(\Omega)} = \delta_{i,i'}\delta_{n,n'}\delta_{j,j'}.$$

Remark 2.5.6. *The previous lemma contains that the set $\left\{y^{(i)}_{n,j}\right\}_{j=-n,\ldots,n}$, $i = 1, 2, 3$, $n \in \mathbb{N}_0$, $n \geq 0_i$, is a complete $\mathrm{l}^2(\Omega)$-orthonormal set in $\left(\mathrm{harm}^{(i)}_n(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{l}^2(\Omega)}\right)$, this means that*

1. $\left\langle y_{n,j}^{(i)}, y_{n,j'}^{(i)} \right\rangle_{\mathrm{l}^2(\Omega)} = \delta_{j,j'}$.

2. *If* $\left\langle f, y_{n,j}^{(i)} \right\rangle_{\mathrm{l}^2(\Omega)} = 0$ *for all* $j = -n, \ldots, n$ *and* $f \in \mathrm{harm}_n^{(i)}(\Omega)$, *then* $f = 0$.

Corollary 2.5.7. *The set* $\left\{ y_{n,j}^{(i)} \right\}_{i=1,2,3, n\in\mathbb{N}_0, n\geq 0_i, j=-n,\ldots,n}$ *is closed in* $\mathrm{c}(\Omega)$, *this means that for all* $f \in \mathrm{c}(\Omega)$ *and for all* $\varepsilon > 0$, *there exists* $\sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} d_{n,j}^{(i)} y_{n,j}^{(i)}$ *such that*

$$\left\| f - \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} d_{n,j}^{(i)} y_{n,j}^{(i)} \right\|_{\mathrm{c}(\Omega)} \leq \varepsilon.$$

Theorem 2.5.8. *The set* $\left\{ y_{n,j}^{(i)} \right\}_{i=1,2,3, n\in\mathbb{N}_0, n\geq 0_i, j=-n,\ldots,n}$ *is closed in* $\mathrm{c}(\Omega)$ *with respect to* $\|\cdot\|_{\mathrm{l}^2(\Omega)}$.

Corollary 2.5.9. *The set* $\left\{ y_{n,j}^{(i)} \right\}_{i=1,2,3 n\in\mathbb{N}_0, n\geq 0_i, j=-n,\ldots,n}$ *is closed in* $\left(\mathrm{l}^2(\Omega), \|\cdot\|_{\mathrm{l}^2(\Omega)}\right)$.

Then, we get the following important theorem.

Theorem 2.5.10. *The set of the vector spherical harmonics* $\left\{ y_{n,j}^{(i)} \right\}_{i=1,2,3, n\in\mathbb{N}_0, n\geq 0_i, j=-n,\ldots,n}$ *is complete in* $\left(\mathrm{l}^2(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{l}^2(\Omega)}\right)$. *Consequently, for* $f, g \in \mathrm{l}^2(\Omega)$, *we get*

$$\lim_{L\to\infty} \left\| f - \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \left\langle f, y_{n,j}^{(i)} \right\rangle_{\mathrm{l}^2(\Omega)} y_{n,j}^{(i)} \right\|_{\mathrm{l}^2(\Omega)} = 0.$$

This means that every $f \in \mathrm{l}^2(\Omega)$ *can be written uniquely in the* $\mathrm{l}^2(\Omega)$*-sense in terms of a Fourier series by*

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=-n}^{n} \left\langle f, y_{n,j}^{(i)} \right\rangle_{\mathrm{l}^2(\Omega)} y_{n,j}^{(i)}.$$

Moreover, the Parseval identity

$$\langle f, g \rangle_{\mathrm{l}^2(\Omega)} = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=-n}^{n} \left\langle f, y_{n,j}^{(i)} \right\rangle_{\mathrm{l}^2(\Omega)} \overline{\left\langle g, y_{n,j}^{(i)} \right\rangle_{\mathrm{l}^2(\Omega)}}$$

holds true.

Consequently, we conclude the following theorem.

Theorem 2.5.11. *The vector spherical harmonics span the function space* $\mathrm{l}^2(\Omega)$. *This means that [74]*

$$\mathrm{l}^2(\Omega) = \bigoplus_{i=1}^{3} \mathrm{l}^2_{(i)}(\Omega).$$

Next, we recapitulate further properties for the vector spherical harmonics [27, 33].

Theorem 2.5.12. *The vector spherical harmonics are the eigenfunctions of the vectorial Beltrami operator* $\boldsymbol{\Delta}^*$, *this means that*

$$\boldsymbol{\Delta}^*_\xi y_n(\xi) = -n(n+1) y_n(\xi)$$

for $y_n \in \mathrm{harm}_n(\Omega)$ *and* $\xi \in \Omega$. *The vectorial Beltrami operator is defined for* $f \in \mathrm{c}^{(2)}(\Omega)$, $\xi \in \Omega$, *by*

$$\boldsymbol{\Delta}^*_\xi f(\xi) := \Delta^*_\xi f(\xi) - 2(\xi \wedge \nabla_\xi) \wedge f(\xi) - 2f(\xi).$$

Theorem 2.5.13 (Addition Theorem For Vector Spherical Harmonics)**.** *For* $i, i' = 1, 2, 3$ *and* $n \in \mathbb{N}_0$, $n \geq 0_i$, *and* $\xi, \eta \in \Omega$, *we get*

$$\sum_{j=-n}^{n} y^{(i)}_{n,j}(\xi) \otimes \overline{y^{(i')}_{n,j}(\eta)} = \frac{2n+1}{4\pi} \left(\mu^{(i)}_n\right)^{-\frac{1}{2}} \left(\mu^{(i')}_n\right)^{-\frac{1}{2}} o^{(i)}_\xi o^{(i')}_\eta P_n(\xi \cdot \eta).$$

Corollary 2.5.14. *Another kind of addition theorem is obviously with Corollary 2.5.3 given by*

$$\sum_{j=-n}^{n} \left| y^{(i)}_{n,j}(\xi) \right|^2 = \frac{2n+1}{4\pi}$$

for $i = 1, 2, 3$, $n \in \mathbb{N}_0$, $n \geq 0_i$ *and* $\xi \in \Omega$.

2.6 Tensor Spherical Harmonics

In this section, we deal with the tensor spherical harmonics by Freeden, Gervens, and Schreiner [27]. Here, we summarize from [24, 27, 28, 33, 74] and use mainly [33].

There are many different definitions of tensor spherical harmonics (for details see [90]). For example, the tensor spherical harmonics of the Newman-Penrose spin-weighted spherical harmonics from 1966 (see [63, 64, 65] and also [9, 34]), the Regge-Wheeler tensor harmonics from 1957 (see [71, 101, 73, 72]) and the pure-orbital tensor spherical harmonics of Mathews from 1962 (see [57]). The last ones are advanced by Zerilli (see [100, 101]) to the pure-spin tensor harmonics. These are the mostly used types of tensor spherical harmonics. For example, the tensor spherical harmonics of both Martinec (see [56]) and of Freeden, Gervens, and Schreiner (see [24, 27, 28, 33, 74]) are based on them.

The tensor fields play an important role in different satellite problems. For example, data from satellite gravity gradiometry like from the satellite mission GOCE are tensor valued, from which we can determine the gravitational field of the Earth. Furthermore, we handle the tensor spherical harmonics consistent to the scalar and vector spherical harmonics for the aim to combine different types of data derived from various sources like from terrestrial, airborne or satellite observations (for further details, see [33]).

We start with the definition of the tensor spherical harmonics of Zerilli [101]. They are defined in a canonical way to the vector spherical harmonics of Hill [40] from Chapter 2.5. Then, we proceed with the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27] and their properties.

Definition 2.6.1. *The tensor spherical harmonics of Zerilli [101] in the notation of Freeden, Gervens, and Schreiner [33] are based on the operator* $\boldsymbol{q}^{(i,k)} : \mathrm{C}^{(2)}(\Omega) \to \mathbf{c}(\Omega)$ *given for* $F : \Omega \to \mathrm{C}^{(2)}(\Omega)$ *by*

$$\boldsymbol{q}^{(1,1)}_\xi F(\xi) := (\xi \otimes \xi)\, F(\xi),$$
$$\boldsymbol{q}^{(1,2)}_\xi F(\xi) := \xi \otimes \nabla^*_\xi F(\xi),$$
$$\boldsymbol{q}^{(1,3)}_\xi F(\xi) := \xi \otimes L^*_\xi F(\xi),$$

$$\begin{aligned}
\boldsymbol{q}_\xi^{(2,1)} F(\xi) &:= \nabla_\xi^* F(\xi) \otimes \xi, \\
\boldsymbol{q}_\xi^{(2,2)} F(\xi) &:= \nabla_\xi^* \otimes \nabla_\xi^* F(\xi), \\
\boldsymbol{q}_\xi^{(2,3)} F(\xi) &:= \nabla_\xi^* \otimes L_\xi^* F(\xi), \\
\boldsymbol{q}_\xi^{(3,1)} F(\xi) &:= L_\xi^* Y_{n,j}(\xi) \otimes \xi, \\
\boldsymbol{q}_\xi^{(3,2)} F(\xi) &:= L_\xi^* \otimes \nabla_\xi^* F(\xi), \\
\boldsymbol{q}_\xi^{(3,3)} F(\xi) &:= L_\xi^* \otimes L_\xi^* F(\xi).
\end{aligned}$$

The pair (i,k) is called the type. Here, type $(1,1)$ is normal, types $(1,2)$ and $(1,3)$ are left normal/right tangential, types $(2,1)$ and $(3,1)$ are left tangential/right normal, and types (i,k), $i,k=2,3$, are left tangential but in general not right tangential.

Therefore, these tensor spherical harmonics are advanced of Freeden, Gervens, and Schreiner [27].

Definition 2.6.2. *The tensor spherical harmonics by Freeden, Gervens, and Schreiner [27] are defined for* $\xi \in \Omega$ *by*

$$\begin{aligned}
\boldsymbol{y}_{n,j}^{(1,1)}(\xi) &:= (\xi \otimes \xi)\, Y_{n,j}(\xi), \\
\boldsymbol{y}_{n,j}^{(1,2)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} \left(\xi \otimes \nabla_\xi^* Y_{n,j}(\xi)\right), \\
\boldsymbol{y}_{n,j}^{(1,3)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} \left(\xi \otimes L_\xi^* Y_{n,j}(\xi)\right), \\
\boldsymbol{y}_{n,j}^{(2,1)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} \left(\nabla_\xi^* Y_{n,j}(\xi) \otimes \xi\right), \\
\boldsymbol{y}_{n,j}^{(2,2)}(\xi) &:= \frac{1}{\sqrt{2}}\, \mathbf{i}_{\tan}(\xi) Y_{n,j}(\xi) \\
&= \frac{1}{\sqrt{2}}\, Y_{n,j}(\xi) \left(\varepsilon^\varphi \otimes \varepsilon^\varphi + \varepsilon^t \otimes \varepsilon^t\right), \\
\boldsymbol{y}_{n,j}^{(2,3)}(\xi) &:= \frac{1}{\sqrt{2n(n+1)(n(n+1)-2)}} \left[\left(\nabla_\xi^* \otimes \nabla_\xi^* - L_\xi^* \otimes L_\xi^*\right) Y_{n,j}(\xi) + 2\nabla_\xi^* Y_{n,j}(\xi) \otimes \xi\right], \\
\boldsymbol{y}_{n,j}^{(3,1)}(\xi) &:= \frac{1}{\sqrt{n(n+1)}} \left(L_\xi^* Y_{n,j}(\xi) \otimes \xi\right), \\
\boldsymbol{y}_{n,j}^{(3,2)}(\xi) &:= \frac{1}{\sqrt{2n(n+1)(n(n+1)-2)}} \left[\left(\nabla_\xi^* \otimes L_\xi^* + L_\xi^* \otimes \nabla_\xi^*\right) Y_{n,j}(\xi) + 2L_\xi^* Y_{n,j}(\xi) \otimes \xi\right], \\
\boldsymbol{y}_{n,j}^{(3,3)}(\xi) &:= \frac{1}{\sqrt{2}}\, \mathbf{j}_{\tan}(\xi) Y_{n,j}(\xi) \\
&= \frac{1}{\sqrt{2}}\, Y_{n,j}(\xi) \left(\varepsilon^t \otimes \varepsilon^\varphi - \varepsilon^\varphi \otimes \varepsilon^t\right)
\end{aligned}$$

for $n \in \mathbb{N}_0$, $n \geq 0_{ik}$, $j = -n, \ldots, n$, *and* $i,k = 1,2,3$ *with*

$$0_{ik} := \begin{cases} 0, & (i,k) = (1,1), (2,2), (3,3) \\ 1, & (i,k) = (1,2), (1,3), (2,1), (3,1) \\ 2, & (i,k) = (2,3), (3,2) \end{cases}.$$

Then, $\boldsymbol{y}_{n,j}^{(1,1)}$ is normal, $\boldsymbol{y}_{n,j}^{(1,2)}$, $\boldsymbol{y}_{n,j}^{(1,3)}$ are left normal/right tangential, $\boldsymbol{y}_{n,j}^{(2,1)}$, $\boldsymbol{y}_{n,j}^{(3,1)}$ are left tangential/ right normal, and $\boldsymbol{y}_{n,j}^{(2,2)}$, $\boldsymbol{y}_{n,j}^{(2,3)}$, $\boldsymbol{y}_{n,j}^{(3,2)}$, $\boldsymbol{y}_{n,j}^{(3,3)}$ are tangential.

Remark 2.6.3. *With the notation from [33], we can write the tensor spherical harmonics also by*

$$\boldsymbol{y}_{n,j}^{(i,k)} = \left(\boldsymbol{\mu}_n^{(i,k)}\right)^{-\frac{1}{2}} \boldsymbol{o}^{(i,k)} Y_{n,j}$$

for $i,k = 1,2,3$, $n \in \mathbb{N}_0$, $n \geq 0_{ik}$, *and* $j = -n, \ldots, n$, *where*

$$\boldsymbol{\mu}_n^{(i,k)} := \begin{cases} 1, & (i,k) = (1,1) \\ 2, & (i,k) = (2,2),(3,3) \\ n(n+1), & (i,k) = (1,2),(1,3),(2,1),(3,1) \\ 2n(n+1)(n(n+1)-2), & (i,k) = (2,3),(3,2) \end{cases}$$

and $\boldsymbol{o}^{(i,k)} : \mathrm{C}^{(0_i)}(\Omega) \to \boldsymbol{c}(\Omega)$ *for* $F : \Omega \to \mathrm{C}^{(2)}(\Omega)$ *with*

$$\begin{aligned}
\boldsymbol{o}_\xi^{(1,1)} F(\xi) &:= \xi \otimes \xi F(\xi), \\
\boldsymbol{o}_\xi^{(1,2)} F(\xi) &:= \xi \otimes \nabla_\xi^* F(\xi), \\
\boldsymbol{o}_\xi^{(1,3)} F(\xi) &:= \xi \otimes L_\xi^* F(\xi), \\
\boldsymbol{o}_\xi^{(2,1)} F(\xi) &:= \nabla_\xi^* F(\xi) \otimes \xi, \\
\boldsymbol{o}_\xi^{(2,2)} F(\xi) &:= \mathbf{i}_{\tan}(\xi) F(\xi), \\
\boldsymbol{o}_\xi^{(2,3)} F(\xi) &:= \left(\nabla_\xi^* \otimes \nabla_\xi^* - L_\xi^* \otimes L_\xi^*\right) F(\xi) + 2\nabla_\xi^* F(\xi) \otimes \xi, \\
\boldsymbol{o}_\xi^{(3,1)} F(\xi) &:= L_\xi^* F(\xi) \otimes \xi, \\
\boldsymbol{o}_\xi^{(3,2)} F(\xi) &:= \left(\nabla_\xi^* \otimes L_\xi^* + L_\xi^* \otimes \nabla_\xi^*\right) F(\xi) + 2L_\xi^* F(\xi) \otimes \xi, \\
\boldsymbol{o}_\xi^{(3,3)} F(\xi) &:= \mathbf{j}_{\tan}(\xi) F(\xi)
\end{aligned}$$

for $\xi \in \Omega$.

Remark 2.6.4. *As intermediate step from the tensor spherical harmonics of Zerilli [101] to those of Freeden, Gervens, and Schreiner [27] in [74] was used instead for* $\xi \in \Omega$

$$\begin{aligned}
\tilde{\boldsymbol{o}}_\xi^{(2,3)} F(\xi) &:= \left(\nabla_\xi^* \otimes \nabla_\xi^* - L_\xi^* \otimes L_\xi^*\right) F(\xi), \\
\tilde{\boldsymbol{o}}_\xi^{(3,2)} F(\xi) &:= \left(\nabla_\xi^* \otimes L_\xi^* + L_\xi^* \otimes \nabla_\xi^*\right) F(\xi).
\end{aligned}$$

Corollary 2.6.5. *We get that for* $\xi \in \Omega$

$$\boldsymbol{y}_{n,j}^{(i,k)}(\xi) : \overline{\boldsymbol{y}_{n,j}^{(i',k')}(\xi)} = 0$$

for $(i,k) \neq (i',k')$.

In analogy to the scalar and the vector case, we can formulate the space of the tensor spherical harmonics.

Definition 2.6.6. *The set* $\mathbf{harm}_n^{(i,k)}(\Omega)$, $n \in \mathbb{N}_0$, $n \geq 0_{ik}$, *denotes the span of the tensor spherical harmonics of degree n and type* (i,k). *This means that for* $i,k = 1,2,3$ *and* $0_{ik} \leq$

$p \le q \le \infty$ that [24]

$$\mathbf{harm}_{p\dots q}(\Omega) := \mathrm{span}\left\{\boldsymbol{y}_{n,j}^{(i,k)}\right\}_{n=p,\dots,q,j=-n,\dots,n}.$$

With

$$\begin{aligned}
\mathbf{harm}_n &:= \mathbf{harm}_n^{(1,1)} \oplus \mathbf{harm}_n^{(2,2)} \oplus \mathbf{harm}_n^{(3,3)}, \qquad n = 0,\\
\mathbf{harm}_n &:= \bigoplus_{\substack{i,k=1\\(i,k)\neq(2,3),(3,2)}}^{3} \mathbf{harm}_n^{(i,k)}, \qquad n = 1,\\
\mathbf{harm}_n &:= \bigoplus_{i,k=1}^{3} \mathbf{harm}_n^{(i,k)}, \qquad n \ge 2,
\end{aligned}$$

we define the space of the tensor spherical harmonics of degree n.

Furthermore, we denote [74]

$$\mathbf{l}^2_{(i,k)}(\Omega) := \overline{\mathbf{harm}^{(i,k)}_{0_{ik}\dots\infty}(\Omega)}^{\|\cdot\|_{\mathbf{l}^2(\Omega)}}.$$

Lemma 2.6.7. *The set of the tensor spherical harmonics $\left\{\boldsymbol{y}_{n,j}^{(i,k)}\right\}_{i,k=1,2,3,n\in\mathbb{N}_0,n\ge 0_{ik},\ j=-n,\dots,n}$ is an orthonormal set in the function space $\left(\mathbf{harm}_{0\dots\infty}(\Omega), \langle\cdot,\cdot\rangle_{\mathbf{l}^2(\Omega)}\right)$, this means that*

$$\left\langle \boldsymbol{y}_{n,j}^{(i,k)}, \boldsymbol{y}_{n',j'}^{(i',k')}\right\rangle_{\mathbf{l}^2(\Omega)} = \delta_{i,i'}\delta_{k,k'}\delta_{n,n'}\delta_{j,j'}.$$

Remark 2.6.8. *From the previous lemma, we can conclude that the set $\left\{\boldsymbol{y}_{n,j}^{(i,k)}\right\}_{j=-n,\dots,n}$, $i,k = 1,2,3$, $n \in \mathbb{N}_0$, $n \ge 0_{ik}$, is a complete $\mathbf{l}^2(\Omega)$-orthonormal set in $\left(\mathbf{harm}_n^{(i,k)}(\Omega), \langle\cdot,\cdot\rangle_{\mathbf{l}^2(\Omega)}\right)$, this means that*

1. $\left\langle \boldsymbol{y}_{n,j}^{(i,k)}, \boldsymbol{y}_{n,j'}^{(i,k)}\right\rangle_{\mathbf{l}^2(\Omega)} = \delta_{j,j'}$.
2. *If $\left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(i,k)}\right\rangle_{\mathbf{l}^2(\Omega)} = 0$ for all $j = -n,\dots,n$ and $\boldsymbol{f} \in \mathbf{harm}_n^{(i,k)}(\Omega)$, then $\boldsymbol{f} = 0$.*

Corollary 2.6.9. *The set $\left\{\boldsymbol{y}_{n,j}^{(i,k)}\right\}_{i,k=1,2,3,n\in\mathbb{N}_0,n\ge 0_{ik},j=-n,\dots,n}$ is closed in $\mathbf{c}(\Omega)$, this means that for all $\boldsymbol{f} \in \boldsymbol{c}(\Omega)$ and for all $\varepsilon > 0$, there exists $\sum_{i,k=1}^{3}\sum_{n=0_{ik}}^{L}\sum_{j=-n}^{n} \boldsymbol{d}_{n,j}^{(i,k)}\boldsymbol{y}_{n,j}^{(i,k)}$ such that*

$$\left\| \boldsymbol{f} - \sum_{i,k=1}^{3}\sum_{n=0_{ik}}^{L}\sum_{j=-n}^{n} \boldsymbol{d}_{n,j}^{(i,k)}\boldsymbol{y}_{n,j}^{(i,k)}\right\|_{\mathbf{c}(\Omega)} \le \varepsilon.$$

Theorem 2.6.10. *The set $\left\{\boldsymbol{y}_{n,j}^{(i,k)}\right\}_{i,k=1,2,3,n\in\mathbb{N}_0,n\ge 0_{ik},j=-n,\dots,n}$ is closed in $\mathbf{c}(\Omega)$ with respect to $\|\cdot\|_{\mathbf{l}^2(\Omega)}$.*

Corollary 2.6.11. *The set $\left\{\boldsymbol{y}_{n,j}^{(i,k)}\right\}_{i,k=1,2,3n\in\mathbb{N}_0,n\ge 0_{ik},j=-n,\dots,n}$ is closed in $\left(\mathbf{l}^2(\Omega), \|\cdot\|_{\mathbf{l}^2(\Omega)}\right)$.*

Then, we get the following important theorem.

Theorem 2.6.12. *The set of the tensor spherical harmonics* $\left\{\boldsymbol{y}_{n,j}^{(i,k)}\right\}_{i,k=1,2,3,n\geq 0_i,\ j=-n,\dots,n}$ *is complete in* $\left(\mathbf{l}^2(\Omega), \langle\cdot,\cdot\rangle_{\mathbf{l}^2(\Omega)}\right)$*. Consequently, for* $\boldsymbol{f},\boldsymbol{g}\in\mathbf{l}^2(\Omega)$*, we obtain*

$$\lim_{L\to\infty}\left\|\boldsymbol{f}-\sum_{i,k=1}^{3}\sum_{n=0_{ik}}^{L}\sum_{j=-n}^{n}\left\langle\boldsymbol{f},\boldsymbol{y}_{n,j}^{(i,k)}\right\rangle_{\mathbf{l}^2(\Omega)}\boldsymbol{y}_{n,j}^{(i,k)}\right\|_{\mathbf{l}^2(\Omega)}=0.$$

This means that every $\boldsymbol{f}\in\mathbf{l}^2(\Omega)$ *can be written uniquely in the* $\mathbf{l}^2(\Omega)$*-sense in terms of a Fourier series by*

$$\boldsymbol{f}=\sum_{i,k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=-n}^{n}\left\langle\boldsymbol{f},\boldsymbol{y}_{n,j}^{(i,k)}\right\rangle_{\mathbf{l}^2(\Omega)}\boldsymbol{y}_{n,j}^{(i,k)}.$$

Moreover, the Parseval identity

$$\langle\boldsymbol{f},\boldsymbol{g}\rangle_{\mathbf{l}^2(\Omega)}=\sum_{i,k=1}^{3}\sum_{n=0_{ik}}^{\infty}\sum_{j=-n}^{n}\left\langle\boldsymbol{f},\boldsymbol{y}_{n,j}^{(i,k)}\right\rangle_{\mathbf{l}^2(\Omega)}\overline{\left\langle\boldsymbol{g},\boldsymbol{y}_{n,j}^{(i,k)}\right\rangle_{\mathbf{l}^2(\Omega)}}$$

holds true.

Then, we can conclude the following theorem.

Theorem 2.6.13. *The tensor spherical harmonics span the function space* $\mathbf{l^2}(\Omega)$*. This means that*

$$\mathbf{l}^2(\Omega)=\bigoplus_{i,k=1}^{3}\mathbf{l}^2_{(i,k)}(\Omega).$$

Next, we recapitulate further properties like the tensorial Beltrami operator and the addition theorem for the tensor spherical harmonics [27, 33].

Theorem 2.6.14. *The tensor spherical harmonics are the eigenfunctions of the tensorial Beltrami operator* $\blacktriangle^*$*, this means that*

$$\blacktriangle^*_\xi\boldsymbol{y}_n(\xi)=-n(n+1)\boldsymbol{y}_n(\xi)$$

for $\boldsymbol{y}_n\in\mathbf{harm}_n(\Omega)$ *and* $\xi\in\Omega$*. The tensorial Beltrami operator is defined for* $\boldsymbol{f}\in\mathbf{c}^{(2)}(\Omega)$ *by*

$$\begin{aligned}\blacktriangle^*_\xi\boldsymbol{f}(\xi):=\ &\boldsymbol{p}_{\mathrm{nor,nor}}(\Delta^*_\xi+4)\boldsymbol{p}_{\mathrm{nor,nor}}\boldsymbol{f}(\xi)+\boldsymbol{p}_{\mathrm{nor,tan}}(\Delta^*_\xi+2)\boldsymbol{p}_{\mathrm{nor,tan}}\boldsymbol{f}(\xi)\\&+\boldsymbol{p}_{\mathrm{tan,nor}}(\Delta^*_\xi+2)\boldsymbol{p}_{\mathrm{tan,nor}}\boldsymbol{f}(\xi)+\boldsymbol{p}_{\mathrm{tan,tan}}(\Delta^*_\xi+2-2\boldsymbol{J}_\xi)\boldsymbol{p}_{\mathrm{tan,tan}}\boldsymbol{f}(\xi)\end{aligned}$$

for $\xi\in\Omega$*, where*

$$\boldsymbol{f}(\xi)=\sum_{i,k=1}^{3}\boldsymbol{f}_{i,k}\varepsilon^i\otimes\varepsilon^k,$$

$\boldsymbol{J}:\mathbf{c}(\Omega)\to\mathbf{c}(\Omega)$ *is defined by*

$$\boldsymbol{J}_\xi\boldsymbol{f}(\xi):=\boldsymbol{f}(\xi)-\sum_{i,k=1}^{3}\boldsymbol{f}_{i,k}\left(\xi\wedge\varepsilon^i\right)\otimes\left(\xi\wedge\varepsilon^k\right)$$

and

$$\begin{aligned}
\boldsymbol{p}_{\mathrm{nor,nor}}\boldsymbol{f}(\xi) &:= \xi \otimes \left(\xi^{\mathrm{T}}\boldsymbol{f}(\xi)\xi \otimes \xi\right),\\
\boldsymbol{p}_{\mathrm{nor,tan}}\boldsymbol{f}(\xi) &:= \xi \otimes \left(\xi^{\mathrm{T}}\boldsymbol{f}(\xi)\right) - \left(\xi^{\mathrm{T}}\boldsymbol{f}(\xi)\cdot\xi\right)\xi \otimes \xi,\\
\boldsymbol{p}_{\mathrm{tan,nor}}\boldsymbol{f}(\xi) &:= \boldsymbol{f}(\xi)\xi \otimes \xi - \xi \otimes \left(\xi^{\mathrm{T}}\boldsymbol{f}(\xi)\xi \otimes \xi\right),\\
\boldsymbol{p}_{\mathrm{tan,tan}}\boldsymbol{f}(\xi) &:= \boldsymbol{f}(\xi) - \xi \otimes \xi^{\mathrm{T}}\cdot\boldsymbol{f}(\xi) - \boldsymbol{f}(\xi)\xi \otimes \xi + \left(\xi^{\mathrm{T}}\boldsymbol{f}(\xi)\cdot\xi\right)\xi \otimes \xi.
\end{aligned}$$

Theorem 2.6.15 (Addition Theorem For Tensor Spherical Harmonics)**.** *For* $i,i',k,k' = 1,2,3$ *and* $n \in \mathbb{N}_0$, $n \geq 0_{ik}$, *and* $\xi,\eta \in \Omega$, *we get*

$$\sum_{j=-n}^{n} \boldsymbol{y}_{n,j}^{(i,k)}(\xi) \otimes \overline{\boldsymbol{y}_{n,j}^{(i',k')}(\eta)} = \frac{2n+1}{4\pi}\left(\boldsymbol{\mu}_n^{(i,k)}\right)^{-\frac{1}{2}}\left(\boldsymbol{\mu}_n^{(i',k')}\right)^{-\frac{1}{2}} \boldsymbol{o}_{\xi}^{(i,k)}\boldsymbol{o}_{\eta}^{(i',k')} P_n(\xi\cdot\eta).$$

Corollary 2.6.16. *Another kind of addition theorem is obviously with Corollary 2.6.5 given by*

$$\sum_{j=-n}^{n} \boldsymbol{y}_{n,j}^{(i,k)}(\xi) : \overline{\boldsymbol{y}_{n,j}^{(i,k)}(\xi)} = \frac{2n+1}{4\pi}$$

for $i,k = 1,2,3$, $n \in \mathbb{N}_0$, $n \geq 0_{ik}$, *and* $\xi \in \Omega$.

Next, we do some calculations for the tensor spherical harmonics. Let $\xi = \xi(t,\varphi) \in \Omega$. We obviously gain with Lemma 2.2.7 and with Definition 2.3.5

$$\begin{aligned}
\boldsymbol{y}_{n,j}^{(1,2)}(\xi) &= \frac{1}{\sqrt{n(n+1)}}\left[\frac{1}{\sqrt{1-t^2}}\,\partial_\varphi Y_{n,j}(\xi)\,\left(\xi\otimes\varepsilon^\varphi\right) + \sqrt{1-t^2}\,\partial_t Y_{n,j}(\xi)\,\left(\xi\otimes\varepsilon^t\right)\right],\\
\boldsymbol{y}_{n,j}^{(1,3)}(\xi) &= \frac{1}{\sqrt{n(n+1)}}\left[-\sqrt{1-t^2}\,\partial_t Y_{n,j}(\xi)\,\left(\xi\otimes\varepsilon^\varphi\right) + \frac{1}{\sqrt{1-t^2}}\,\partial_\varphi Y_{n,j}(\xi)\,\left(\xi\otimes\varepsilon^t\right)\right],\\
\boldsymbol{y}_{n,j}^{(2,1)}(\xi) &= \frac{1}{\sqrt{n(n+1)}}\left[\frac{1}{\sqrt{1-t^2}}\,\partial_\varphi Y_{n,j}(\xi)\,\left(\varepsilon^\varphi\otimes\xi\right) + \sqrt{1-t^2}\,\partial_t Y_{n,j}(\xi)\,\left(\varepsilon^t\otimes\xi\right)\right],\\
\boldsymbol{y}_{n,j}^{(3,1)}(\xi) &= \frac{1}{\sqrt{n(n+1)}}\left[-\sqrt{1-t^2}\,\partial_t Y_{n,j}(\xi)\,\left(\varepsilon^\varphi\otimes\xi\right) + \frac{1}{\sqrt{1-t^2}}\,\partial_\varphi Y_{n,j}(\xi)\,\left(\varepsilon^t\otimes\xi\right)\right],
\end{aligned}$$

and

$$\partial_\varphi\varepsilon^\varphi = \begin{pmatrix}-\cos(\varphi)\\ -\sin(\varphi)\\ 0\end{pmatrix},\qquad \partial_\varphi\varepsilon^t = -t\varepsilon^\varphi,\qquad \partial_t\varepsilon^\varphi = 0,\qquad \partial_t\varepsilon^t = -\frac{1}{\sqrt{1-t^2}}\,\xi.$$

Consequently, analogous formulae can also be derived for the remaining tensor spherical harmonics. We get for the tensor product of the surface gradient with the surface gradient of the spherical harmonics

$$\begin{aligned}
&\nabla_\xi^*\otimes\nabla_\xi^* Y_{n,j}(\xi)\\
&= \left(\varepsilon^\varphi\,\frac{1}{\sqrt{1-t^2}}\,\partial_\varphi + \varepsilon^t\sqrt{1-t^2}\partial_t\right)\otimes\left(\varepsilon^\varphi\,\frac{1}{\sqrt{1-t^2}}\,\partial_\varphi Y_{n,j} + \varepsilon^t\sqrt{1-t^2}\partial_t Y_{n,j}\right)\\
&= \frac{1}{1-t^2}\,\left(\varepsilon^\varphi\otimes\partial_\varphi\varepsilon^\varphi\right)\partial_\varphi Y_{n,j} + \frac{1}{1-t^2}\,\left(\varepsilon^\varphi\otimes\varepsilon^\varphi\right)\partial_\varphi^2 Y_{n,j} - t\left(\varepsilon^\varphi\otimes\varepsilon^\varphi\right)\partial_t Y_{n,j}\\
&\quad + \left(\varepsilon^\varphi\otimes\varepsilon^t\right)\partial_\varphi\partial_t Y_{n,j} + \left(\varepsilon^t\otimes\varepsilon^\varphi\right)\partial_t\partial_\varphi Y_{n,j} + \frac{t}{1-t^2}\,\left(\varepsilon^t\otimes\varepsilon^\varphi\right)\partial_\varphi Y_{n,j}
\end{aligned}$$

$$- \sqrt{1-t^2}\left(\varepsilon^t \otimes \xi\right) \partial_t Y_{n,j} + \left(1-t^2\right)\left(\varepsilon^t \otimes \varepsilon^t\right) \partial_t^2 Y_{n,j} - t\left(\varepsilon^t \otimes \varepsilon^t\right) \partial_t Y_{n,j},$$

and for the tensor product of the surface curl gradient with the surface curl gradient of the spherical harmonics, we obtain

$$\begin{aligned}
&L_\xi^* \otimes L_\xi^* Y_{n,j}(\xi) \\
&= \left(-\varepsilon^\varphi \sqrt{1-t^2}\partial_t + \varepsilon^t \frac{1}{\sqrt{1-t^2}}\partial_\varphi\right) \otimes \left(-\varepsilon^\varphi \sqrt{1-t^2}\partial_t Y_{n,j} + \varepsilon^t \frac{1}{\sqrt{1-t^2}}\partial_\varphi Y_{n,j}\right) \\
&= (1-t^2)\left(\varepsilon^\varphi \otimes \varepsilon^\varphi\right) \partial_t^2 Y_{n,j} - t\left(\varepsilon^\varphi \otimes \varepsilon^\varphi\right) \partial_t Y_{n,j} + \frac{1}{\sqrt{1-t^2}}\left(\varepsilon^\varphi \otimes \xi\right) \partial_\varphi Y_{n,j} \\
&\quad - \left(\varepsilon^\varphi \otimes \varepsilon^t\right) \partial_t \partial_\varphi Y_{n,j} - \frac{t}{1-t^2}\left(\varepsilon^\varphi \otimes \varepsilon^t\right) \partial_\varphi Y_{n,j} - \left(\varepsilon^t \otimes \partial_\varphi \varepsilon^\varphi\right) \partial_t Y_{n,j} \\
&\quad - \left(\varepsilon^t \otimes \varepsilon^\varphi\right) \partial_\varphi \partial_t Y_{n,j} - \frac{t}{1-t^2}\left(\varepsilon^t \otimes \varepsilon^\varphi\right) \partial_\varphi Y_{n,j} + \frac{1}{1-t^2}\left(\varepsilon^t \otimes \varepsilon^t\right) \partial_\varphi^2 Y_{n,j}.
\end{aligned}$$

Furthermore, for the mixed tensor products, we get

$$\begin{aligned}
&\nabla_\xi^* \otimes L_\xi^* Y_{n,j}(\xi) \\
&= \left(\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}}\partial_\varphi + \varepsilon^t \sqrt{1-t^2}\partial_t\right) \otimes \left(-\varepsilon^\varphi \sqrt{1-t^2}\partial_t Y_{n,j} + \varepsilon^t \frac{1}{\sqrt{1-t^2}}\partial_\varphi Y_{n,j}\right) \\
&= -\left(\varepsilon^\varphi \otimes \partial_\varphi \varepsilon^\varphi\right) \partial_t Y_{n,j} - \left(\varepsilon^\varphi \otimes \varepsilon^\varphi\right) \partial_\varphi \partial_t Y_{n,j} - \frac{t}{1-t^2}\left(\varepsilon^\varphi \otimes \varepsilon^\varphi\right) \partial_\varphi Y_{n,j} \\
&\quad + \frac{1}{1-t^2}\left(\varepsilon^\varphi \otimes \varepsilon^t\right) \partial_\varphi^2 Y_{n,j} - \left(1-t^2\right)\left(\varepsilon^t \otimes \varepsilon^\varphi\right) \partial_t^2 Y_{n,j} + t\left(\varepsilon^t \otimes \varepsilon^\varphi\right) \partial_t Y_{n,j} \\
&\quad - \frac{1}{\sqrt{1-t^2}}\left(\varepsilon^t \otimes \xi\right) \partial_\varphi Y_{n,j} + \left(\varepsilon^t \otimes \varepsilon^t\right) \partial_t \partial_\varphi Y_{n,j} + \frac{t}{1-t^2}\left(\varepsilon^t \otimes \varepsilon^t\right) \partial_\varphi Y_{n,j}
\end{aligned}$$

and

$$\begin{aligned}
&L_\xi^* \otimes \nabla_\xi^* Y_{n,j}(\xi) \\
&= \left(-\varepsilon^\varphi \sqrt{1-t^2}\partial_t + \varepsilon^t \frac{1}{\sqrt{1-t^2}}\partial_\varphi\right) \otimes \left(\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}}\partial_\varphi Y_{n,j} + \varepsilon^t \sqrt{1-t^2}\partial_t Y_{n,j}\right) \\
&= -\left(\varepsilon^\varphi \otimes \varepsilon^\varphi\right) \partial_t \partial_\varphi Y_{n,j} - \frac{t}{1-t^2}\left(\varepsilon^\varphi \otimes \varepsilon^\varphi\right) \partial_\varphi Y_{n,j} + \sqrt{1-t^2}\left(\varepsilon^\varphi \otimes \xi\right) \partial_t Y_{n,j} \\
&\quad - \left(1-t^2\right)\left(\varepsilon^\varphi \otimes \varepsilon^t\right) \partial_t^2 Y_{n,j} + t\left(\varepsilon^\varphi \otimes \varepsilon^t\right) \partial_t Y_{n,j} + \frac{1}{1-t^2}\left(\varepsilon^t \otimes \partial_\varphi \varepsilon^\varphi\right) \partial_\varphi Y_{n,j} \\
&\quad + \frac{1}{1-t^2}\left(\varepsilon^t \otimes \varepsilon^\varphi\right) \partial_\varphi^2 Y_{n,j} - t\left(\varepsilon^t \otimes \varepsilon^\varphi\right) \partial_t Y_{n,j} + \left(\varepsilon^t \otimes \varepsilon^t\right) \partial_\varphi \partial_t Y_{n,j}.
\end{aligned}$$

Note that $\nabla_\xi^* \otimes L_\xi^* Y_{n,j}(\xi)$ is *not* the transpose of $L_\xi^* \otimes \nabla_\xi^* Y_{n,j}(\xi)$. Here, '$\otimes$' only formally represents the rules of a dyadic product, transferred from the (complex) multiplication to the application of differential operators, where the commutativity gets lost. In the same manner, $\nabla_\xi^* \otimes \nabla_\xi^* Y_{n,j}(\xi)$ and $L_\xi^* \otimes L_\xi^* Y_{n,j}(\xi)$ are *not* symmetric tensors.

Chapter 3

Spin-Weighted Spherical Harmonics

This chapter deals with the spin-weighted spherical harmonics defined by Newman and Penrose [63]. An important fact of the definition is that the spin-weighted spherical harmonics of spin weight zero are the well-known fully normalized scalar spherical harmonics. So, in some literature, the spin-weighted spherical harmonics are also called the generalized spherical harmonics (except for some notations) [12]. The generalized spherical harmonics are previously mentioned in [88].
Another particular case are the so-called monopole harmonics [15]. The monopole harmonics solve the Schrödinger equation for a charged particle (for example an electron) in the field of a Dirac magnetic monopole [15, 88, 91].
This results in a vast field of applications of spin-weighted spherical harmonics in physics. They are used for the theory of gravitation [63], for example, for the study of the asymptotic behavior of the gravitational field in null directions [15, 91], and in early universe and classical cosmology, for example, for the analysis of the temperature and polarization anisotropies of the cosmic microwave background [97]. Additionally, they are frequently used in geophysics [12].
The spin-weighted spherical harmonics have a huge spectrum of versatility, because they can be formulated in terms of the Wigner D-functions, the Jacobi polynomials, the generalized associated Legendre functions, and the hypergeometric function [91]. So, they are expedient to describe scalar, vector, and tensor fields.
Furthermore, they can be used to define a basis transformation of the vector spherical harmonics of Hill [40] and the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27] with the advantage that their inner dot products are zero for different and one for equal basis elements [90]. We want to use this to define tensor Slepian functions for concentration problems on the sphere.
We do not only recapitulate the definition of the spin-weighted spherical harmonics, but also show and summarize several important properties in a unified mathematical notation. In this context, we prove novel recursion relations and determine a differential equation that leads to an operator of which they are the eigenfunctions. Moreover, we show the formulation by Wigner D-functions and the addition theorem. Furthermore, we characterize the relation between the spin-weighted spherical harmonics and the scalar, vector, and tensor spherical harmonics as described above.

3.1 Spin Weight and Function of Spin Weight

First, we explain the notation of the spin weight and of a function of spin weight $N \in \mathbb{Z}$.

Definition 3.1.1. *The coefficients $d_{i_1 i_2 \ldots i_{2n}} \in \mathbb{R}$ for $n \in \mathbb{N}_0$ are called totally symmetric, if they are equal for every permutation of the index $(i_1, i_2, \ldots, i_{2n}) \in \mathbb{N}_0^{2n}$.*

Definition 3.1.2. *A function* ${}_NF_n \in \mathrm{L}^2(\Omega)$ *is called a function of spin weight* $N \in \mathbb{Z}$ *and degree* $n \in \mathbb{N}_0$*, if it can be written as [91]*

$$ {}_NF_n = \sum_{i_1,\dots,i_{2n}=1}^{2} d_{i_1 i_2 \dots i_{2n}} \underbrace{o^{i_1} o^{i_2} \dots o^{i_{n+N}}}_{n+N} \underbrace{\hat{o}^{i_{n+N+1}} \dots \hat{o}^{i_{2n}}}_{n-N}, $$

where $|N| \le n$*, the coefficients* $d_{i_1 i_2 \dots i_{2n}} \in \mathbb{R}$ *are totally symmetric, and for* $\xi = \xi(t,\varphi) \in \Omega$*, we define*

$$ o^1(\xi) := o^1_\xi := e^{-i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}}, \qquad o^2(\xi) := o^2_\xi := e^{i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}}, $$
$$ \hat{o}^1(\xi) := \hat{o}^1_\xi := -e^{-i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}}, \qquad \hat{o}^2(\xi) := \hat{o}^2_\xi := e^{i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}}. $$

It is obvious that

$$ \overline{o^i} = (-1)^j \hat{o}^j $$

for $i, j = 1, 2$ with $i \neq j$.
Note that the expansion in the functions o^i and $\hat{o}^i$ for $i = 1, 2$ is not unique, if the spin weight is unknown (see Remark 3.5.7).

Definition 3.1.3. *We define a function* ${}_NF \in \mathrm{L}^2(\Omega)$ *of spin weight* $N \in \mathbb{Z}$ *as a finite linear combination of functions of spin weight* N *and arbitrary degrees.*

Remark 3.1.4. *In the previous definition, we use a finite linear combination. Later, we will see that the closure leads to the whole* $\mathrm{L}^2(\Omega)$*.*

Definition 3.1.5. *With* $\mathrm{sw}(\cdot)$*, we denote the spin weight of a function. For the functions from the definition above, we have*

$$ \mathrm{sw}\left(o^k\right) = \frac{1}{2}, \qquad k = 1, 2, $$

and

$$ \mathrm{sw}\left(\hat{o}^k\right) = -\frac{1}{2}, \qquad k = 1, 2. $$

Remark 3.1.6. *Another definition for a function* ${}_NF \in \mathrm{L}^2(\Omega)$ *of spin weight* $N \in \mathbb{Z}$ *is , if it transforms under a right-handed rotation of* $(\varepsilon^\varphi, \varepsilon^t)$ *by an angle* ψ *as*

$$ {}_NF' = e^{-iN\psi}\ {}_NF. $$

For further details, see [63, 64, 65, 99].

Lemma 3.1.7. *With Definition 3.1.2, the following properties of* $\mathrm{sw}(\cdot)$ *can be easily proved. Let* ${}_NF_n$ *be a function of spin weight* $N \in \mathbb{Z}$ *and degree* $n \in \mathbb{N}_0$ *and* ${}_MG_m$ *be a function of spin weight* $M \in \mathbb{Z}$ *and degree* $m \in \mathbb{N}_0$*. Then, the following holds true.*

- $\mathrm{sw}\left(\overline{{}_NF_n}\right) = -\mathrm{sw}\left({}_NF_n\right) = -N$,
- $\mathrm{sw}\left({}_NF_n\ {}_MG_m\right) = \mathrm{sw}({}_NF_n) + \mathrm{sw}({}_MG_m) = N + M$,
- $\mathrm{sw}\left({}_NF_n + {}_MG_n\right) = N$*, if* $N = M$,
- $\mathrm{sw}\left({}_NF_n^k\right) = k\ \mathrm{sw}\left({}_NF_n\right) = kN$ *for* $k \in \mathbb{Z}$.

Remark 3.1.8. *So, with the previous lemma, we see that Definition 3.1.2 with help of Definition 3.1.5 defines for certain a function ${}_N F_n$ of spin weight $N \in \mathbb{Z}$, because*

$$\mathrm{sw}\left({}_N F_n\right) = (n+N)\,\mathrm{sw}\left(o^k\right) + (n-N)\,\mathrm{sw}\left(\hat{o}^k\right) = \frac{1}{2}(n+N) + \left(-\frac{1}{2}\right)(n-N) = N.$$

3.2 Definition of the Spin-Weighted Spherical Harmonics

In this chapter, the spin-weighted spherical harmonics of spin weight N, mainly for $N = 0, \pm 1, \pm 2$, are introduced. We start with basic notations.

Definition 3.2.1. *We define the unit sphere without the poles by*

$$\Omega_0 := \Omega / \left\{\xi = \xi(t,\varphi) \mid t = \pm 1\right\},$$

where $\xi = \xi(t,\varphi)$ is the polar coordinate representation of $\xi \in \Omega$.

Remark 3.2.2. *It is well known that the following identities hold true*

$$n(n+1) - N(N+1) = (n-N)(n+N+1), \tag{3.1}$$
$$\sum_{k=0}^{L}(2k+1) = (L+1)^2,$$
$$\sum_{k=n+1}^{L}(2k+1) = (L+1)^2 - (n+1)^2 = L(L+2) - n(n+2).$$

Now, we define spin-weighted differential operators.

Definition 3.2.3. *We define the spin-weighted differential operators $\eth_N : \mathrm{C}^{(1)}(\Omega_0) \to \mathrm{C}(\Omega_0)$ and $\bar{\eth}_N : \mathrm{C}^{(1)}(\Omega_0) \to \mathrm{C}(\Omega_0)$ of spin weight $N \in \mathbb{Q}$ by [63]*

$$\eth_N\ F(\xi) := \left(\sqrt{1-t^2}\ \partial_t + \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\right) F(\xi),$$
$$\bar{\eth}_N\ F(\xi) := \left(\sqrt{1-t^2}\ \partial_t - \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\right) F(\xi),$$

where $\xi = \xi(t,\varphi) \in \Omega_0$ and $F \in \mathrm{C}^{(1)}(\Omega_0)$.

Remark 3.2.4. *Note that $\bar{\eth}_N$ is not the complex conjugation of $\eth_N$. It is the complex conjugation of $\eth_{-N}$. This means that*

$$\bar{\eth}_N = \overline{\eth_{-N}}.$$

Definition 3.2.5. *The notation $\eth_N^M$ for $M \in \mathbb{Q}_0^+$ and $N \in \mathbb{Q}$ means the iterative use of the operator $\eth$ on a function $F \in \mathrm{C}^{(M)}(\Omega_0)$ such that*

$$\eth_N^M\ F := \eth_{N+M-1}\eth_{N+M-2}\dots\eth_{N+1}\eth_N\ F.$$

In the same way, we define $\bar{\eth}_N^M$ for the iterative use of the operator $\bar{\eth}$ by

$$\bar{\eth}_N^M\ F := \bar{\eth}_{N-M+1}\bar{\eth}_{N-M+2}\dots\bar{\eth}_{N-1}\bar{\eth}_N\ F.$$

The case $M = 0$ means that

$$\eth_N^0 = \mathrm{Id} = \overline{\eth}_N^0.$$

Next, we define the spin-weighted spherical harmonics of Newman and Penrose [63].

Definition 3.2.6. *The spin-weighted spherical harmonics of Newman and Penrose [63] (see also [15, 34, 53, 91, 96, 97]) are defined for $n \in \mathbb{N}_0$, $N \in \mathbb{Q}$, $n \geq |N|$, and $j = -n, \dots, n$ by*

$$_NY_{n,j} := \begin{cases} \sqrt{\frac{(n-N)!}{(n+N)!}}\, \eth_0^N\ Y_{n,j}, & 0 \leq N \leq n \\ (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}}\, \overline{\eth}_0^{-N}\ Y_{n,j}, & -n \leq N \leq 0 \\ 0, & n < |N| \end{cases}.$$

Note that we define the spin-weighted spherical harmonics on Ω_0, because the operators $\eth$ and $\overline{\eth}$ have singularities at the poles.

Remark 3.2.7. *Here, we need mostly $N \in \mathbb{Z}$. Mainly, we use $N = 0, \pm 1, \pm 2$. Additionally, in physics there is use of $N = \pm\frac{1}{2}, \pm\frac{3}{2}$ see also [18, 93].*

Lemma 3.2.8. *The spin-weighted spherical harmonics fulfill the following properties for $n \in \mathbb{N}_0$ and $N \in \mathbb{Z}$ [15, 34, 53, 63, 91, 96, 97]*

- $_0Y_{n,j} = Y_{n,j}$ *for $n \geq 0$, $j = -n, \dots, n$,*
- $\eth_N\ {}_NY_{n,j} = \sqrt{n(n+1) - N(N+1)}\ {}_{N+1}Y_{n,j}$ *for $N \geq 0$, $n \geq N+1$, $j = -n, \dots, n$,*
- $\overline{\eth}_N\ {}_NY_{n,j} = -\sqrt{n(n+1) - N(N-1)}\ {}_{N-1}Y_{n,j}$ *for $N \leq 0$, $n \geq -N+1$, $j = -n, \dots, n$.*

Proof. The lemma was mentioned in [15, 34, 53, 63, 91, 96, 97] without proof. The proof follows directly from Definition 3.2.6. So, the first property is obvious, if $N = 0$. Then, we get for $n \geq 0$ and $j = -n, \dots, n$

$$_0Y_{n,j} = \sqrt{\frac{n!}{n!}}\ \eth_0^0\ Y_{n,j} = Y_{n,j}.$$

For $N \geq 0$, $n \geq N+1$, and $j = -n, \dots, n$, we obtain with (3.1)

$$\begin{aligned} \eth_N\ {}_NY_{n,j} &= \sqrt{\frac{(n-N)!}{(n+N)!}}\ \eth_N \eth_0^N\ Y_{n,j} \\ &= \sqrt{(n-N)(n+N+1)}\sqrt{\frac{(n-(N+1))!}{(n+N+1)!}}\ \eth_0^{N+1}\ Y_{n,j} \\ &= \sqrt{n(n+1) - N(N+1)}\ {}_{N+1}Y_{n,j}. \end{aligned}$$

For $N \leq 0$, $n \geq -N+1$, and $j = -n, \dots, n$, we get again with (3.1)

$$\begin{aligned} \overline{\eth}_N\ {}_NY_{n,j} &= (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}}\ \overline{\eth}_N \overline{\eth}_0^{-N}\ Y_{n,j} \\ &= -\sqrt{(n+N)(n-N+1)}(-1)^{N-1}\sqrt{\frac{(n+N-1)!}{(n-(N-1))!}}\ \overline{\eth}_0^{-(N-1)}\ Y_{n,j} \end{aligned}$$

$$= -\sqrt{n(n+1) - N(N-1)}\ {}_{N-1}Y_{n,j}.$$

□

We will see in Lemma 3.6.2 that the previous lemma holds true for all $N \in \mathbb{Z}$. From the previous lemma, we can already conclude the following lemma.

Lemma 3.2.9. *Definition 3.2.6 is equivalent to the following definition of the spin-weighted spherical harmonics*

$$ {}_NY_{n,j} := \begin{cases} Y_{n,j}, & N = 0 \\ \frac{1}{\sqrt{n(n+1)-N(N-1)}}\ \eth_{N-1}\ {}_{N-1}Y_{n,j}, & 0 \le N \le n \\ \frac{-1}{\sqrt{n(n+1)-N(N+1)}}\ \overline{\eth}_{N+1}\ {}_{N+1}Y_{n,j}, & -n \le N \le 0 \\ 0, & n < |N| \end{cases}$$

for all $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, and $j = -n, \ldots, n$.

Note that the three definitions for the case $N = 0$ are equal.

Proof. With the previous lemma, one direction is obvious. We show the other direction in this proof.
Let $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \ge |N|$, $j = -n, \ldots, n$. The case $N = 0$ is trivial.

We use Lemma 3.2.8 and (3.1). Then, we get for the first case, for $0 \le N \le n$,

$$\begin{aligned} {}_NY_{n,j} &= \frac{1}{\sqrt{(n-N+1)(n+N)}}\ \eth^1_{N-1}\ {}_{N-1}Y_{n,j} \\ &= \ldots \\ &= \frac{1}{\sqrt{(n-N+1)(n-N+2)\ldots(n-1+1)(n+N)(n+N-1)\ldots(n+1)}}\ \eth^N_0\ {}_0Y_{n,j} \\ &= \sqrt{\frac{(n-N)!n!}{n!(n+N)!}}\ \eth^N_0\ Y_{n,j} \\ &= \sqrt{\frac{(n-N)!}{(n+N)!}}\ \eth^N_0\ Y_{n,j}. \end{aligned}$$

For the second case, for $-n \le N \le 0$, we obtain

$$\begin{aligned} {}_NY_{n,j} &= \frac{-1}{\sqrt{(n-N)(n+N+1)}}\ \overline{\eth}^1_{N+1}\ {}_{N+1}Y_{n,j} \\ &= \ldots \\ &= \frac{(-1)^N}{\sqrt{(n-N)(n-N-1)\ldots(n+1)(n+N+1)(n+N+2)\ldots(n-1+1)}}\ \overline{\eth}^{|N|}_0\ {}_0Y_{n,j} \\ &= (-1)^N \sqrt{\frac{n!(n+N)!}{(n-N)!n!}}\ \overline{\eth}^{-N}_0\ Y_{n,j} \\ &= (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}}\ \overline{\eth}^{-N}_0\ Y_{n,j}. \end{aligned}$$

Finally, ${}_NY_{n,j} = 0$ for $n < |N|$ is given by definition. □

Corollary 3.2.10. *Since we know from Definition 2.4.37 that* $Y_{n,j}(\xi(t,\varphi)) = X_{n,j}(t)e^{ij\varphi}$, *the spin-weighted spherical harmonics obviously depend solely on* φ *by a factor* $e^{ij\varphi}$ *for all* $\xi = \xi(t,\varphi) \in \Omega_0$. *This means that we can write*

$$ {}_NY_{n,j}(\xi(t,\varphi)) = {}_NX_{n,j}(t)e^{ij\varphi} $$

for $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, *and* $j = -n, \ldots, n$. *We will later see what* ${}_NX_{n,j}$ *looks like.*

Corollary 3.2.11. *From Lemma 3.2.9, we can directly conclude that the spin-weighted spherical harmonics are infinitely differentiable on* Ω_0. *This means that* ${}_NY_{n,j} \in \mathrm{C}^{(\infty)}(\Omega_0)$ *for* $N \in \mathbb{Z}$, $n \geq |N|$, *and* $j = -n, \ldots, n$.

So, the spin-(±1) harmonics are given for $\xi = \xi(t,\varphi) \in \Omega_0$, for $n \in \mathbb{N}_0$, $n \geq 1$, and for $j = -n, \ldots, n$ by

$$ \begin{aligned} {}_1Y_{n,j}(\xi) &= \frac{1}{\sqrt{n(n+1)}}\ \eth_0\ {}_0Y_{n,j}(\xi) \\ &= \frac{1}{\sqrt{n(n+1)}}\ \left(\sqrt{1-t^2}\ \partial_t - \frac{i}{\sqrt{1-t^2}}\ \partial_\varphi\right) Y_{n,j}(\xi), \\ {}_{-1}Y_{n,j}(\xi) &= \frac{-1}{\sqrt{n(n+1)}}\ \overline{\eth}_0\ {}_0Y_{n,j}(\xi) \\ &= \frac{-1}{\sqrt{n(n+1)}}\ \left(\sqrt{1-t^2}\ \partial_t + \frac{i}{\sqrt{1-t^2}}\ \partial_\varphi\right) Y_{n,j}(\xi). \end{aligned} $$

Consequently, we obtain

$$ {}_{\pm1}Y_{n,j}(\xi) = \frac{\pm1}{\sqrt{n(n+1)}}\ \left(\sqrt{1-t^2}\ \partial_t - \frac{i}{\sqrt{1-t^2}}\ (\pm\partial_\varphi)\right) Y_{n,j}(\xi). \tag{3.2} $$

Furthermore, the spin-(±2) harmonics are given for $\xi = \xi(t,\varphi) \in \Omega_0$, for $n \in \mathbb{N}_0$, $n \geq 2$, and for $j = -n, \ldots, n$ by

$$ \begin{aligned} {}_2Y_{n,j}(\xi) &= \frac{1}{\sqrt{n(n+1)-2}}\ \eth_1\ {}_1Y_{n,j}(\xi) \\ &= \frac{1}{\sqrt{n(n+1)-2}}\ \left(\sqrt{1-t^2}\ \partial_t + \frac{t-i\partial_\varphi}{\sqrt{1-t^2}}\right) {}_1Y_{n,j}(\xi) \\ &= \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}}\ \left(\sqrt{1-t^2}\ \partial_t + \frac{t-i\partial_\varphi}{\sqrt{1-t^2}}\right) \\ &\quad \times \left(\sqrt{1-t^2}\ \partial_t - \frac{i}{\sqrt{1-t^2}}\ \partial_\varphi\right) Y_{n,j}(\xi) \\ &= \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}}\ \Bigg[\sqrt{1-t^2}\ \underbrace{\left(\partial_t\sqrt{1-t^2}\right)}_{=\frac{-t}{\sqrt{1-t^2}}} \partial_t + \left(1-t^2\right)\partial_t^2 - i\sqrt{1-t^2} \\ &\quad \times \underbrace{\left(\partial_t \frac{1}{\sqrt{1-t^2}}\right)}_{=\frac{t}{(1-t^2)^{\frac{3}{2}}}} \partial_\varphi - i\partial_t\partial_\varphi - i\partial_\varphi\partial_t - \frac{1}{1-t^2}\ \partial_\varphi^2 + t\partial_t - i\ \frac{t}{1-t^2}\ \partial_\varphi\Bigg] Y_{n,j}(\xi) \\ &= \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}}\ \Bigg[(1-t^2)\partial_t^2 - \frac{1}{1-t^2}\ \partial_\varphi^2 \end{aligned} $$

$$-i \underbrace{\left(\partial_t \partial_\varphi + \partial_\varphi \partial_t + \frac{2t}{1-t^2}\ \partial_\varphi \right)}_{=-2i\left(\partial_\varphi \partial_t + \frac{t}{1-t^2}\ \partial_\varphi\right)} \Bigg] Y_{n,j}(\xi)$$

and

$$\begin{aligned}
{}_{-2}Y_{n,j}(\xi) &= \frac{-1}{\sqrt{n(n+1)-2}}\ \bar{\eth}_{-1}\ {}_{-1}Y_{n,j}(\xi) \\
&= \frac{-1}{\sqrt{n(n+1)-2}} \left(\sqrt{1-t^2}\ \partial_t + \frac{t+i\partial_\varphi}{\sqrt{1-t^2}} \right) {}_{-1}Y_{n,j}(\xi) \\
&= \frac{(-1)\cdot(-1)}{\sqrt{n(n+1)(n(n+1)-2)}} \left(\sqrt{1-t^2}\ \partial_t + \frac{t+i\partial_\varphi}{\sqrt{1-t^2}} \right) \\
&\quad \times \left(\sqrt{1-t^2}\ \partial_t + \frac{i}{\sqrt{1-t^2}}\ \partial_\varphi \right) Y_{n,j}(\xi) \\
&= \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}} \Bigg[\sqrt{1-t^2} \underbrace{\left(\partial_t \sqrt{1-t^2} \right)}_{=\frac{-t}{\sqrt{1-t^2}}} \partial_t + \left(1-t^2\right) \partial_t^2 + i\sqrt{1-t^2} \\
&\quad \times \underbrace{\left(\partial_t \frac{1}{\sqrt{1-t^2}} \right)}_{=\frac{t}{(1-t^2)^{\frac{3}{2}}}} \partial_\varphi + i\partial_t\partial_\varphi + i\partial_\varphi\partial_t - \frac{1}{1-t^2}\ \partial_\varphi^2 + t\partial_t + i\ \frac{t}{1-t^2}\ \partial_\varphi \Bigg] Y_{n,j}(\xi) \\
&= \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}} \Bigg[(1-t^2)\partial_t^2 - \frac{1}{1-t^2}\ \partial_\varphi^2 \\
&\qquad + i \underbrace{\left(\partial_t\partial_\varphi + \partial_\varphi\partial_t + \frac{2t}{1-t^2}\ \partial_\varphi \right)}_{=2i\left(\partial_\varphi\partial_t + \frac{t}{1-t^2}\ \partial_\varphi\right)} \Bigg] Y_{n,j}(\xi).
\end{aligned}$$

This yields eventually

$$\begin{aligned}
{}_{\pm 2}Y_{n,j}(\xi) &= \frac{\pm 1}{\sqrt{n(n+1)-2}} \left(\sqrt{1-t^2}\ \partial_t + \frac{t - i\left(\pm\partial_\varphi\right)}{\sqrt{1-t^2}} \right) {}_{\pm 1}Y_{n,j}(\xi) \\
&= \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}} \Bigg[(1-t^2)\partial_t^2 - \frac{1}{1-t^2}\ \partial_\varphi^2 \\
&\qquad \mp 2i \left(\partial_\varphi\partial_t + \frac{t}{1-t^2}\ \partial_\varphi \right) \Bigg] Y_{n,j}(\xi). \qquad (3.3)
\end{aligned}$$

Note that we use $\mathrm{L}^2(\Omega)$ instead of $\mathrm{L}^2(\Omega_0)$, because the poles span a (Lebesgue) null set such that the integrals over both regions are equal, if the integrals exist.

Remark 3.2.12. *It is well known that every function $F \in \mathrm{L}^2(\Omega)$ can be written in the basis of the spherical harmonics by*

$$F = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} c_{n,j} Y_{n,j}.$$

So, this also holds true for the function

$$F := \sqrt{1-t^2} \in \mathrm{L}^2(\Omega).$$

But

$$F = \sum_{n=0}^{L} \sum_{j=-n}^{n} c_{n,j} Y_{n,j}$$

with a bandlimit L is not exactly possible.

With Definition 2.4.37 and Lemma 3.2.9, we get that

$$\begin{aligned} Y_{1,0}(\xi(t,\varphi)) &= (-1)^0 \sqrt{\frac{2\cdot 1+1}{4\pi}} \sqrt{\frac{(1-0)!}{(1+0)!}} \frac{1}{2^1 1!} \left(1-t^2\right)^0 \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{1+0} \left(t^2-1\right)^1 \\ &= \sqrt{\frac{3}{4\pi}} t \end{aligned}$$

and therefore,

$$\begin{aligned} {}_1Y_{1,0}(\xi(t,\varphi)) &= \frac{1}{\sqrt{1(1+1)-0(0+1)}} \eth_0 \, {}_0Y_{1,0}(\xi) \\ &= \frac{1}{\sqrt{2}} \left(\sqrt{1-t^2}\partial_t + \frac{0\cdot t - i\partial_\varphi}{\sqrt{1-t^2}} \right) Y_{1,0}(\xi) \\ &= \frac{1}{\sqrt{2}} \left(\sqrt{1-t^2}\partial_t - \frac{i\partial_\varphi}{\sqrt{1-t^2}} \right) \sqrt{\frac{3}{4\pi}} t \\ &= \sqrt{\frac{3}{8\pi}} \sqrt{1-t^2}, \end{aligned}$$

where $\xi = \xi(t,\varphi) \in \Omega_0$. *Under the basis transformation to the spin-weighted spherical harmonics of spin weight* 1, *F can clearly be written in a bandlimited basis, this means that*

$$F = \sum_{n=1}^{L} \sum_{j=-n}^{n} {}_1c_{n,j} \; {}_1Y_{n,j},$$

because

$$F = \sqrt{\frac{4\pi}{3}} \; {}_1Y_{1,0}.$$

3.3 Properties of the Spin-Weighted Spherical Harmonics

For the spin-weighted spherical harmonics, we formulate novel recursion relations and a Christoffel-Darboux formula. Furthermore, a differential equation leads to an operator, a spin-weighted version of the Beltrami operator, for which the spin-weighted spherical harmonics are the eigenfunctions.

In analogy to Theorem 2.4.38, we can formulate recursion relations for the spin-weighted spherical harmonics of spin weight $N \in \mathbb{Z}$.

Theorem 3.3.1. *The spin-weighted spherical harmonics satisfy the following recursion relations for* $\xi = \xi(t,\varphi) \in \Omega_0$

$$\left(t^2-1\right) \partial_t \; {}_NY_{n,j}(\xi) = \left(nt + \frac{Nj}{n}\right) {}_NY_{n,j}(\xi) - (2n+1)\alpha_{n,j}^N \; {}_NY_{n-1,j}(\xi), \tag{3.4}$$

$$= -\left((n+1)t + \frac{Nj}{n+1}\right) {}_NY_{n,j}(\xi) + (2n+1)\alpha^N_{n+1,j}\ {}_NY_{n+1,j}(\xi), \tag{3.5}$$

$$\left(t + \frac{Nj}{n(n+1)}\right) {}_NY_{n,j}(\xi) = \alpha^N_{n,j}\ {}_NY_{n-1,j}(\xi) + \alpha^N_{n+1,j}\ {}_NY_{n+1,j}(\xi), \tag{3.6}$$

where

$$\alpha^N_{n,j} := \frac{\sqrt{(n-N)(n+N)}}{n} c_{n,j} = \frac{\sqrt{(n-N)(n+N)}}{n} \sqrt{\frac{(n-j)(n+j)}{(2n-1)(2n+1)}},$$

$N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N+1|$, and $j = -n, \ldots, n$. Furthermore, we denote ${}_NY_{n,j} := 0$ for $n < |j|$.

To the knowledge of the author the first two relations are new. The third relation was previously mentioned in [93] for the Wigner D-function without proof. For proving Theorem 3.3.1, we first have to introduce additional formulations for the spin-weighted spherical harmonics and various other properties.

Remark 3.3.2. *For the coefficients in Theorem 3.3.1, we obviously get*

$$\begin{aligned}
\alpha^N_{N,j} &= \frac{\sqrt{(N-N)(N+N)}}{N} c_{N,j} = 0, \\
\alpha^N_{-N,j} &= \frac{\sqrt{(-N-N)(-N+N)}}{N} c_{-N,j} = 0, \\
\alpha^N_{j,j} &= \frac{\sqrt{(j-N)(j+N)}}{j} \sqrt{\frac{(j-j)(j+j)}{(2j-1)(2j+1)}} = 0, \\
\alpha^N_{-j,j} &= \frac{\sqrt{(-j-N)(-j+N)}}{j} \sqrt{\frac{(-j-j)(-j+j)}{(-2j-1)(-2j+1)}} = 0
\end{aligned}$$

for all $N, j \in \mathbb{Z}$.

Remark 3.3.3. *We know from Corollary 3.2.10 that the spin-weighted spherical harmonics only depend on φ in form of a factor $e^{ij\varphi}$. So, we obtain for $N \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, $n \geq N+1$, and $j = -n, \ldots, n$ with $\xi = \xi(t,\varphi) \in \Omega_0$*

$$\begin{aligned}
{}_{N+1}Y_{n,j}(\xi) &= \frac{1}{\sqrt{n(n+1) - N(N+1)}}\ \eth_N\ {}_NY_{n,j}(\xi) \\
&= \frac{1}{\sqrt{n(n+1) - N(N+1)}} \left(\sqrt{1-t^2}\partial_t + \frac{-i\partial_\varphi + Nt}{\sqrt{1-t^2}}\right) {}_NY_{n,j}(\xi) \\
&= \frac{1}{\sqrt{n(n+1) - N(N+1)}} \left(\sqrt{1-t^2}\partial_t + \frac{j + Nt}{\sqrt{1-t^2}}\right) {}_NY_{n,j}(\xi), \\
{}_{-(N+1)}Y_{n,j}(\xi) &= \frac{-1}{\sqrt{n(n+1) - N(N+1)}}\ \bar{\eth}_{-N}\ {}_{-N}Y_{n,j}(\xi) \\
&= \frac{-1}{\sqrt{n(n+1) - N(N+1)}} \left(\sqrt{1-t^2}\partial_t - \frac{-i\partial_\varphi - Nt}{\sqrt{1-t^2}}\right) {}_{-N}Y_{n,j}(\xi) \\
&= \frac{-1}{\sqrt{n(n+1) - N(N+1)}} \left(\sqrt{1-t^2}\partial_t - \frac{j - Nt}{\sqrt{1-t^2}}\right) {}_{-N}Y_{n,j}(\xi).
\end{aligned}$$

Combined, this leads to

$$_{\pm(N+1)}Y_{n,j}(\xi) = \frac{\pm 1}{\sqrt{n(n+1)-N(N+1)}\sqrt{1-t^2}} \left(-\left(t^2-1\right)\partial_t \pm j + Nt\right) {}_{\pm N}Y_{n,j}(\xi).$$

If the recursion relations (3.4) and (3.5) are fulfilled for $N \in \mathbb{N}_0$, then with $\alpha_{n,j}^{-N} = \alpha_{n,j}^{N}$, we get

$$\begin{aligned} {}_{\pm(N+1)}Y_{n,j}(\xi) &\overset{(3.4)}{=} \frac{\pm 1}{\sqrt{n(n+1)-N(N+1)}\sqrt{1-t^2}} \left(-(n-N)\left(t-\frac{(\pm j)}{n}\right) {}_{\pm N}Y_{n,j}(\xi)\right. \\ &\qquad \left. + (2n+1)\alpha_{n,j}^{N} \; {}_{\pm N}Y_{n-1,j}(\xi)\right), \end{aligned} \tag{3.7}$$

$$\begin{aligned} {}_{\pm(N+1)}Y_{n,j}(\xi) &\overset{(3.5)}{=} \frac{\pm 1}{\sqrt{n(n+1)-N(N+1)}\sqrt{1-t^2}} \left((n+N+1)\left(t+\frac{(\pm j)}{n+1}\right) {}_{\pm N}Y_{n,j}(\xi)\right. \\ &\qquad \left. - (2n+1)\alpha_{n+1,j}^{N} \; {}_{\pm N}Y_{n+1,j}(\xi)\right). \end{aligned} \tag{3.8}$$

Now, Theorem 3.3.1 can be proved.

Proof. We use induction for the proof. Here, we skip the argument of the spin-weighted spherical harmonics for reasons of readability. This means that ${}_NY_{n,j} := {}_NY_{n,j}(\xi)$ for $\xi = \xi(t,\varphi) \in \Omega_0$, $N \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, $n \geq N+1$, and $j = -n, \ldots, 0$.
Base case: For $N = 0$, we get obviously the recursion relations of the fully normalized spherical harmonics (Theorem 2.4.38).
Induction hypothesis: Let us assume that (3.4), (3.5), and (3.6) are satisfied for a parameter $\pm N$, $N \in \mathbb{N}_0$.
Induction step: Now, we do the induction step $\pm N \to \pm(N+1)$, $N \in \mathbb{N}_0$ with $n \geq N+1$.

- We start by proving (3.4) for $\pm(N+1)$. With Remark 3.3.3, we obtain from the validity of (3.4) for $\pm N$ that (3.7) implies, in combination with (3.4) and (3.5), the result

$$\begin{aligned} &\left(t^2-1\right)\partial_t \; {}_{\pm(N+1)}Y_{n,j} \\ &= \frac{-\left(1-t^2\right)\cdot(\pm 1)t}{\sqrt{n(n+1)-N(N+1)}\,(1-t^2)^{\frac{3}{2}}} \left[-(n-N)\left(t-\frac{(\pm j)}{n}\right) {}_{\pm N}Y_{n,j}\right. \\ &\qquad\qquad \left. + (2n+1)\alpha_{n,j}^{N} \; {}_{\pm N}Y_{n-1,j}\right] \\ &\quad + \frac{\pm 1}{\sqrt{n(n+1)-N(N+1)}\sqrt{1-t^2}} \left[-(n-N)\left(t^2-1\right) {}_{\pm N}Y_{n,j}\right. \\ &\quad - (n-N)\left(t-\frac{(\pm j)}{n}\right) \underbrace{\left(\left(nt+\frac{(\pm j)N}{n}\right) {}_{\pm N}Y_{n,j} - (2n+1)\alpha_{n,j}^{N} \; {}_{\pm N}Y_{n-1,j}\right)}_{=(t^2-1)\partial_t \; {}_{\pm N}Y_{n,j}} \\ &\quad \left. + (2n+1)\alpha_{n,j}^{N} \underbrace{\left(-\left(nt+\frac{(\pm j)N}{n}\right) {}_{\pm N}Y_{n-1,j} + (2n-1)\alpha_{n,j}^{N} \; {}_{\pm N}Y_{n,j}\right)}_{=(t^2-1)\partial_t \; {}_{\pm N}Y_{n-1,j}}\right] \\ &= \frac{\pm 1}{\sqrt{n(n+1)-N(N+1)}\sqrt{1-t^2}} \left(-(n-N)\left[-t\left(t-\frac{(\pm j)}{n}\right) + t^2 - 1\right.\right. \end{aligned}$$

$$+\left(t-\frac{(\pm j)}{n}\right)\left(nt+\frac{(\pm j)N}{n}\right)-(2n+1)(2n-1)\underbrace{\frac{(n+N)\left(n^2-j^2\right)}{n^2(2n+1)(2n-1)}}_{=\frac{1}{n-N}\left(\alpha_{n,j}^N\right)^2}\Bigg]\,{}_{\pm N}Y_{n,j}$$

$$+(2n+1)\alpha_{n,j}^N\left[-t+(n-N)\left(t-\frac{(\pm j)}{n}\right)-\left(nt+\frac{(\pm j)N}{n}\right)\right]{}_{\pm N}Y_{n-1,j}\Bigg)$$

$$=\frac{\pm 1}{\sqrt{n(n+1)-N(N+1)}\sqrt{1-t^2}}\Bigg(-(n-N)\Bigg[\left(t-\frac{(\pm j)}{n}\right)\left(nt+\frac{(\pm j)N}{n}\right)$$

$$-t^2+\frac{(\pm j)}{n}t+t^2-1+\frac{j^2}{n^2}-\frac{j^2}{n^2}-\frac{(n+N)\left(n^2-j^2\right)}{n^2}\Bigg]{}_{\pm N}Y_{n,j}+(2n+1)\alpha_{n,j}^N$$

$$\times\left[-t+nt-(\pm j)-Nt+\frac{(\pm j)N}{n}-nt-\frac{(\pm j)N}{n}\right]{}_{\pm N}Y_{n-1,j}\Bigg)$$

$$=\frac{\pm 1}{\sqrt{n(n+1)-N(N+1)}\sqrt{1-t^2}}\Bigg(-(n-N)\Bigg[\left(t-\frac{(\pm j)}{n}\right)\left(nt+\frac{(\pm j)N}{n}\right)$$

$$+\left(t-\frac{(\pm j)}{n}\right)\frac{(\pm j)}{n}-\frac{(n+N)\left(n^2-j^2\right)}{n^2}-\frac{n^2-j^2}{n^2}\Bigg]{}_{\pm N}Y_{n,j}+(2n+1)\alpha_{n,j}^N$$

$$\times\left[nt+\frac{(\pm j)(N+1)}{n}-nt-t-Nt-(\pm j)-\frac{(\pm j)(N+1)}{n}\right]{}_{\pm N}Y_{n-1,j}\Bigg)$$

$$=\frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}}\Bigg(-(n-N)\Bigg[\left(t-\frac{(\pm j)}{n}\right)\left(nt+\frac{(\pm j)(N+1)}{n}\right)$$

$$-\underbrace{\frac{(n+N+1)\left(n^2-j^2\right)}{n^2}}_{=\frac{(2n-1)(2n+1)(n+N+1)}{(n-N)(n\mp N)}\left(\alpha_{n,j}^N\right)^2}\Bigg]{}_{\pm N}Y_{n,j}+(2n+1)\alpha_{n,j}^N\Bigg[nt+\frac{(\pm j)(N+1)}{n}$$

$$-\frac{n+1+N}{n+N}(n+N)\left(t+\frac{(\pm j)}{n}\right)\Bigg]{}_{\pm N}Y_{n-1,j}\Bigg)$$

$$=\frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}}\left(nt+\frac{(\pm j)(N+1)}{n}\right)$$

$$\times\left[-(n-N)\left(t-\frac{(\pm j)}{n}\right){}_{\pm N}Y_{n,j}+(2n+1)\alpha_{n,j}^N\,{}_{\pm N}Y_{n-1,j}\right]$$

$$-\frac{\pm 1}{\sqrt{(n-N-1)(n+N)}\sqrt{1-t^2}}(2n+1)\underbrace{\alpha_{n,j}^N\sqrt{\frac{(n-1-N)(n+1+N)}{(n-N)(n+N)}}}_{=\alpha_{n,j}^{N+1}}$$

$$\times\left[(n+N)\left(t+\frac{(\pm j)}{n}\right){}_{\pm N}Y_{n-1,j}-(2n-1)\alpha_{n,j}^N\,{}_{\pm N}Y_{n,j}\right].$$

Finally, we use again the assumption of the induction and Remark 3.3.3, which yields the validity of (3.7) and (3.8) such that we obtain with (3.1) the identity

$$\left(t^2-1\right)\partial_t\,{}_{\pm(N+1)}Y_{n,j}=\left(nt+\frac{(\pm(N+1))j}{n}\right){}_{\pm(N+1)}Y_{n,j}-(2n+1)\alpha_{n,j}^{N+1}\,{}_{\pm(N+1)}Y_{n-1,j},$$

which completes the proof of (3.4) for $\pm(N+1)$.

- We now proceed with a proof of (3.6) for $\pm(N+1)$. With Remark 3.3.3, we also obtain with (3.7), (3.8), and (3.1)

$$
\begin{aligned}
&\alpha_{n,j}^{N+1}\ {}_{\pm(N+1)}Y_{n-1,j} + \alpha_{n+1,j}^{N+1}\ {}_{\pm(N+1)}Y_{n+1,j} \\
&= \frac{\pm 1}{\sqrt{(n-1-N)(n+N)}\sqrt{1-t^2}} \sqrt{\frac{(n-1-N)(n+N+1)}{(n-N)(n+N)}}\ \alpha_{n,j}^{N} \\
&\quad \times \left[(n+N)\left(t+\frac{(\pm j)}{n}\right) {}_{\pm N}Y_{n-1,j} - (2n-1)\alpha_{n,j}^{N}\ {}_{\pm N}Y_{n,j}\right] \\
&\quad + \frac{\pm 1}{\sqrt{(n+1-N)(n+N+2)}\sqrt{1-t^2}} \sqrt{\frac{(n-N)(n+N+2)}{(n+1-N)(n+N+1)}}\ \alpha_{n+1,j}^{N} \\
&\quad \times \left[-(n+1-N)\left(t-\frac{(\pm j)}{n+1}\right) {}_{\pm N}Y_{n+1,j} + (2n+3)\alpha_{n+1,j}^{N}\ {}_{\pm N}Y_{n,j}\right] \\
&= \frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}} \left(-\frac{n-N}{n+1-N}\ (n+1-N)\left(t-\frac{(\pm j)}{n+1}\right)\right. \\
&\quad \times \alpha_{n+1,j}^{N}\ {}_{\pm N}Y_{n+1,j} + \frac{n+N+1}{n+N}\ (n+N)\left(t+\frac{(\pm j)}{n}\right)\alpha_{n,j}^{N}\ {}_{\pm N}Y_{n-1,j} \\
&\quad - \left[\frac{n+N+1}{n+N}\ \frac{(n-N)(n+N)\,(n^2-j^2)}{n^2(2n-1)(2n+1)}\ (2n-1)\right. \\
&\quad \left.\left. -\frac{n-N}{n+1-N}\ \frac{(n+1-N)(n+1+N)\,((n+1)^2-j^2)}{(n+1)^2(2n+1)(2n+3)}\ (2n+3)\right] {}_{\pm N}Y_{n,j}\right) \\
&= \frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}} \left(-(n-N)\left(t-\frac{(\pm j)}{n+1}\right)\left[\alpha_{n+1,j}^{N}\ {}_{\pm N}Y_{n+1,j}\right.\right. \\
&\quad \left. +\alpha_{n,j}^{N}\ {}_{\pm N}Y_{n-1,j}\right] + \left[(n+N+1)\left(t+\frac{(\pm j)}{n}\right) + (n-N)\left(t-\frac{(\pm j)}{n+1}\right)\right] \\
&\quad \times \alpha_{n,j}^{N}\ {}_{\pm N}Y_{n-1,j} - \frac{(n+N+1)(n-N)}{n^2(n+1)^2(2n+1)}\left[n^2(n+1)^2 - n^2j^2 - 2nj^2 - j^2\right. \\
&\quad \left.\left. -n^2(n+1)^2 + n^2j^2\right] {}_{\pm N}Y_{n,j}\right).
\end{aligned}
$$

Hence, the induction hypothesis yields that

$$
\begin{aligned}
&\alpha_{n,j}^{N+1}\ {}_{\pm(N+1)}Y_{n-1,j} + \alpha_{n+1,j}^{N+1}\ {}_{\pm(N+1)}Y_{n+1,j} \\
&= \frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}} \left(-(n-N)\left[\left(t-\frac{(\pm j)}{n+1}\right)\left(t+\frac{(\pm j)N}{n(n+1)}\right)\right.\right. \\
&\quad \left. -\frac{(n+N+1)j^2}{n^2(n+1)^2}\right] {}_{\pm N}Y_{n,j} + \left[nt + (\pm j) + Nt + \frac{(\pm j)N}{n} + t + \frac{(\pm j)}{n} + nt\right. \\
&\quad \left.\left. -Nt - (\pm j) + \frac{(\pm j)N}{n+1} + \frac{(\pm j)}{n+1}\right] \alpha_{n,j}^{N}\ {}_{\pm N}Y_{n-1,j}\right) \\
&= \frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}} \left(-(n-N)\left[\frac{n}{n+1}\left(t+\frac{t}{n}-\frac{(\pm j)}{n}\right)\right.\right. \\
&\quad \left. \times \left(t+\frac{(\pm j)(N+1)}{n(n+1)} - \frac{(\pm j)}{n(n+1)}\right) - \frac{(n+N+1)j^2}{n^2(n+1)^2}\right] {}_{\pm N}Y_{n,j} \\
&\quad \left. + \left[2nt + t + (\pm j)\left(\frac{N}{n} + \frac{1}{n} + \frac{N+1}{n+1}\right)\right] \alpha_{n,j}^{N}\ {}_{\pm N}Y_{n-1,j}\right)
\end{aligned}
$$

$$= \frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}} \left(-(n-N) \left[\frac{n}{n+1} \left(t - \frac{(\pm j)}{n} \right) \right.\right.$$
$$\times \left(t + \frac{(\pm j)(N+1)}{n(n+1)} \right) + \frac{t}{n+1} \left(t + \frac{(\pm j)N}{n(n+1)} \right) - \frac{(\pm j)}{(n+1)^2} \left(t - \frac{(\pm j)}{n} \right)$$
$$\left.\left. - \frac{(n+N+1)j^2}{n^2(n+1)^2} \right] {}_{\pm N}Y_{n,j} + \left(t + \frac{(\pm j)(N+1)}{n(n+1)} \right) (2n+1) \alpha_{n,j}^N \; {}_{\pm N}Y_{n-1,j} \right).$$

With

$$\frac{t}{n+1} \left(t + \frac{(\pm j)N}{n(n+1)} \right) - \frac{(\pm j)}{(n+1)^2} \left(t - \frac{(\pm j)}{n} \right) - \frac{(n+N+1)j^2}{n^2(n+1)^2}$$
$$= \frac{1}{n+1} \left(t^2 + \frac{(\pm j)N}{n(n+1)} t - \underbrace{\frac{(\pm j)}{n+1} t}_{\substack{= \frac{(\pm j)t(n+1-1)}{n(n+1)} \\ = \frac{(\pm j)t}{n} - \frac{(\pm j)t}{n(n+1)}}} + \frac{j^2}{n(n+1)} - \frac{j^2}{n(n+1)} \right.$$
$$\left. - \frac{Nj^2}{n^2(n+1)} - \frac{j^2}{n^2(n+1)} \right)$$
$$= \frac{1}{n+1} \left(t^2 + \frac{(\pm j)N}{n(n+1)} t + \frac{(\pm j)}{n(n+1)} t - \frac{(\pm j)}{n} t - \frac{j^2 N}{n^2(n+1)} - \frac{j^2}{n^2(n+1)} \right)$$
$$= \frac{1}{n+1} \left(t - \frac{(\pm j)}{n} \right) \left(t + \frac{(\pm j)(N+1)}{n(n+1)} \right),$$

all in all, we get, by using (3.7),

$$\alpha_{n,j}^{N+1} \; {}_{\pm(N+1)}Y_{n-1,j} + \alpha_{n+1,j}^{N+1} \; {}_{\pm(N+1)}Y_{n+1,j}$$
$$= \frac{\pm 1}{\sqrt{(n-N)(n+N+1)}\sqrt{1-t^2}} \left(t + \frac{(\pm j)(N+1)}{n(n+1)} \right)$$
$$\times \left[-(n-N) \frac{n+1}{n+1} \left(t - \frac{(\pm j)}{n} \right) {}_{\pm N}Y_{n,j} + (2n+1) \alpha_{n,j}^N \; {}_{\pm N}Y_{n-1,j} \right]$$
$$= \left(t + \frac{(\pm(N+1))j}{n(n+1)} \right) {}_{\pm(N+1)}Y_{n,j}.$$

So, (3.6) is proved.

- Finally, we prove (3.5). Above, we showed that the recursion relations (3.4) and (3.6) are satisfied for all $N \in \mathbb{Z}$. Hence, we can use them to prove (3.5). Then, we get

$$\left(t^2 - 1\right) \partial_t \; {}_NY_{n,j} \overset{(3.4)}{=} \left(nt + \frac{Nj}{n} \right) {}_NY_{n,j} - (2n+1) \alpha_{n,j}^N \; {}_NY_{n-1,j}$$
$$\overset{(3.6)}{=} \left(nt + \frac{Nj}{n} \right) {}_NY_{n,j} - \underbrace{(2n+1)}_{=n+(n+1)} \left(t + \frac{Nj}{n(n+1)} \right) {}_NY_{n,j}$$
$$+ (2n+1) \alpha_{n+1,j}^N \; {}_NY_{n+1,j}$$
$$= \left[nt + \frac{Nj}{n} - 2nt - t - \frac{Nj}{n+1} - \frac{Nj}{n} \right] {}_NY_{n,j}$$
$$+ (2n+1) \alpha_{n+1,j}^N \; {}_NY_{n+1,j}$$

$$= -\left((n+1)t + \frac{Nj}{n+1}\right) {}_NY_{n,j} + (2n+1)\alpha^N_{n+1,j} \ {}_NY_{n+1,j}.$$

□

Before we can formulate the Christoffel-Darboux formula for the spin-weighted spherical harmonics, we have to make a remark on the coefficients $\alpha^N_{n_j,j}$.

Remark 3.3.4. *For the coefficients in Theorem 3.3.1, we obviously get from Remark 3.3.2 for all $N, j \in \mathbb{Z}$*

$$\alpha^N_{n_j,j} = 0,$$

where $n_j = \max\{|N|, |j|\}$.

To the knowledge of the author, the Christoffel-Darboux formula for the spin-weighted spherical harmonics is not mentioned so far.

Theorem 3.3.5 (Christoffel-Darboux Formula)**.** *For all $N \in \mathbb{Z}$, we obtain the Christoffel-Darboux formula for the spin-weighted spherical harmonics*

$$(t_1 - t_2) \sum_{n=n_j}^{L-1} \overline{{}_NY_{n,j}(\xi)} \ {}_NY_{n,j}(\eta) = \alpha^N_{L,j} \left(\overline{{}_NY_{L,j}(\xi)} \ {}_NY_{L-1,j}(\eta) - \overline{{}_NY_{L-1,j}(\xi)} \ {}_NY_{L,j}(\eta) \right),$$

where

$$n_j := \max\{|N|, |j|\},$$

$\xi = \xi(t_1, \varphi_1)$, $\eta = \eta(t_2, \varphi_2)$ is the polar coordinate representation of $\xi, \eta \in \Omega_0$, $L > n_j$ is the bandlimit, and $j = -L, \ldots, L$.

Proof. With recursion relation (3.6), we get the equation system for $\xi = \xi(t_1, \varphi_1)$, $\eta = \eta(t_1, \varphi_1) \in \Omega_0$ and for $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, $j = -n, \ldots, n$

$$\left(t_1 + \frac{Nj}{n(n+1)}\right) {}_NY_{n,j}(\xi) = \alpha^N_{n,j} \ {}_NY_{n-1,j}(\xi) + \alpha^N_{n+1,j} \ {}_NY_{n+1,j}(\xi),$$
$$\left(t_2 + \frac{Nj}{n(n+1)}\right) {}_NY_{n,j}(\eta) = \alpha^N_{n,j} \ {}_NY_{n-1,j}(\eta) + \alpha^N_{n+1,j} \ {}_NY_{n+1,j}(\eta).$$

First, we complex conjugate the first equation and then multiply it with ${}_NY_{n,j}(\eta)$. Furthermore, we multiply the second equation with $\overline{{}_NY_{n,j}(\xi)}$. Then, we subtract the second equation from the first one and sum up over n from n_j to $(L-1)$. So, we obtain

$$\sum_{n=n_j}^{L-1} \left(t_1 + \frac{Nj}{n(n+1)} - t_2 - \frac{Nj}{n(n+1)}\right) \overline{{}_NY_{n,j}(\xi)} \ {}_NY_{n,j}(\eta)$$
$$= \sum_{n=n_j}^{L-1} \alpha^N_{n,j} \ \overline{{}_NY_{n-1,j}(\xi)} \ {}_NY_{n,j}(\eta) + \sum_{n=n_j}^{L-1} \alpha^N_{n+1,j} \ \overline{{}_NY_{n+1,j}(\xi)} \ {}_NY_{n,j}(\eta)$$
$$- \sum_{n=n_j}^{L-1} \alpha^N_{n,j} \ \overline{{}_NY_{n,j}(\xi)} \ {}_NY_{n-1,j}(\eta) - \sum_{n=n_j}^{L-1} \alpha^N_{n+1,j} \ \overline{{}_NY_{n,j}(\xi)} \ {}_NY_{n+1,j}(\eta)$$

and consequently,

$$\sum_{n=n_j}^{L-1} (t_1 - t_2) \overline{{}_NY_{n,j}(\xi)} \ {}_NY_{n,j}(\eta)$$

$$= \sum_{n=n_j}^{L-1} \alpha_{n,j}^N \, \overline{{}_NY_{n-1,j}(\xi)} \, {}_NY_{n,j}(\eta) + \sum_{n=n_j+1}^{L} \alpha_{n,j}^N \, \overline{{}_NY_{n,j}(\xi)} \, {}_NY_{n-1,j}(\eta)$$
$$- \sum_{n=n_j}^{L-1} \alpha_{n,j}^N \, \overline{{}_NY_{n,j}(\xi)} \, {}_NY_{n-1,j}(\eta) - \sum_{n=n_j+1}^{L} \alpha_{n,j}^N \, \overline{{}_NY_{n-1,j}(\xi)} \, {}_NY_{n,j}(\eta).$$

From Remark 3.3.4, we use that $\alpha_{n_j,j}^N = 0$. Then, we get

$$\sum_{n=n_j}^{L-1} (t_1 - t_2) \, \overline{{}_NY_{n,j}(\xi)} \, {}_NY_{n,j}(\eta)$$
$$= \sum_{n=n_j}^{L-1} \alpha_{n,j}^N \, \overline{{}_NY_{n-1,j}(\xi)} \, {}_NY_{n,j}(\eta) + \sum_{n=n_j}^{L-1} \alpha_{n,j}^N \, \overline{{}_NY_{n,j}(\xi)} \, {}_NY_{n-1,j}(\eta)$$
$$+ \alpha_{L,j}^N \, \overline{{}_NY_{L,j}(\xi)} \, {}_NY_{L-1,j}(\eta) - \sum_{n=n_j}^{L-1} \alpha_{n,j}^N \, \overline{{}_NY_{n,j}(\xi)} \, {}_NY_{n-1,j}(\eta)$$
$$- \sum_{n=n_j}^{L-1} \alpha_{n,j}^N \, \overline{{}_NY_{n-1,j}(\xi)} \, {}_NY_{n,j}(\eta) - \alpha_{L,j}^N \, \overline{{}_NY_{L-1,j}(\xi)} \, {}_NY_{L,j}(\eta)$$
$$= \alpha_{L,j}^N \left(\overline{{}_NY_{L,j}(\xi)} \, {}_NY_{L-1,j}(\eta) - \overline{{}_NY_{L-1,j}(\xi)} \, {}_NY_{L,j}(\eta) \right).$$

□

Now, we derive a differential equation for the spin-weighted spherical harmonics ${}_NY_{n,j}$ and automatically define an operator for which they are eigenfunctions (see [12, 49] without proof).

Theorem 3.3.6. *The spin-weighted spherical harmonics satisfy the following Sturm-Liouville differential equation [49] for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, $n \geq |N|$, and all $j = -n, \ldots, n$*

$$\partial_t \left(\left(1 - t^2\right) \partial_t \; {}_NY_{n,j}(\xi) \right) = \left(-n(n+1) + \frac{-\partial_\varphi^2 - 2itN\partial_\varphi + N^2}{1 - t^2} \right) {}_NY_{n,j}(\xi),$$

where $\xi = \xi(t, \varphi) \in \Omega_0$.

Proof. With the recursion relations (3.4) and (3.5) and with Corollary 3.2.10, the proof is straight forward. For $\xi = \xi(t, \varphi) \in \Omega_0$, we get for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, $n \geq |N|$, and all $j = -n, \ldots, n$

$$\partial_t \left(\left(1 - t^2\right) \partial_t \; {}_NY_{n,j}(\xi) \right)$$
$$= -\partial_t \left(\left(t^2 - 1\right) \partial_t \; {}_NY_{n,j}(\xi) \right)$$
$$\overset{(3.4)}{=} -\partial_t \left(\left(nt + \frac{Nj}{n} \right) {}_NY_{n,j}(\xi) - (2n+1)\alpha_{n,j}^N \; {}_NY_{n-1,j}(\xi) \right)$$
$$= \frac{1}{1 - t^2} \Bigg(-n \left(1 - t^2\right) {}_NY_{n,j}(\xi) + \left(nt + \frac{Nj}{n} \right) \left(t^2 - 1\right) \partial_t \; {}_NY_{n,j}(\xi)$$
$$- (2n+1)\alpha_{n,j}^N \left(t^2 - 1\right) \partial_t \; {}_NY_{n-1,j}(\xi) \Bigg)$$
$$\overset{(3.4),(3.5)}{=} \frac{1}{1 - t^2} \Bigg(-n \left(1 - t^2\right) {}_NY_{n,j}(\xi)$$

$$+\left(nt+\frac{Nj}{n}\right)\left[\left(nt+\frac{Nj}{n}\right)\ {}_NY_{n,j}(\xi)-(2n+1)\alpha_{n,j}^N\ {}_NY_{n-1,j}(\xi)\right]$$
$$-(2n+1)\alpha_{n,j}^N\left[-\left(nt+\frac{Nj}{n}\right)\ {}_NY_{n-1,j}(\xi)+(2n-1)\alpha_{n,j}^N\ {}_NY_{n,j}(\xi)\right]\Bigg)$$
$$=\frac{1}{1-t^2}\Bigg(\Bigg[-n\left(1-t^2\right)+n^2t^2+2Njt+\frac{N^2j^2}{n^2}-(2n+1)(2n-1)\underbrace{\frac{(n^2-N^2)(n^2-j^2)}{n^2(2n+1)(2n-1)}}_{=\left(\alpha_{n,j}^N\right)^2}\Bigg]$$
$$\times\ {}_NY_{n,j}(\xi)-(2n+1)\alpha_{n,j}^N\left[\left(nt+\frac{Nj}{n}\right)-\left(nt+\frac{Nj}{n}\right)\right]\ {}_NY_{n-1,j}(\xi)\Bigg)$$
$$=\frac{1}{1-t^2}\left(-n\left(1-t^2\right)+n^2t^2+2Njt+\frac{N^2j^2}{n^2}-n^2+j^2+N^2-\frac{N^2j^2}{n^2}\right)\ {}_NY_{n,j}(\xi)$$
$$=\left(-n(n+1)+\frac{j^2+2jtN+N^2}{1-t^2}\right)\ {}_NY_{n,j}(\xi)$$
$$=\left(-n(n+1)+\frac{-\partial_\varphi^2-2itN\partial_\varphi+N^2}{1-t^2}\right)\ {}_NY_{n,j}(\xi).$$

□

Corollary 3.3.7. *From the previous theorem, Theorem 3.3.6, we conclude directly that the spin-weighted spherical harmonics are eigenfunctions of a differential operator. This means that for all $\xi=\xi(t,\varphi)\in\Omega_0$, all $N\in\mathbb{Z}$, all $n\in\mathbb{N}_0$, $n\geq|N|$, and all $j=-n,\ldots,n$*

$$\Delta_\xi^{*,N}\ {}_NY_{n,j}(\xi)=-n(n+1)\ {}_NY_{n,j}(\xi)$$

and we define this spin-weighted Beltrami operator by

$$\Delta_\xi^{*,N}:=\Delta_\xi^*-\frac{N^2-2itN\partial_\varphi}{1-t^2},$$

where $\Delta_\xi^*=\partial_t\left((1-t^2)\partial_t\right)+\frac{1}{1-t^2}\ \partial_\varphi^2$ is the Beltrami operator.

Note that this operator contains a factor $\frac{1}{1-t^2}$ like all differential operators in polar coordinates. Therefore, we consider it only outside the x_3-axis as usual for similar cases.
Note also that in literature this operator is denoted by Δ^N [49, 93]. We decide to use the notation $\Delta^{*,N}$ instead, because we want to name it similar to the spin weight zero case.

It is obvious that we get Theorem 2.4.27 for $N=0$ and $\xi=\xi(t,\varphi)\in\Omega_0$

$$\Delta_\xi^*Y_{n,j}(\xi)=-n(n+1)Y_{n,j}(\xi).$$

3.4 The Addition Theorem for the Spin-Weighted Spherical Harmonics

By formulating the spin-weighted spherical harmonics with the help of the Wigner D-function, we can show the orthonormality and the addition theorem. This relation is useful, because the Wigner D-function is well-examined and a great quantity of properties is known (see [18, 93] for further details).

Definition 3.4.1. *$D^n_{j,N}$ is called the Wigner D-function (see [93]) with*

$$D^n_{j,N}(\alpha,\beta,\gamma) := e^{-ij\alpha} d^n_{j,N}(\beta) e^{-iN\gamma}$$

and $d^n_{j,N}$ can be written as

$$\begin{aligned} d^n_{j,N}(\beta) :&= (-1)^{n+j}\,\frac{1}{2^n}\sqrt{\frac{(n-j)!}{(n+j)!(n-N)!(n+N)!}}\,(1-\cos\beta)^{\frac{j-N}{2}}(1+\cos\beta)^{\frac{j+N}{2}} \\ &\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{n+j}\left[(1-\cos\beta)^{n+N}(1+\cos\beta)^{n-N}\right], \qquad (3.9) \\ &= (-1)^{n-N}\,\frac{1}{2^n}\sqrt{\frac{(n+j)!}{(n-j)!(n-N)!(n+N)!}}\,(1-\cos\beta)^{-\frac{j-N}{2}}(1+\cos\beta)^{-\frac{j+N}{2}} \\ &\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{n-j}\left[(1-\cos\beta)^{n-N}(1+\cos\beta)^{n+N}\right], \qquad (3.10) \\ &= (-1)^{n-N}\,\frac{1}{2^n}\sqrt{\frac{(n+N)!}{(n-N)!(n-j)!(n+j)!}}\,(1-\cos\beta)^{\frac{j-N}{2}}(1+\cos\beta)^{-\frac{j+N}{2}} \\ &\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{n-N}\left[(1-\cos\beta)^{n-j}(1+\cos\beta)^{n+j}\right], \qquad (3.11) \\ &= (-1)^{n+j}\,\frac{1}{2^n}\sqrt{\frac{(n-N)!}{(n+N)!(n-j)!(n+j)!}}\,(1-\cos\beta)^{-\frac{j-N}{2}}(1+\cos\beta)^{\frac{j+N}{2}} \\ &\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{n+N}\left[(1-\cos\beta)^{n+j}(1+\cos\beta)^{n-j}\right] \qquad (3.12) \end{aligned}$$

for $n \in \mathbb{N}_0$, $j,N \in \mathbb{Z}$ with $n \geq |N|$ and $n \geq |j|$. Here, α, β, and γ are the Euler angles with $\alpha,\gamma \in [0,2\pi]$ and $\beta \in [0,\pi]$.
We take the formula for $d^n_{j,N}$, where we have to calculate the lowest number of derivatives and, where the part without the derivatives has no singularities for $\cos\beta = \pm 1$. Therefore, $d^n_{j,N}$ has no singularities for $\cos\beta = \pm 1$, too.

Next, we formulate two new lemmas for the Wigner D-function that are needed in the following.

Lemma 3.4.2. *Let $n \in \mathbb{N}_0$, $j,N \in \mathbb{Z}$ with $n \geq |N|$ and $n \geq |j|$. From the previous definition, we get that for $\beta \in [0,\pi]$ and for spin weight zero*

$$\begin{aligned} d^n_{j,0}(\beta) &= \frac{(-1)^n}{2^n}\sqrt{\frac{1}{(n+j)!(n-j)!}}\,(1-\cos\beta)^{\frac{j}{2}}(1+\cos\beta)^{-\frac{j}{2}} \\ &\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{n}\left[(1-\cos\beta)^{n-j}(1+\cos\beta)^{n+j}\right] \\ &= \frac{(-1)^{n+j}}{2^n}\sqrt{\frac{1}{(n+j)!(n-j)!}}\,(1-\cos\beta)^{-\frac{j}{2}}(1+\cos\beta)^{\frac{j}{2}} \\ &\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{n}\left[(1-\cos\beta)^{n+j}(1+\cos\beta)^{n-j}\right]. \end{aligned}$$

Then, we can write $d^n_{j,N}$ *with help of* $d^n_{j,0}$ *for* $N \geq 0$ *as*

$$d^n_{j,N}(\beta) = \sqrt{\frac{(n-N)!}{(n+N)!}}\,(1-\cos\beta)^{-\frac{j-N}{2}}(1+\cos\beta)^{\frac{j+N}{2}} \times \left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{N}\left[(1-\cos\beta)^{\frac{j}{2}}(1+\cos\beta)^{-\frac{j}{2}}d^n_{j,0}(\beta)\right]$$

and for $N \leq 0$ *as*

$$d^n_{j,N}(\beta) = (-1)^N\sqrt{\frac{(n+N)!}{(n-N)!}}\,(1-\cos\beta)^{\frac{j-N}{2}}(1+\cos\beta)^{-\frac{j+N}{2}} \times \left(\frac{\mathrm{d}}{\mathrm{d}(\cos\beta)}\right)^{-N}\left[(1-\cos\beta)^{-\frac{j}{2}}(1+\cos\beta)^{\frac{j}{2}}d^n_{j,0}(\beta)\right].$$

Lemma 3.4.3. *Let* $n \in \mathbb{N}_0$, $j, N \in \mathbb{Z}$ *with* $n \geq |j|$ *and let* $\beta \in (0,\pi)$. *Then, we get for* $N \geq 0$ *and* $n \geq N+1$

$$d^n_{j,-(N+1)}(\beta) = \frac{-1}{\sqrt{n(n+1)-N(N+1)}}\left(\sqrt{1-\cos^2\beta}\frac{\mathrm{d}}{\mathrm{d}\cos\beta}d^n_{j,-N}(\beta) + \frac{N\cos\beta + j}{\sqrt{1-\cos^2\beta}}d^n_{j,-N}(\beta)\right).$$

Furthermore, we obtain for $N \leq 0$ *and* $n \geq -N+1$

$$d^n_{j,-(N-1)}(\beta) = \frac{1}{\sqrt{n(n+1)-N(N-1)}}\left(\sqrt{1-\cos^2\beta}\frac{\mathrm{d}}{\mathrm{d}\cos\beta}d^n_{j,-N}(\beta) - \frac{N\cos\beta + j}{\sqrt{1-\cos^2\beta}}d^n_{j,-N}(\beta)\right).$$

Proof. Let $n \in \mathbb{N}_0$, $j, N \in \mathbb{Z}$ with $n \geq |j|$ and let $\beta \in (0,\pi)$.

- For $N \geq 0$ and $n \geq N+1$, we get with (3.11) and with (3.1)

$$\begin{aligned}
&d^n_{j,-(N+1)}(\beta) \\
&= (-1)^{n+N+1}\,\frac{1}{2^n}\sqrt{\frac{(n-N-1)!}{(n+N+1)!(n+j)!(n-j)!}}\,(1-\cos\beta)^{\frac{j+N+1}{2}}(1+\cos\beta)^{-\frac{j-N-1}{2}} \\
&\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}\cos\beta}\right)^{n+N+1}\left[(1-\cos\beta)^{n-j}(1+\cos\beta)^{n+j}\right] \\
&= \frac{-1}{\sqrt{(n-N)(n+N+1)}}(1-\cos\beta)^{\frac{j+N+1}{2}}(1+\cos\beta)^{-\frac{j-N-1}{2}} \\
&\quad\times\frac{\mathrm{d}}{\mathrm{d}\cos\beta}\Bigg((1-\cos\beta)^{-\frac{j+N}{2}}(1+\cos\beta)^{\frac{j-N}{2}}\,\frac{(-1)^{n+N}}{2^n}\sqrt{\frac{(n-N)!}{(n+N)!(n+j)!(n-j)!}} \\
&\quad\times(1-\cos\beta)^{\frac{j+N}{2}}(1+\cos\beta)^{-\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}\cos\beta}\right)^{n+N}\left[(1-\cos\beta)^{n-j}(1+\cos\beta)^{n+j}\right]\Bigg) \\
&= \frac{-1}{\sqrt{(n-N)(n+N+1)}}(1-\cos\beta)^{\frac{j+N+1}{2}}(1+\cos\beta)^{-\frac{j-N-1}{2}} \\
&\quad\times\frac{\mathrm{d}}{\mathrm{d}\cos\beta}\left((1-\cos\beta)^{-\frac{j+N}{2}}(1+\cos\beta)^{\frac{j-N}{2}}d^n_{j,-N}(\beta)\right) \\
&= \frac{-1}{\sqrt{n(n+1)-N(N+1)}}(1-\cos\beta)^{\frac{j+N}{2}+\frac{1}{2}}(1+\cos\beta)^{-\frac{j-N}{2}+\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[(1-\cos\beta)^{-\frac{j+N}{2}} (1+\cos\beta)^{\frac{j-N}{2}} \frac{\mathrm{d}}{\mathrm{d}\cos\beta} d^n_{j,-N}(\beta) \right.\\
&+ \left(-\frac{j+N}{2}\right) (-1)(1-\cos\beta)^{-\frac{j+N}{2}-1} (1+\cos\beta)^{\frac{j-N}{2}} d^n_{j,-N}(\beta)\\
&\left. + \frac{j-N}{2} (1-\cos\beta)^{-\frac{j+N}{2}} (1+\cos\beta)^{\frac{j-N}{2}-1} d^n_{j,-N}(\beta) \right]\\
&= \frac{-1}{\sqrt{n(n+1)-N(N+1)}} \left[\sqrt{1-\cos^2\beta}\, \frac{\mathrm{d}}{\mathrm{d}\cos\beta} d^n_{j,-N}(\beta) \right.\\
&\left. + \underbrace{\left(\frac{j+N}{2} \sqrt{\frac{1+\cos\beta}{1-\cos\beta}} + \frac{j-N}{2} \sqrt{\frac{1-\cos\beta}{1+\cos\beta}} \right)}_{= \frac{(j+N)(1+\cos\beta)+(j-N)(1-\cos\beta)}{2\sqrt{1-\cos^2\beta}} = \frac{N\cos\beta + j}{\sqrt{1-\cos^2\beta}}} d^n_{j,-N}(\beta) \right]\\
&= \frac{-1}{\sqrt{n(n+1)-N(N+1)}} \left[\sqrt{1-\cos^2\beta}\, \frac{\mathrm{d}}{\mathrm{d}\cos\beta} d^n_{j,-N}(\beta) + \frac{N\cos\beta + j}{\sqrt{1-\cos^2\beta}} d^n_{j,-N}(\beta) \right].
\end{aligned}
$$

- For $N \le 0$ and $n \ge -N+1$, we get with (3.12) and with (3.1) analogously

$$
\begin{aligned}
&d^n_{j,-(N-1)}(\beta)\\
&= (-1)^{n+j} \frac{1}{2^n} \sqrt{\frac{(n+N-1)!}{(n-N+1)!(n+j)!(n-j)!}} (1-\cos\beta)^{-\frac{j+N-1}{2}} (1+\cos\beta)^{\frac{j-N+1}{2}}\\
&\times \left(\frac{\mathrm{d}}{\mathrm{d}\cos\beta}\right)^{n-N+1} \left[(1-\cos\beta)^{n+j}(1+\cos\beta)^{n-j}\right]\\
&= \frac{1}{\sqrt{(n+N)(n-N+1)}} (1-\cos\beta)^{-\frac{j+N-1}{2}} (1+\cos\beta)^{\frac{j-N+1}{2}}\\
&\times \frac{\mathrm{d}}{\mathrm{d}\cos\beta} \left((1-\cos\beta)^{\frac{j+N}{2}} (1+\cos\beta)^{-\frac{j-N}{2}} \frac{(-1)^{n+j}}{2^n} \sqrt{\frac{(n+N)!}{(n-N)!(n+j)!(n-j)!}} \right.\\
&\left. \times (1-\cos\beta)^{-\frac{j+N}{2}} (1+\cos\beta)^{\frac{j-N}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}\cos\beta}\right)^{n-N} \left[(1-\cos\beta)^{n+j}(1+\cos\beta)^{n-j}\right] \right)\\
&= \frac{1}{\sqrt{(n+N)(n-N+1)}} (1-\cos\beta)^{-\frac{j+N-1}{2}} (1+\cos\beta)^{\frac{j-N+1}{2}}\\
&\times \frac{\mathrm{d}}{\mathrm{d}\cos\beta} \left((1-\cos\beta)^{\frac{j+N}{2}} (1+\cos\beta)^{-\frac{j-N}{2}} d^n_{j,-N}(\beta) \right)\\
&= \frac{1}{\sqrt{n(n+1)-N(N-1)}} (1-\cos\beta)^{-\frac{j+N}{2}+\frac{1}{2}} (1+\cos\beta)^{\frac{j-N}{2}+\frac{1}{2}}\\
&\times \left[(1-\cos\beta)^{\frac{j+N}{2}} (1+\cos\beta)^{-\frac{j-N}{2}} \frac{\mathrm{d}}{\mathrm{d}\cos\beta} d^n_{j,-N}(\beta) \right.\\
&- \frac{j+N}{2} (1-\cos\beta)^{\frac{j+N}{2}-1} (1+\cos\beta)^{-\frac{j-N}{2}} d^n_{j,-N}(\beta)\\
&\left. - \frac{j-N}{2} (1-\cos\beta)^{\frac{j+N}{2}} (1+\cos\beta)^{-\frac{j-N}{2}-1} d^n_{j,-N}(\beta) \right]\\
&= \frac{1}{\sqrt{n(n+1)-N(N-1)}} \left[\sqrt{1-\cos^2\beta}\, \frac{\mathrm{d}}{\mathrm{d}\cos\beta} d^n_{j,-N}(\beta) \right.
\end{aligned}
$$

$$- \underbrace{\left(\frac{j+N}{2} \sqrt{\frac{1+\cos\beta}{1-\cos\beta}} + \frac{j-N}{2} \sqrt{\frac{1-\cos\beta}{1+\cos\beta}} \right)}_{= \frac{N\cos\beta + j}{\sqrt{1-\cos^2\beta}}} d^n_{j,-N}(\beta) \Bigg]$$

$$= \frac{1}{\sqrt{n(n+1)-N(N-1)}} \left[\sqrt{1-\cos^2\beta}\, \frac{\mathrm{d}}{\mathrm{d}\cos\beta} d^n_{j,-N}(\beta) - \frac{N\cos\beta + j}{\sqrt{1-\cos^2\beta}} d^n_{j,-N}(\beta) \right].$$

□

Remark 3.4.4. *We collect the required properties of the Wigner D-function from [93]. For further details, see [18, 93]. Let* $\beta, \beta_1, \beta_2 \in [0,\pi]$, $\alpha, \gamma \in [0, 2\pi]$, $n, n' \in \mathbb{N}_0$, *and* $j, j', N, N', N_1, N_2 \in \mathbb{Z}$ *with* $n \geq |j|$, $n \geq |N|$, $n \geq |N_1|$, $n \geq |N_2|$, $n' \geq |j'|$, *and* $n' \geq |N'|$.

1. $-\cos\beta = \cos(\pi - \beta)$,
2. $d^n_{j,N}(\pi - \beta) = (-1)^{n-N} d^n_{-j,N}(\beta) = (-1)^{n+j} d^n_{j,-N}(\beta)$,
3. $d^n_{j,N}(\beta \pm 2k\pi) = d^n_{j,N}(\beta)$ *for* $k \in \mathbb{Z}$ *and* $n \in \mathbb{N}_0$, $n \geq |N|$,
4. $d^n_{j,N}(\beta) = d^n_{N,j}(-\beta) = (-1)^{j-N} d^n_{N,j}(\beta)$,
5. $\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{-ij\varphi} = e^{-iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{-iN_2\gamma}$ *for* $\sin\varphi \neq 0$ *with the Euler angles* α, β, *and* γ *given by*

$$\cot\alpha = \cos\beta_1 \cot\varphi + \cot\beta_2 \frac{\sin\beta_1}{\sin\varphi},$$
$$\cos\beta = \cos\beta_1 \cos\beta_2 - \sin\beta_1 \sin\beta_2 \cos\varphi,$$
$$\cot\gamma = \cos\beta_2 \cot\varphi + \cot\beta_1 \frac{\sin\beta_2}{\sin\varphi},$$

6. $\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) = d^n_{N_1,N_2}(\beta_1 + \beta_2)$, *if* $\beta_1 + \beta_2 \in [0, \pi]$,
7. $D^n_{j,N}(0,0,0) = \delta_{j,N}$ *and consequently* $d^n_{j,N}(0) = \delta_{j,N}$,
8. $\int_0^\pi d^n_{j,N}(\vartheta) d^{n'}_{j,N}(\vartheta) \sin\vartheta \, \mathrm{d}\vartheta = \frac{2}{2n+1}\, \delta_{n,n'}$,
9. $\int_0^{2\pi} \int_0^\pi \int_0^{2\pi} \overline{D^n_{j,N}(\alpha,\beta,\gamma)} D^{n'}_{j',N'}(\alpha,\beta,\gamma) \, \mathrm{d}\gamma \sin\beta \, \mathrm{d}\beta \, \mathrm{d}\alpha = \frac{8\pi^2}{2n+1}\, \delta_{n,n'} \delta_{j,j'} \delta_{N,N'}$,
10. $D^n_{j,N}(\alpha,\beta,\gamma) = (-1)^{N-j}\, \overline{D^n_{-j,-N}(\alpha,\beta,\gamma)}$,
11. $D^n_{j,N}(\alpha,\beta,\gamma) = (-1)^{N+j} D^n_{N,j}(\gamma,\beta,\alpha)$.

Lemma 3.4.5. *The function* $d^n_{j,-N}$ *can be written for* $\beta \in [0,\pi]$, $n \in \mathbb{N}_0$, $j, N \in \mathbb{Z}$ *with* $n \geq |j|$ *and* $n \geq |N|$ *by*

$$d^n_{j,-N}(\beta) = (-1)^{n+j} \frac{1}{2^n} \sqrt{(n-j)!(n+j)!(n-N)!(n+N)!}$$
$$\times \sum_{k=\max\{0,j-N\}}^{\min\{n+j,n-N\}} (-1)^k \frac{(1-\cos\beta)^{n-k+\frac{j-N}{2}} (1+\cos\beta)^{k-\frac{j-N}{2}}}{k!(n+j-k)!(n-N-k)!(N-j+k)!}. \tag{3.13}$$

This is equivalent to the formulation given in [34, 42, 63, 93, 97] without proof

$$d^n_{j,-N}(\beta) = (-1)^{n+j} \sqrt{(n-j)!(n+j)!(n-N)!(n+N)!}$$

$$\times \sum_{k=\max\{0,j-N\}}^{\min\{n+j,n-N\}} (-1)^k \frac{\left(\sin\frac{\beta}{2}\right)^{2n-2k+j-N}\left(\cos\frac{\beta}{2}\right)^{2k-j+N}}{k!(n+j-k)!(n-N-k)!(N-j+k)!}.$$

Proof. From Definition 3.4.1, we get with $t=\cos\beta$, $\beta\in[0,\pi]$,

$$d^n_{j,-N}(\beta) = (-1)^{n+j}\frac{1}{2^n}\sqrt{\frac{(n-j)!}{(n+j)!(n-N)!(n+N)!}}(1-t)^{\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}} \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j}\left[(1-t)^{n-N}(1+t)^{n+N}\right]$$

for $n\in\mathbb{N}_0$, $j,N\in\mathbb{Z}$ with $n\geq|j|$ and $n\geq|N|$. With the Leibniz rule, we obtain for the derivative

$$\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j}\left[(1-t)^{n-N}(1+t)^{n+N}\right]\\
&=\sum_{k=0}^{n+j}\binom{n+j}{k}\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k}(1-t)^{n-N}\right)\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j-k}(1+t)^{n+N}\right)\\
&=\sum_{k=\max\{0,j-N\}}^{\min\{n+j,n-N\}}\frac{(n+j)!}{k!(n+j-k)!}(-1)^k\frac{(n-N)!}{(n-N-k)!}\frac{(n+N)!}{(N-j+k)!}(1-t)^{n-N-k}(1+t)^{N-j+k}.
\end{aligned}$$

Then, we get the first proposition. The second one follows directly, because of $\sin\frac{\beta}{2}=\sqrt{\frac{1-\cos\beta}{2}}$ and $\cos\frac{\beta}{2}=\sqrt{\frac{1+\cos\beta}{2}}$. □

Corollary 3.4.6. *With 5. from Remark 3.4.4, we can conclude that for $n\in\mathbb{N}_0$, $j,N_1,N_2\in\mathbb{Z}$ with $n\geq|j|$, $n\geq|N_1|$, and $n\geq|N_2|$*

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)e^{ij\varphi}=e^{iN_1\alpha}d^n_{N_1,N_2}(\beta)e^{iN_2\gamma}$$

for $\sin\varphi\neq0$ with the Euler angles α, β, and γ given by

$$\begin{aligned}
\cot\alpha&=\cos\beta_1\cot\varphi+\cot\beta_2\frac{\sin\beta_1}{\sin\varphi},\\
\cos\beta&=\cos\beta_1\cos\beta_2-\sin\beta_1\sin\beta_2\cos\varphi,\\
\cot\gamma&=\cos\beta_2\cot\varphi+\cot\beta_1\frac{\sin\beta_2}{\sin\varphi}.
\end{aligned}$$

Furthermore, by using 6. from Remark 3.4.4, we can formulate the general relation

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)e^{\pm ij\varphi}=e^{\pm iN_1\alpha}d^n_{N_1,N_2}(\beta)e^{\pm iN_2\gamma}$$

with the Euler angles α, β, and γ given by

- *if $\sin\varphi\neq0$*

$$\cot\alpha=\cos\beta_1\cot\varphi+\cot\beta_2\frac{\sin\beta_1}{\sin\varphi},$$

$$\cos\beta = \cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2\cos\varphi,$$
$$\cot\gamma = \cos\beta_2\cot\varphi + \cot\beta_1\frac{\sin\beta_2}{\sin\varphi}.$$

- *if* $\sin\varphi = 0$*, so* $\varphi = k\pi$*,* $k\in\mathbb{Z}$*, then for all* $l\in\mathbb{Z}$

$$\begin{cases} \alpha=0,\beta=-2l\pi+(\beta_1+\beta_2),\gamma=0 & ,\text{ if } k \text{ even}, \beta_1+\beta_2\in[2l\pi,(2l+1)\pi) \\ \alpha=\pi,\beta=2l\pi-(\beta_1+\beta_2),\gamma=\pi & ,\text{ if } k \text{ even}, \beta_1+\beta_2\in[(2l-1)\pi,2l\pi) \\ \alpha=0,\beta=-2l\pi+(\beta_1-\beta_2),\gamma=\pi & ,\text{ if } k \text{ odd}, \beta_1\geq\beta_2, \beta_1-\beta_2\in[2l\pi,(2l+1)\pi) \\ \alpha=\pi,\beta=2l\pi-(\beta_1-\beta_2),\gamma=0 & ,\text{ if } k \text{ odd}, \beta_1\geq\beta_2, \beta_1-\beta_2\in[(2l-1)\pi,2l\pi) \\ \alpha=0,\beta=2l\pi-(\beta_2-\beta_1),\gamma=\pi & ,\text{ if } k \text{ odd}, \beta_2\geq\beta_1, \beta_2-\beta_1\in[(2l-1)\pi,2l\pi) \\ \alpha=\pi,\beta=-2l\pi+(\beta_2-\beta_1),\gamma=0 & ,\text{ if } k \text{ odd}, \beta_2\geq\beta_1, \beta_2-\beta_1\in[2l\pi,(2l+1)\pi) \end{cases}.$$

[93] use such a formula but do not deal with all of the special cases in the way we do.

Proof. We get the first result with 5. from Remark 3.4.4 by complex conjugation. Together with 5. from Remark 3.4.4 itself, this leads us to

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)e^{\pm ij\varphi} = e^{\pm iN_1\alpha}d^n_{N_1,N_2}(\beta)e^{\pm iN_2\gamma}$$

for $\sin\varphi\neq 0$ with the Euler angles α, β, and γ given by

$$\cot\alpha = \cos\beta_1\cot\varphi + \cot\beta_2\frac{\sin\beta_1}{\sin\varphi},$$
$$\cos\beta = \cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2\cos\varphi,$$
$$\cot\gamma = \cos\beta_2\cot\varphi + \cot\beta_1\frac{\sin\beta_2}{\sin\varphi}.$$

So, for the second result, we only have to look at the cases, where $\sin\varphi = 0$, so $\varphi = k\pi$ for $k\in\mathbb{Z}$. Here, β also fulfills

$$\cos\beta = \cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2\cos\varphi$$

for reasons of continuity.

1. If k is even, then $\cos\varphi = 1$, so

$$\begin{aligned}\cos\beta &= \cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2\\ &= \cos(\beta_1+\beta_2)\end{aligned}$$

with $\beta\in[0,\pi]$. Therefore, for all $l\in\mathbb{Z}$,

$$\beta = \begin{cases} -2l\pi+(\beta_1+\beta_2) & ,\beta_1+\beta_2\in[2l\pi,(2l+1)\pi) \\ 2l\pi-(\beta_1+\beta_2) & ,\beta_1+\beta_2\in[(2l-1)\pi,2l\pi) \end{cases}.$$

Now, we look at these two cases for β.

- If k is even and $\beta_1+\beta_2\in[2l\pi,(2l+1)\pi)$ with $l\in\mathbb{Z}$, then $(-2l\pi+(\beta_1+\beta_2))\in[0,\pi)$.

We get with 3. and 6. of Remark 3.4.4

$$\begin{aligned}\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)e^{\pm ij\varphi} &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)\\ &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1-2l\pi)d^n_{j,N_2}(\beta_2)\\ &= d^n_{N_1,N_2}(-2l\pi+\beta_1+\beta_2).\end{aligned}$$

We know from above that for this case $\beta = -2l\pi + (\beta_1+\beta_2)$. Then, on the whole, we can write

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)e^{\pm ij\varphi} = e^{\pm iN_1\alpha}d^n_{N_1,N_2}(\beta)e^{\pm iN_2\gamma},$$

where $\alpha=\gamma=0$ and $\beta = -2l\pi + (\beta_1+\beta_2)$.

- If k is even and $\beta_1+\beta_2 \in [(2l-1)\pi, 2l\pi)$ with $l \in \mathbb{Z}$, then $(2l\pi-(\beta_1+\beta_2)) \in [0,\pi)$. We get with 4., 3., and 6. of Remark 3.4.4

$$\begin{aligned}\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)e^{\pm ij\varphi} &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)\\ &= \sum_{j=-n}^{n} (-1)^{j-N_1}d^n_{N_1,j}(-\beta_1)(-1)^{N_2-j}d^n_{j,N_2}(-\beta_2)\\ &= (-1)^{N_1+N_2}\sum_{j=-n}^{n} d^n_{N_1,j}(-\beta_1+2l\pi)d^n_{j,N_2}(-\beta_2)\\ &= (-1)^{N_1+N_2}d^n_{N_1,N_2}(2l\pi-\beta_1-\beta_2).\end{aligned}$$

We know from above that for this case $\beta = 2l\pi-(\beta_1+\beta_2)$. Then, on the whole, we can write

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1)d^n_{j,N_2}(\beta_2)e^{\pm ij\varphi} = e^{\pm iN_1\alpha}d^n_{N_1,N_2}(\beta)e^{\pm iN_2\gamma},$$

where $\alpha=\gamma=\pi$ and $\beta = 2l\pi-(\beta_1+\beta_2)$, because $-1=e^{\pm i\pi}$.

- For $\beta_1+\beta_2 = 2l\pi$, we get for both cases the identical results. Here, we have $\beta=0$. Then, we obtain for the first case $\alpha=\gamma=0$ and with 7. from Remark 3.4.4

$$e^{\pm iN_1\alpha}d^n_{N_1,N_2}(\beta)e^{\pm iN_2\gamma} = d^n_{N_1,N_2}(0) = \delta_{N_1,N_2}.$$

For the second case, we get $\alpha=\gamma=\pi$ again with 7. from Remark 3.4.4

$$e^{\pm iN_1\alpha}d^n_{N_1,N_2}(\beta)e^{\pm iN_2\gamma} = (-1)^{N_1+N_2}d^n_{N_1,N_2}(0) = (-1)^{2N_1}\delta_{N_1,N_2} = \delta_{N_1,N_2}.$$

2. If k is odd, then $\cos\varphi=-1$, so

$$\begin{aligned}\cos\beta &= \cos\beta_1\cos\beta_2+\sin\beta_1\sin\beta_2\\ &= \cos(\beta_1-\beta_2)\\ &= \cos(\beta_2-\beta_1)\end{aligned}$$

with $\beta \in [0, \pi]$. Therefore, for all $l \in \mathbb{Z}$

$$\beta = \begin{cases} -2l\pi + |\beta_1 - \beta_2| & , |\beta_1 - \beta_2| \in [2l\pi, (2l+1)\pi) \\ 2l\pi - |\beta_1 - \beta_2| & , |\beta_1 - \beta_2| \in [(2l-1)\pi, 2l\pi) \end{cases}.$$

- If k is odd, $\beta_1 \geq \beta_2$, and $\beta_1 - \beta_2 \in [2l\pi, (2l+1)\pi)$ with $l \in \mathbb{Z}$, then $(-2l\pi + (\beta_1 - \beta_2)) \in [0, \pi)$. We get with 3., 4., and 6. of Remark 3.4.4

$$\begin{aligned} \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2)(-1)^j \\ &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1 - 2l\pi)(-1)^{N_2-j} d^n_{j,N_2}(-\beta_2)(-1)^j \\ &= (-1)^{N_2} \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1 - 2l\pi) d^n_{j,N_2}(-\beta_2) \\ &= (-1)^{N_2} d^n_{N_1,N_2}(-2l\pi + \beta_1 - \beta_2). \end{aligned}$$

We know from above that for this case $\beta = -2l\pi + (\beta_1 - \beta_2)$. Then, on the whole, we can write

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} = e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma},$$

where $\alpha = 0$, $\beta = -2l\pi + (\beta_1 - \beta_2)$, and $\gamma = \pi$.

- If k is odd, $\beta_1 \geq \beta_2$, and $\beta_1 - \beta_2 \in [(2l-1)\pi, 2l\pi)$ with $l \in \mathbb{Z}$, then $(2l\pi - (\beta_1 - \beta_2)) \in [0, \pi)$. We get with 3., 4., and 6. of Remark 3.4.4

$$\begin{aligned} \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2)(-1)^j \\ &= \sum_{j=-n}^{n} (-1)^{j-N_1} d^n_{N_1,j}(-\beta_1) d^n_{j,N_2}(\beta_2 + 2l\pi)(-1)^j \\ &= (-1)^{N_1} \sum_{j=-n}^{n} d^n_{N_1,j}(-\beta_1) d^n_{j,N_2}(\beta_2 + 2l\pi) \\ &= (-1)^{N_1} d^n_{N_1,N_2}(2l\pi - \beta_1 + \beta_2). \end{aligned}$$

We know from above that for this case $\beta = 2l\pi - (\beta_1 - \beta_2)$. Then, on the whole, we can write

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} = e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma},$$

where $\alpha = \pi$, $\beta = 2l\pi - (\beta_1 - \beta_2)$, and $\gamma = 0$.

- For $\beta_1 - \beta_2 = 2l\pi$, we get for both cases the identical results. Then, we have $\beta = 0$. Therefore, we obtain for the first case $\alpha = 0$ and $\gamma = \pi$. With 7. from Remark 3.4.4, we get

$$e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma} = (-1)^{N_2} d^n_{N_1,N_2}(0) = (-1)^{N_2} \delta_{N_1,N_2} = (-1)^{N_1} \delta_{N_1,N_2}.$$

For the second case, we get $\alpha = \pi$ and $\gamma = 0$. Again with 7. from Remark 3.4.4, we obtain

$$e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma} = (-1)^{N_1} d^n_{N_1,N_2}(0) = (-1)^{N_1} \delta_{N_1,N_2}.$$

- If k is odd, $\beta_2 \geq \beta_1$, and $\beta_2 - \beta_1 \in [(2l-1)\pi, 2l\pi)$ with $l \in \mathbb{Z}$, then $(2l\pi - (\beta_2 - \beta_1)) \in [0, \pi)$. We get with 4., 3., and 6. of Remark 3.4.4

$$\begin{aligned}\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2)(-1)^j \\ &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1 + 2l\pi)(-1)^{N_2-j} d^n_{j,N_2}(-\beta_2)(-1)^j \\ &= (-1)^{N_2} \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1 + 2l\pi) d^n_{j,N_2}(-\beta_2) \\ &= (-1)^{N_2} d^n_{N_1,N_2}(2l\pi + \beta_1 - \beta_2).\end{aligned}$$

We know from above that for this case $\beta = 2l\pi - (\beta_2 - \beta_1)$. Then, on the whole, we can write

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} = e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma},$$

where $\alpha = 0$, $\beta = 2l\pi - (\beta_2 - \beta_1)$, and $\gamma = \pi$.

- If k is odd, $\beta_2 \geq \beta_1$, and $\beta_2 - \beta_1 \in [2l\pi, (2l+1)\pi)$ with $l \in \mathbb{Z}$, then $(-2l\pi + (\beta_2 - \beta_1)) \in [0, \pi)$. We get with 3., 4., and 6. of Remark 3.4.4

$$\begin{aligned}\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} &= \sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2)(-1)^j \\ &= \sum_{j=-n}^{n} (-1)^{j-N_1} d^n_{N_1,j}(-\beta_1) d^n_{j,N_2}(\beta_2 - 2l\pi)(-1)^j \\ &= (-1)^{N_1} \sum_{j=-n}^{n} d^n_{N_1,j}(-\beta_1) d^n_{j,N_2}(\beta_2 - 2l\pi) \\ &= (-1)^{N_1} d^n_{N_1,N_2}(-2l\pi - \beta_1 + \beta_2).\end{aligned}$$

We know from above that for this case $\beta = -2l\pi + (\beta_2 - \beta_1)$. Then, we can write on the whole

$$\sum_{j=-n}^{n} d^n_{N_1,j}(\beta_1) d^n_{j,N_2}(\beta_2) e^{\pm ij\varphi} = e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma},$$

where $\alpha = \pi$, $\beta = -2l\pi + (\beta_2 - \beta_1)$, and $\gamma = 0$, because $-1 = e^{\pm i\pi}$.

- For $\beta_2 - \beta_1 = 2l\pi$, we also get for both cases the identical results. Then, we have $\beta = 0$. Therefore, we obtain for the first case $\alpha = 0$ and $\gamma = \pi$. With 7. from Remark 3.4.4, we get

$$e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma} = (-1)^{N_2} \delta_{N_1,N_2} = (-1)^{N_1} \delta_{N_1,N_2}.$$

For the second case, we get $\alpha = \pi$ and $\gamma = 0$. With 7. from Remark 3.4.4, we obtain

$$e^{\pm iN_1\alpha} d^n_{N_1,N_2}(\beta) e^{\pm iN_2\gamma} = (-1)^{N_1} d^n_{N_1,N_2}(0) = (-1)^{N_1}\delta_{N_1,N_2}.$$

□

Lemma 3.4.7. *Let $n, b \in \mathbb{N}_0$ and $j, N \in \mathbb{Z}$ with $n \geq b$, $n \geq |j|$, and $n \geq |N|$. Then, the recursion for $k \geq 0$*

$$a_k = -\frac{(n-b-k+1)(n+b+k)}{2k(k+|j-N|)}\, a_{k-1}, \qquad k \geq 1,$$

with

$$a_0 = \frac{(n+b)!}{(n-b)!(|j-N|)!}$$

can be written in the explicit form given by

$$a_k = \frac{(-1)^k}{2^k k!}\, \frac{(n+k+b)!}{(n-b-k)!(k+|j-N|)!}, \tag{3.14}$$

where

$$a_k = 0 \qquad \forall k \geq n-b+1.$$

Proof. We use induction for the proof.
Base case: Let $k = 0$. Then, we obviously obtain

$$a_0 = \frac{(-1)^0}{2^0 \cdot 0!}\, \frac{(n+b)!}{(n-b)!(|j-N|)!} = \frac{(n+b)!}{(n-b)!(|j-N|)!}.$$

Induction hypothesis: Let us assume that (3.14) is satisfied for one $k \in \mathbb{N}_0$.
Induction step: The induction step $k \to k+1$ is given by

$$\begin{aligned} a_{k+1} &= -\frac{(n-b-k)(n+k+b+1)}{2(k+1)(k+|j-N|+1)}\, a_k \\ &= \frac{-1}{2(k+1)}\, \frac{(n-b-k)(n+k+b+1)}{k+|j-N|+1}\, \frac{(-1)^k}{2^k k!}\, \frac{(n+k+b)!}{(n-b-k)!(k+|j-N|)!} \\ &= \frac{(-1)^{k+1}}{2^{k+1}(k+1)!}\, \frac{(n+k+1+b)!}{(n-b-(k+1))!(k+1+|j-N|)!}. \end{aligned}$$

Furthermore, we see that one coefficient of the recursion relation vanishes for $k = n-b$. So, we see directly that

$$a_k = 0 \qquad \forall k \geq n-b+1.$$

□

Lemma 3.4.8. *We can show that the identity*

$$\sum_{k=0}^{n-b} \frac{(-1)^k}{2^k k!}\, \frac{(n+b+k)!}{(n-b-k)!(k+b+a)!}\, z^k = \frac{(-1)^{n+b}}{2^{n-b}(n+a)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n+b} (z-2)^{n+a} z^{n-a}$$

holds true for $z \in \mathbb{C}$, where $n, b \in \mathbb{N}_0$ and $a \in \mathbb{Z}$ with $n \geq b$, $n \geq |a|$, and $a + b \geq 0$.

Proof. With the well-known binomial formula, this proof is straight forward. Let

$$\begin{aligned}
V(z) &:= \frac{(-1)^{n+b}}{2^{n-b}(n+a)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n+b} (z-2)^{n+a} z^{n-a} \\
&= \frac{(-1)^{n+b}}{2^{n-b}(n+a)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n+b} \sum_{k=0}^{n+a} \frac{(n+a)!}{k!(n+a-k)!} z^k (-2)^{n+a-k} z^{n-a} \\
&= \frac{(-1)^{n+b}}{2^{n-b}(n+a)!} (-1)^{n+a}\, 2^{n+a}(n+a)! \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n+b} \sum_{k=0}^{n+a} \frac{(-1)^k}{2^k k!(n+a-k)!} z^{n-a+k} \\
&= \frac{(-1)^{a+b}}{2^{-(a+b)}} \sum_{k=a+b}^{n+a} \frac{(-1)^k}{2^k k!} \frac{(n-a+k)!}{(n+a-k)!(k-(a+b))!} z^{k-(a+b)}.
\end{aligned}$$

Now, we substitute $l = k-(a+b)$ such that

$$\begin{aligned}
V(z) &= \frac{(-1)^{a+b}}{2^{-(a+b)}} \sum_{l=0}^{n-b} \frac{(-1)^{l+a+b}}{2^{l+a+b}(l+a+b)!} \frac{(n-a+l+a+b)!}{(n+a-l-a-b)!(l+a+b-(a+b))!} z^{l+a+b-(a+b)} \\
&= \sum_{l=0}^{n-b} \frac{(-1)^l}{2^l l!} \frac{(n+b+l)!}{(n-b-l)!(l+a+b)!} z^l.
\end{aligned}$$

□

Theorem 3.4.9. *The spin-weighted spherical harmonics also satisfy [53, 96, 97]*

$$\begin{aligned}
{}_N Y_{n,j}(\xi) &= (-1)^N \sqrt{\frac{2n+1}{4\pi}}\, e^{ij\varphi} d^n_{j,-N}(\vartheta) \\
&= (-1)^N \sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{j,-N}(\varphi,\vartheta,0)},
\end{aligned}$$

where $\xi = \xi(t,\varphi) \in \Omega_0$*,* $t = \cos\vartheta$*,* $n \in \mathbb{N}_0$*,* $N \in \mathbb{Z}$*,* $n \geq |N|$*,* $j = -n, \ldots, n$*, and* $D^n_{j,N}$ *is the Wigner D-function.*

It is well known that the spin-weighted spherical harmonics can be written by the Wigner D-function (see [53, 96, 97]) but to the knowledge of the author without proof. Here, we borrow several methods of proof which were used for a related proposition in [88]. Furthermore, the formulation of the spin-weighted spherical harmonics by the Wigner D-function is not unique in literature (see Remark 3.4.10). We show a uniqueness such that this formulation is equal to the definition of the spin-weighted spherical harmonics from Lemma 3.2.9.

Proof. We divide this proof into six parts.
(1) Separation of variables:
We use that the spin-weighted spherical harmonics satisfy the differential equation in Theorem 3.3.6. So, we get for $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, and $j = -n, \ldots, n$

$$\left[\partial_t \left(\left(1-t^2\right)\partial_t\right) - \frac{-\partial_\varphi^2 - 2iNt\partial_\varphi + N^2}{1-t^2}\right] {}_N Y_{n,j}(\xi) = -n(n+1)\, {}_N Y_{n,j}(\xi),$$

where $\xi = \xi(t,\varphi) \in \Omega_0$. With separation of variables by

$${}_N Y_{n,j}(\xi(t,\varphi)) = {}_N\tilde{c}_{n,j}\, {}_N\tilde{P}_{n,j}(t) e^{ij\varphi},$$

we obtain the differential equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(1-t^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\right)-\frac{j^2+2jNt+N^2}{1-t^2}\right]\ {}_N\tilde{P}_{n,j}(t)=-n(n+1)\ {}_N\tilde{P}_{n,j}(t)$$

$$\Leftrightarrow\left[\left(1-t^2\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}-2t\frac{\mathrm{d}}{\mathrm{d}t}-\frac{j^2+2jNt+N^2}{1-t^2}\right]\ {}_N\tilde{P}_{n,j}(t)=-n(n+1)\ {}_N\tilde{P}_{n,j}(t).$$

By substituting $z=1+t$ (then $1-t=2-z$ and $1-t^2=z(2-z)$), we get

$$z(2-z)\ {}_N\tilde{P}''_{n,j}(z)-2(z-1)\ {}_N\tilde{P}'_{n,j}(z)+\left[n(n+1)-\frac{j^2+2jN(z-1)+N^2}{z(2-z)}\right]\ {}_N\tilde{P}_{n,j}(z)=0. \tag{3.15}$$

So, there are two singularities. One for $z=0$ and one for $z=2$.

(2) Solving a differential equation with singularities:
From [5], we know that differential equations with singularities for $z=z_k$, $k=1,2\ldots,$ of the form

$$P''(z)+p_1(z)P'(z)+p_2(z)P(z)=0$$

with

$$p_1(z)=\sum_k\frac{A_k}{z-z_k}$$

$$p_2(z)=\sum_k\left(\frac{B_k}{(z-z_k)^2}+\frac{C_k}{z-z_k}\right)$$

have the fundamental equations

$$\lambda(\lambda-1)+A_k\lambda+B_k=0,$$

where each k refers to one singularity. So, the solution of the differential equation is given by

$$P(z)=\prod_k(z-z_k)^{\lambda_k}\sum_{n=0}^{\infty}a_nz^n \tag{3.16}$$

for coefficients $a_n\in\mathbb{R}$.

Therefore, we look at (3.15) and get

$${}_N\tilde{P}''_{n,j}(z)+\frac{2(z-1)}{z(z-2)}\ {}_N\tilde{P}'_{n,j}(z)+\left[-\frac{n(n+1)}{z(z-2)}-\frac{j^2+2jN(z-1)+N^2}{z^2(2-z)^2}\right]\ {}_N\tilde{P}_{n,j}(z)=0.$$

Then, by comparison of the coefficients, we obtain

$$p_1(z)=\frac{A_1}{z}+\frac{A_2}{z-2}=\frac{(A_1+A_2)z-2A_1}{z(z-2)}\overset{!}{=}\frac{2(z-1)}{z(z-2)}$$

$$\Rightarrow A_1=A_2=1$$

and

$$p_2(z)=\frac{B_1}{z^2}+\frac{B_2}{(z-2)^2}+\frac{C_1}{z}+\frac{C_2}{z-2}$$

$$
\begin{aligned}
&= \frac{(C_1+C_2)z^3 + (B_1+B_2-4C_1-2C_2)z^2 + 4(C_1-B_1)z + 4B_1}{z^2(z-2)^2} \\
&\overset{!}{=} -\frac{n(n+1)}{z(z-2)} - \frac{j^2+2jN(z-1)+N^2}{z^2(2-z)^2} \\
&= \frac{-n(n+1)z^2 + 2(n(n+1)-jN)z - (j^2-2jN+N^2)}{z^2(z-2)^2} \\
\Rightarrow B_1 &= -\left(\frac{j-N}{2}\right)^2, \quad B_2 = -\left(\frac{j+N}{2}\right)^2 \\
C_1 &= \frac{n(n+1)}{2} - \frac{j^2+N^2}{4}, \quad C_2 = -\frac{n(n+1)}{2} + \frac{j^2+N^2}{4}.
\end{aligned}
$$

So, the fundamental equations are for $z_1 = 0$ given by

$$
\begin{aligned}
0 &= \lambda(\lambda-1) + 1\cdot\lambda - \left(\frac{j-N}{2}\right)^2 \\
\Rightarrow \lambda_1 &= \pm\frac{j-N}{2}
\end{aligned}
$$

and for $z_2 = 2$

$$
\begin{aligned}
0 &= \lambda(\lambda-1) + 1\cdot\lambda - \left(\frac{j+N}{2}\right)^2 \\
\Rightarrow \lambda_2 &= \pm\frac{j+N}{2}.
\end{aligned}
$$

For reasons of continuity, we only use the non-negative exponents. Then, we set by (3.16) like in [88]

$$
{}_N\tilde{P}_{n,j}(z,V) = z^{\frac{|j-N|}{2}}(2-z)^{\frac{|j+N|}{2}}V(z),
$$

where V denotes a power series ansatz.

(3) Using this ansatz:
For the derivatives, we get

$$
{}_N\tilde{P}'_{n,j}(z,V) = \frac{|j-N|}{2}\,\frac{1}{z}\,{}_N\tilde{P}_{n,j}(z,V) + \frac{|j+N|}{2}\,\frac{1}{z-2}\,{}_N\tilde{P}_{n,j}(z,V) + {}_N\tilde{P}_{n,j}(z,V')
$$

and

$$
\begin{aligned}
&{}_N\tilde{P}''_{n,j}(z,V) \\
&= \frac{|j-N|}{2}\left(\frac{|j-N|}{2}-1\right)\frac{1}{z^2}\,{}_N\tilde{P}_{n,j}(z,V) + 2\,\frac{|j-N|}{2}\,\frac{|j+N|}{2}\,\frac{1}{z(z-2)}\,{}_N\tilde{P}_{n,j}(z,V) \\
&\quad + 2\,\frac{|j-N|}{2}\,\frac{1}{z}\,{}_N\tilde{P}_{n,j}(z,V') + \frac{|j+N|}{2}\left(\frac{|j+N|}{2}-1\right)\frac{1}{(z-2)^2}\,{}_N\tilde{P}_{n,j}(z,V) \\
&\quad + 2\,\frac{|j+N|}{2}\,\frac{1}{z-2}\,{}_N\tilde{P}_{n,j}(z,V') + {}_N\tilde{P}_{n,j}(z,V''),
\end{aligned}
$$

where

$$
\begin{aligned}
{}_N\tilde{P}_{n,j}(z,V') &= z^{\frac{|j-N|}{2}}(2-z)^{\frac{|j+N|}{2}}V'(z), \\
{}_N\tilde{P}_{n,j}(z,V'') &= z^{\frac{|j-N|}{2}}(2-z)^{\frac{|j+N|}{2}}V''(z).
\end{aligned}
$$

So, we obtain from (3.15)

$$\begin{aligned}0 &= z(2-z)V''(z) + \left[z(2-z)\ \frac{|j-N|}{z} - z(2-z)\ \frac{|j+N|}{2-z} - 2(z-1)\right] V'(z) \\ &+ \left[n(n+1) - \frac{j^2+2jN(z-1)+N^2}{z(2-z)} + z(2-z)\ \frac{|j-N|}{2}\left(\frac{|j-N|}{2}-1\right)\ \frac{1}{z^2}\right. \\ &\quad + z(2-z)\frac{|j-N||j+N|}{2z(z-2)} + z(2-z)\ \frac{|j+N|}{2}\left(\frac{|j+N|}{2}-1\right)\ \frac{1}{(z-2)^2} \\ &\quad \left. - 2(z-1)\ \frac{|j-N|}{2z} - 2(z-1)\ \frac{|j+N|}{2(z-2)}\right] V(z).\end{aligned}$$

The coefficient of $V'(z)$ can be summarized as follows

$$\begin{aligned}(2-z)|j-N| - z|j+N| - 2(z-1) &= -\left(|j-N|+|j+N|\right)z - 2z + 2|j-N| + 2 \\ &= -2\left(z\left(\frac{|j-N|+|j+N|}{2}+1\right) - (|j-N|+1)\right).\end{aligned}$$

Moreover, for the coefficient of $V(z)$, we obtain

$$\begin{aligned}&n(n+1) - \frac{j^2+2jN(z-1)+N^2}{z(2-z)} + \frac{|j-N|}{2}\left(\frac{|j-N|}{2}-1\right)\ \frac{2-z}{z} - \frac{|j-N||j+N|}{2} \\ &\quad + \frac{|j+N|}{2}\left(\frac{|j+N|}{2}-1\right)\ \frac{z}{2-z} - |j-N|\ \frac{z-1}{z} + |j+N|\ \frac{z-1}{2-z} \\ &= n(n+1) - \frac{|j-N||j+N|}{2} - \frac{j^2+2jN(z-1)+N^2}{z(2-z)} \\ &\quad + \frac{|j-N|}{2z}\left(\frac{|j-N|}{2}\ (2-z) - 2 + z - 2z + 2\right) + \frac{|j+N|}{2(2-z)}\left(\frac{|j+N|}{2}\ z - z + 2z - 2\right) \\ &= n(n+1) - \frac{|j-N||j+N|}{2} - \frac{j^2+2jN(z-1)+N^2}{z(2-z)} + \left(\frac{j-N}{2}\right)^2\ \frac{2-z}{z} - \frac{|j-N|}{2} \\ &\quad + \left(\frac{j+N}{2}\right)^2\ \frac{z}{2-z} - \frac{|j+N|}{2} \\ &= n(n+1) - \frac{|j-N||j+N| + |j-N| + |j+N|}{2} \\ &\quad - \frac{1}{4z(2-z)}\left[4\left(j^2+2jN(z-1)+N^2\right) - (j-N)^2(2-z)^2 - (j+N)^2z^2\right] \\ &= n(n+1) - \frac{|j-N||j+N| + |j-N| + |j+N|}{2} \\ &\quad - \frac{1}{4z(2-z)}\left[4j^2 + 8jNz - 8jN + 4N^2 - 4j^2 + 4j^2z - j^2z^2 + 8jN - 8jNz + 2jNz^2\right. \\ &\quad \left. -4N^2 + 4N^2z - N^2z^2 - j^2z^2 - 2jNz^2 - N^2z^2\right] \\ &= n(n+1) - \frac{|j-N||j+N| + |j-N| + |j+N|}{2} - \frac{1}{2z(2-z)}\left[2j^2z - j^2z^2 + 2N^2z - N^2z^2\right] \\ &= n(n+1) - \frac{|j-N||j+N| + |j-N| + |j+N|}{2} - \frac{(2-z)\left(j^2+N^2\right)}{2(2-z)} \\ &= n(n+1) - \frac{|j-N| + |j+N|}{2} - \frac{j^2+N^2+|j-N||j+N|}{2}\end{aligned}$$

$$
\begin{aligned}
&= n(n+1) - \frac{|j-N|+|j+N|}{2} - \frac{2j^2+2N^2+2|j-N||j+N|}{4} \\
&= n(n+1) - \frac{|j-N|+|j+N|}{2} - \frac{j^2-2jN+N^2+2|j-N||j+N|+j^2+2jN+N^2}{4} \\
&= n(n+1) - \frac{|j-N|+|j+N|}{2} - \frac{(j-N)^2+2|j-N||j+N|+(j+N)^2}{4} \\
&= n(n+1) - \frac{|j-N|+|j+N|}{2} - \left(\frac{|j-N|+|j+N|}{2}\right)^2 \\
&= n(n+1) - \frac{|j-N|+|j+N|}{2}\left(\frac{|j-N|+|j+N|}{2}+1\right).
\end{aligned}
$$

Now, we define

$$b := \frac{|j-N|+|j+N|}{2}.$$

Obviously, we see that $b \geq 0$ and $n \geq b$.

All in all, we obtain the differential equation

$$0 = z(2-z)V''(z) - 2(z(b+1) - (|j-N|+1))V'(z) + (n(n+1) - b(b+1))V(z).$$

(4) Using a power series ansatz:
With a power series ansatz $V(z) = \sum_{k=0}^{\infty} a_k z^k$, we get

$$
\begin{aligned}
0 &= \sum_{k=0}^{\infty} a_k \big[z(2-z)k(k-1)z^{k-2} - 2(z(b+1) - (|j-N|+1))kz^{k-1} \\
&\quad + (n(n+1) - b(b+1))z^k\big] \\
&= \sum_{k=0}^{\infty} a_k \left[2k(k-1) + 2(|j-N|+1)k\right] z^{k-1} \\
&\quad + \sum_{k=0}^{\infty} a_k \left[n(n+1) - b(b+1) - k(k-1) - 2(b+1)k\right] z^k \\
&= \sum_{k=0}^{\infty} \big[a_{k+1}\left(2k(k+1) + 2(|j-N|+1)(k+1)\right) \\
&\quad + a_k \left(n(n+1) - b(b+1) - k(k-1) - 2(b+1)k\right)\big] z^k.
\end{aligned}
$$

Comparison of the coefficients delivers

$$
\begin{aligned}
2(k+1)(k+|j-N|+1)a_{k+1} &= -\left(n(n+1) - b(b+1) - k^2 + k - 2kb - 2k\right) a_k \\
&= -\left(n(n+1) - b(b+1) - k(k+1) - 2kb\right) a_k
\end{aligned}
$$

such that

$$a_{k+1} = -\frac{n(n+1) - b(b+1) - k(k+1) - 2kb}{2(k+1)(k+|j-N|+1)}\, a_k,$$

where the denominator is obviously positive and we can reformulate

$$
\begin{aligned}
n(n+1) - b(b+1) - k(k+1) - 2kb &= n(n+1) - b(b+1) - k(k+1) - 2kb + nb - nb \\
&= n(n+b+1) - b(n+b+1) - k(k+2b+1) \\
&= (n-b)(n+b+1) - k(k+b+1) - bk + kn - kn
\end{aligned}
$$

$$= (n-b)(n+b+k+1) - k(n+b+k+1)$$
$$= (n-b-k)(n+b+k+1).$$

So, we get

$$a_{k+1} = -\frac{(n-b-k)(n+b+k+1)}{2(k+1)(k+|j-N|+1)}\ a_k$$
$$\Rightarrow a_k = -\frac{(n-b-k+1)(n+b+k)}{2k(k+|j-N|)}\ a_{k-1}.$$

We have two degrees of freedom. Therefore, we choose the start coefficient of this recursion arbitrary as one degree of freedom by

$$a_0 := \frac{(n+b)!}{(n-b)!(|j-N|)!}.$$

Then, we get with Lemma 3.4.7

$$a_k = \frac{(-1)^k}{2^k k!}\ \frac{(n+b+k)!}{(n-b-k)!(k+|j-N|)!}, \tag{3.17}$$

where

$$a_k = 0 \qquad \forall k \geq n-b+1.$$

Now, we define

$$a := \frac{|j-N| - |j+N|}{2}$$

such that $|j-N| = a+b$, where $n \geq |a|$. So, we obtain with Lemma 3.4.8

$$V(z) = \sum_{k=0}^{n-b} \frac{(-1)^k}{2^k k!}\ \frac{(n+b+k)!}{(n-b-k)!(k+a+b)!}\ z^k.$$
$$= \frac{(-1)^{n+b}}{2^{n-b}(n+a)!}\ \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n+b} (z-2)^{n+a} z^{n-a}.$$

(5) Reconstruction of the solution:
Then, we see that

$$_N\tilde{P}_{n,j}(z) := {_N\tilde{P}_{n,j}}(z,V)$$
$$= z^{\frac{|j-N|}{2}} (2-z)^{\frac{|j+N|}{2}} V(z)$$
$$= \frac{(-1)^{n+b}}{2^{n-b}(n+a)!} z^{\frac{|j-N|}{2}} (2-z)^{\frac{|j+N|}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n+b} (z-2)^{n+a} z^{n-a}.$$

We resubstitute $z = 1+t$ to get

$$_N\tilde{P}_{n,j}(t) = \frac{(-1)^{n+b}}{2^{n-b}(n+a)!}\ (1+t)^{\frac{|j-N|}{2}} (1-t)^{\frac{|j+N|}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+b} (t-1)^{n+a}(t+1)^{n-a}$$
$$= \frac{(-1)^{a+b}}{2^{n-b}(n+a)!}\ (1-t)^{\frac{|j+N|}{2}} (1+t)^{\frac{|j-N|}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+b} (1-t)^{n+a}(1+t)^{n-a}.$$

Now, we have to consider each case separately. We use $t = \cos\vartheta$.

1. For $j-N\geq 0$, $j+N\geq 0$, we get $b=j$ and $a=-N$. Then, we obtain with (3.9)

$$\begin{aligned} {}_N\tilde{P}_{n,j}(t) &= \frac{(-1)^{j+N}}{2^{n-j}(n-N)!}\,(1-t)^{\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j}(1-t)^{n-N}(1+t)^{n+N} \\ &= (-1)^{n+N}2^{j}\sqrt{\frac{(n+j)!(n+N)!}{(n-j)!(n-N)!}}\; d^n_{j,-N}(\vartheta). \end{aligned}$$

2. For $j-N\geq 0$, $j+N<0$, we get $b=-N$ and $a=j$. Then, we obtain with (3.12)

$$\begin{aligned} {}_N\tilde{P}_{n,j}(t) &= \frac{(-1)^{j+N}}{2^{n+N}(n+j)!}\,(1-t)^{-\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-N}(1-t)^{n+j}(1+t)^{n-j} \\ &= (-1)^{n+N}2^{-N}\sqrt{\frac{(n-j)!(n-N)!}{(n+j)!(n+N)!}}\; d^n_{j,-N}(\vartheta). \end{aligned}$$

3. For $j-N<0$, $j+N\geq 0$, we get $b=N$ and $a=-j$. Then, we obtain with (3.11)

$$\begin{aligned} {}_N\tilde{P}_{n,j}(t) &= \frac{(-1)^{j+N}}{2^{n-N}(n-j)!}\,(1-t)^{\frac{j+N}{2}}(1+t)^{-\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+N}(1-t)^{n-j}(1+t)^{n+j} \\ &= (-1)^{n+j}2^{N}\sqrt{\frac{(n+j)!(n+N)!}{(n-j)!(n-N)!}}\; d^n_{j,-N}(\vartheta). \end{aligned}$$

4. For $j-N<0$, $j+N<0$, we get $b=-j$ and $a=N$. Then, we obtain with (3.10)

$$\begin{aligned} {}_N\tilde{P}_{n,j}(t) &= \frac{(-1)^{j+N}}{2^{n+j}(n+N)!}\,(1-t)^{-\frac{j+N}{2}}(1+t)^{-\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-j}(1-t)^{n+N}(1+t)^{n-N} \\ &= (-1)^{n+j}2^{-j}\sqrt{\frac{(n-j)!(n-N)!}{(n+j)!(n+N)!}}\; d^n_{j,-N}(\vartheta). \end{aligned}$$

Altogether, we get

$${}_N\tilde{P}_{n,j}(t) = (-1)^{n+\frac{a+b-(j+N)}{2}}\;2^{b}\sqrt{\frac{(n-a)!(n+b)!}{(n+a)!(n-b)!}}\; d^n_{j,-N}(\vartheta).$$

Then, with $t=\cos\vartheta$, we obtain

$$\begin{aligned} {}_NY_{n,j}(\xi) &= {}_N\tilde{c}_{n,j}\;{}_N\tilde{P}_{n,j}(t)e^{ij\varphi} \\ &= {}_N\tilde{c}_{n,j}\;(-1)^{n+\frac{a+b-(j+N)}{2}}\;2^{b}\sqrt{\frac{(n-a)!(n+b)!}{(n+a)!(n-b)!}}\; e^{ij\varphi}d^n_{j,-N}(\vartheta). \end{aligned}$$

Now, we choose as second degree of freedom the coefficient ${}_Nc_{n,j}$ by

$${}_Nc_{n,j} := (-1)^{n+N+\frac{a+b-(j+N)}{2}}\;\frac{1}{2^b}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n+a)!(n-b)!}{(n-a)!(n+b)!}}.$$

All in all, we get

$$\begin{aligned} {}_N Y_{n,j}(\xi) &= (-1)^N \sqrt{\frac{2n+1}{4\pi}}\, e^{ij\varphi} d^n_{j,-N}(\vartheta) \\ &= (-1)^N \sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{j,-N}(\varphi,\vartheta,0)}. \end{aligned} \tag{3.18}$$

(6) Verification of the chosen coefficients:
We started this proof with the differential equation from Theorem 3.3.6. We know that the spin-weighted spherical harmonics satisfy this differential equation. So, we show that the representation of (3.18) is equal to the definition of the spin-weighted spherical harmonics in Lemma 3.2.9. For that, we use induction.
Base case: For $N = 0$, we get the definition of the fully normalized spherical harmonics, Definition 2.4.37. For spin weight zero, we see directly from Definition 3.4.1 that

$$d^n_{j,0}(\vartheta) = \frac{(-1)^{n+j}}{2^n n!} \sqrt{\frac{(n-j)!}{(n+j)!}} \left(1-\cos^2\vartheta\right)^{\frac{j}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}\cos\vartheta}\right)^{n+j} \left(1-\cos^2\vartheta\right)^n .$$

With $t = \cos\vartheta$, with Theorem 2.4.2, and with Definition 2.4.10, we get for $j \geq 0$

$$d^n_{j,0}(\vartheta) = (-1)^j \sqrt{\frac{(n-j)!}{(n+j)!}}\, P_{n,j}(\cos\vartheta).$$

Then, we see directly with Definition 2.4.37 that for $j \geq 0$

$$\sqrt{\frac{2n+1}{4\pi}} e^{ij\varphi} d^n_{j,0}(\vartheta) = Y_{n,j}(\xi).$$

Furthermore, we obtain with Definition 2.4.37 and with 2. from Remark 3.4.4 for $j < 0$

$$\begin{aligned} Y_{n,j}(\xi) &= (-1)^j\, \overline{Y_{n,-j}(\xi)} \\ &= (-1)^j \sqrt{\frac{2n+1}{4\pi}}\, e^{ij\varphi} \underbrace{d^n_{-j,0}(\vartheta)}_{= (-1)^{n+j+n-0} d^n_{j,0}(\vartheta) = (-1)^j d^n_{j,0}(\vartheta)} \\ &= \sqrt{\frac{2n+1}{4\pi}}\, e^{ij\varphi} d^n_{j,0}(\vartheta). \end{aligned}$$

Consequently, the assumption holds true for spin weight zero.
Induction hypothesis: Let us assume that the representation of (3.18) is equal to the definition of the spin-weighted spherical harmonics in Lemma 3.2.9 up to spin weight N, $N \in \mathbb{Z}$ with $n \geq |N|$.
Induction step: Now, we do the induction step. Here, we have to look at two cases.

- The first case is $N \geq 0$. Then, we do the induction step $N \to N+1$ with $n \geq N+1$. With Lemma 3.4.3, with the assumption of the induction, and with Lemma 3.2.9, we get for $t = \cos\vartheta$

$$\begin{aligned} &(-1)^{N+1} \sqrt{\frac{2n+1}{4\pi}}\, e^{ij\varphi} d^n_{j,-(N+1)}(\vartheta) \\ &= \frac{1}{\sqrt{n(n+1)-N(N+1)}} \left(\sqrt{1-\cos^2\vartheta}\, \frac{\mathrm{d}}{\mathrm{d}\cos\vartheta} + \frac{N\cos\vartheta + j}{\sqrt{1-\cos^2\vartheta}} \right) (-1)^N \sqrt{\frac{2n+1}{4\pi}} \end{aligned}$$

$$
\begin{aligned}
&\times e^{ij\varphi} d^n_{j,-N}(\vartheta)\\
&= \frac{1}{\sqrt{n(n+1)-N(N+1)}} \left(\sqrt{1-t^2}\partial_t + \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\right) {}_N Y_{n,j}(\xi)\\
&= \frac{1}{\sqrt{n(n+1)-N(N+1)}} \eth_N \, {}_N Y_{n,j}(\xi)\\
&= {}_{N+1}Y_{n,j}(\xi).
\end{aligned}
$$

- The second case is $N \leq 0$. Then, we do the induction step $N \to N-1$ with $n \geq -N+1$. With Lemma 3.4.3, with the assumption of the induction, and with Lemma 3.2.9, we get again for $t = \cos\vartheta$

$$
\begin{aligned}
&(-1)^{N-1}\sqrt{\frac{2n+1}{4\pi}}\, e^{ij\varphi} d^n_{j,-(N-1)}(\vartheta)\\
&= \frac{-1}{\sqrt{n(n+1)-N(N-1)}} \left(\sqrt{1-\cos^2\vartheta}\,\frac{\mathrm{d}}{\mathrm{d}\cos\vartheta} - \frac{N\cos\vartheta + j}{\sqrt{1-\cos^2\vartheta}}\right)(-1)^N\sqrt{\frac{2n+1}{4\pi}}\\
&\times e^{ij\varphi} d^n_{j,-N}(\vartheta)\\
&= \frac{-1}{\sqrt{n(n+1)-N(N-1)}} \left(\sqrt{1-t^2}\partial_t - \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\right) {}_N Y_{n,j}(\xi)\\
&= \frac{-1}{\sqrt{n(n+1)-N(N-1)}} \bar{\eth}_N \, {}_N Y_{n,j}(\xi)\\
&= {}_{N-1}Y_{n,j}(\xi).
\end{aligned}
$$

□

Remark 3.4.10. *The description of the spin-weighted spherical harmonics by the Wigner D-function from Theorem 3.4.9 is unique except for a constant. We get the uniqueness by using Lemma 3.2.9. However, in literature different formulations in this regard can be found. Let* $\xi = \xi(\cos\vartheta, \varphi) \in \Omega_0$, $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, *and* $j = -n, \ldots, n$. *For example, in [9, 34, 42] the formulation is given by*

$$
{}_N Y_{n,j}(\xi) = \sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{-N,j}(0,\vartheta,\varphi)} = (-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{j,-N}(\varphi,\vartheta,0)}
$$

and in [15] by

$$
{}_N Y_{n,j}(\xi) = (-1)^N \sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{-N,j}(0,\vartheta,\varphi)} = (-1)^j\sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{j,-N}(\varphi,\vartheta,0)},
$$

where we used in both cases 11. from Remark 3.4.4.

The generalized spherical harmonics of Dahlen and Tromp [12] are given by

$$
Y^N_{n,j}(\xi) = \sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{j,N}(\varphi,\vartheta,0)} = (-1)^N {}_{-N}Y_{n,j}(\xi).
$$

They are also used in [8, 14, 66]. Furthermore, a different kind of generalized spherical harmonics are previously mentioned in [88].

The monopole harmonics are defined with help of 2. from Remark 3.4.4 and with Defini-

tion 3.4.1 by [98]

$$
{}_{N}Y_{n,j}(\xi) = \sqrt{\frac{2n+1}{4\pi}}\; e^{i(N+j)\varphi}\, \underbrace{d^n_{-j,N}(\vartheta)}_{=(-1)^{N+j} d^n_{j,-N}(\vartheta)} = (-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}\; \overline{D^n_{j,-N}(\varphi,\vartheta,0)}\; e^{iN\varphi}.
$$

We will keep the formulae in Theorem 3.4.9.

Remark 3.4.11. *From Theorem 3.4.9 and from the definition of the Wigner D-function, Definition 3.4.1, we see that we can define the spin-weighted spherical harmonics also at the poles. For reasons of continuity, we will keep Ω_0 as domain of definition, where we need continuity or continuous differentiability.*

Now, there is another representation of the spin-weighted spherical harmonics.

Lemma 3.4.12. *Let $\xi = \xi(t,\varphi) \in \Omega$, $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, and $j = -n, \ldots, n$. We know from Theorem 3.4.9 with (3.13) that*

$$
\begin{aligned}
{}_{N}Y_{n,j}(\xi) = {} & (-1)^{n+j+N}\sqrt{\frac{2n+1}{4\pi}}\sqrt{(n-j)!(n+j)!(n-N)!(n+N)!} \\
& \times \sum_{k=\max\{0,j-N\}}^{\min\{n+j,n-N\}} \frac{(-1)^k}{2^n}\, \frac{(1-t)^{n-k+\frac{j-N}{2}}(1+t)^{k-\frac{j-N}{2}}}{k!(n+j-k)!(n-N-k)!(N-j+k)!}\; e^{ij\varphi}.
\end{aligned}
$$

Corollary 3.4.13. *The function $d^n_{j,-N}$ and consequently, the spin-weighted spherical harmonics are bounded on Ω.*

Proof. From the previous lemma and with Theorem 3.4.9, we see that $d^n_{j,-N}$ and consequently, the spin-weighted spherical harmonics are bounded on Ω, because they depend only on bounded functions. We can see this, because the exponents of the finite series from the previous lemma are all non-negative. This means that

$$
\begin{aligned}
n-k+\frac{j-N}{2} &\geq n-\min\{n+j,n-N\}+\frac{j-N}{2} \\
&= \max\left\{-\frac{j+N}{2}, \frac{j+N}{2}\right\} \\
&= \left|\frac{j+N}{2}\right| \\
&\geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
k-\frac{j-N}{2} &\geq \max\{0,j-N\}-\frac{j-N}{2} \\
&= \max\left\{-\frac{j-N}{2}, \frac{j-N}{2}\right\} \\
&= \left|\frac{j-N}{2}\right| \\
&\geq 0.
\end{aligned}
$$

□

Corollary 3.4.14. *All derivatives for φ of the spin-weighted spherical harmonics are bounded in Ω_0. For the derivative for t, we obtain*

$$\partial_t^M \; {}_N Y_{n,j}(\xi) = \mathcal{O}\left(\left(1-t^2\right)^{\frac{1}{2}-M}\right) \qquad \textit{as } t \to \pm 1,$$

where $M \in \mathbb{N}$.

Proof. Let $\xi = \xi(t, \varphi) \in \Omega_0$, $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, and $j = -n, \ldots, n$. Obviously, we can conclude from Lemma 3.4.12 that all derivatives for φ of the spin-weighted spherical harmonics are bounded in Ω_0. Furthermore, for the derivative for t, we obtain

$$\partial_t \; {}_N Y_{n,j}(\xi) = \mathcal{O}\left(\frac{1}{\sqrt{1-t^2}}\right) \qquad \text{as } t \to \pm 1.$$

This is obvious, because we know already from the proof of the previous corollary that in Lemma 3.4.12 all exponents are non-negative but they can be $\frac{1}{2}$. This can be continued inductively such that

$$\partial_t^M \; {}_N Y_{n,j}(\xi) = \mathcal{O}\left(\left(1-t^2\right)^{\frac{1}{2}-M}\right) \qquad \text{as } t \to \pm 1,$$

where $M \in \mathbb{N}$. □

Corollary 3.4.15. *Let $\xi = \xi(t, \varphi) \in \Omega_0$, $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, and $j = -n, \ldots, n$. For the spin-weighted Beltrami operator, we get*

$$\Delta_\xi^{*,N} \; {}_N Y_{n,j}(\xi) = \mathcal{O}(1) \qquad \textit{as } t \to \pm 1.$$

Proof. Let $\xi = \xi(t, \varphi) \in \Omega_0$, $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, and $j = -n, \ldots, n$. We know from Corollary 3.3.7 that

$$\Delta_\xi^{*,N} \; {}_N Y_{n,j}(\xi) = -n(n+1) \; {}_N Y_{n,j}(\xi).$$

Additionally, we know from Corollary 3.4.13 that the spin-weighted spherical harmonics are bounded. So, we get directly the proposition. □

Lemma 3.4.16. *Let $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$ with $n \geq |N|$, and $j = -n, \ldots, n$. Then, we get for $N \geq 0$*

$$\eth_0^N \; Y_{n,j} = \mathcal{O}(1)$$

and for $N \leq 0$

$$\overline{\eth}_0^N \; Y_{n,j} = \mathcal{O}(1)$$

as $t \to \pm 1$.

Proof. We conclude the lemma directly from the definition of the spin-weighted spherical harmonics, Definition 3.2.6, and from Corollary 3.4.13. □

Definition 3.4.17. *We denote by $\mathrm{X}^k(\Gamma)$, $k \in \mathbb{N}_0$, the set of functions $F \in \mathrm{C}^{(k)}(\Gamma) \cap \mathrm{L}^2\left(\overline{\Gamma}\right)$ that satisfies the following conditions. Let $\xi = \xi(t, \varphi) \in \Gamma \subset \Omega$.*

- *F has the form $H(t)e^{ij\varphi}$ for $j \in \mathbb{Z}$,*
- *F is bounded on $\overline{\Gamma}$,*
- *$\partial_t^k F(\xi) = \mathcal{O}\left((1-t^2)^{\frac{1}{2}-k}\right)$ as $t \to \pm 1$,*
- *$\Delta_\xi^{*,N} F(\xi) = \mathcal{O}(1)$ for $N \in \mathbb{Z}$ as $t \to \pm 1$.*

Corollary 3.4.18. *With Corollary 3.2.10, Corollary 3.2.11, Corollary 3.4.13, Corollary 3.4.14, and Corollary 3.4.15, we see directly that*

$$ {}_N Y_{n,j} \in \mathrm{X}^k(\Omega_0) $$

with $\overline{\Omega_0} = \Omega$, *for all* $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, $j = -n, \ldots, n$, *and all* $k \in \mathbb{N}_0$. *Particularly, this holds true for the span of the spin-weighted spherical harmonics.*

Lemma 3.4.19. *The spin-weighted spherical harmonics can be written in the functions* o^i *and* $\hat{o}^i$, $i = 1, 2$, *from Definition 3.1.2 for* $\xi = \xi(t, \varphi) \in \Omega$ *by*

$$ {}_N Y_{n,j}(\xi) = (-1)^j \sqrt{\frac{2n+1}{4\pi}} \sqrt{(n-j)!(n+j)!(n-N)!(n+N)!} \times \sum_{k=\max\{0,j-N\}}^{\min\{n+j,n-N\}} \frac{\left(o^1_\xi\right)^{k+N-j} \left(o^2_\xi\right)^{n-k+j} \left(\hat{o}^1_\xi\right)^{n-k-N} \left(\hat{o}^2_\xi\right)^{k}}{k!(n+j-k)!(n-N-k)!(N-j+k)!} $$

for $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, *and* $j = -n, \ldots, n$.

Proof. Let $\xi = \xi(t, \varphi) \in \Omega$, $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, and $j = -n, \ldots, n$. We know from Lemma 3.4.12 that

$$ {}_N Y_{n,j}(\xi) = (-1)^{n+j+N} \sqrt{\frac{2n+1}{4\pi}} \sqrt{(n-j)!(n+j)!(n-N)!(n+N)!} \times \sum_{k=\max\{0,j-N\}}^{\min\{n+j,n-N\}} \frac{(-1)^k}{2^n} \frac{(1-t)^{n-k+\frac{j-N}{2}} (1+t)^{k-\frac{j-N}{2}}}{k!(n+j-k)!(n-N-k)!(N-j+k)!} e^{ij\varphi}. $$

Now, we take a look at

$$ \begin{aligned} &\frac{1}{2^n}(1-t)^{n-k+\frac{j-N}{2}} (1+t)^{k-\frac{j-N}{2}} e^{ij\varphi} \\ &= \left(\sqrt{\frac{1-t}{2}}\right)^{2n-2k+j-N} \left(\sqrt{\frac{1+t}{2}}\right)^{2k-j+N} e^{ij\varphi} \\ &= \left(\sqrt{\frac{1-t}{2}} e^{i\frac{\varphi}{2}}\right)^{n-k} \left(\sqrt{\frac{1-t}{2}} e^{-i\frac{\varphi}{2}}\right)^{n-k} \left(\sqrt{\frac{1-t}{2}} e^{i\frac{\varphi}{2}}\right)^{j} \left(\sqrt{\frac{1-t}{2}} e^{\pm i\frac{\varphi}{2}}\right)^{-N} \\ &\quad \times \left(\sqrt{\frac{1+t}{2}} e^{i\frac{\varphi}{2}}\right)^{k} \left(\sqrt{\frac{1+t}{2}} e^{-i\frac{\varphi}{2}}\right)^{k} \left(\sqrt{\frac{1+t}{2}} e^{-i\frac{\varphi}{2}}\right)^{-j} \left(\sqrt{\frac{1+t}{2}} e^{\pm i\frac{\varphi}{2}}\right)^{N}. \end{aligned} $$

So, there are two possible representations. The first one is

$$ \begin{aligned} &\frac{1}{2^n}(1-t)^{n-k+\frac{j-N}{2}} (1+t)^{k-\frac{j-N}{2}} e^{ij\varphi} \\ &= \left(o^2_\xi\right)^{n-k} (-1)^{n-k} \left(\hat{o}^1_\xi\right)^{n-k} \left(o^2_\xi\right)^{j} \left(o^2_\xi\right)^{-N} \left(\hat{o}^2_\xi\right)^{k} \left(o^1_\xi\right)^{k} \left(o^1_\xi\right)^{-j} \left(\hat{o}^2_\xi\right)^{N} \\ &= (-1)^{n-k} \left(o^1_\xi\right)^{k-j} \left(o^2_\xi\right)^{n-k+j-N} \left(\hat{o}^1_\xi\right)^{n-k} \left(\hat{o}^2_\xi\right)^{k+N}. \end{aligned} $$

We see that there are $k-j+n-k+j-N = n-N$ times a function o^i and $n-k+k+N = n+N$ times a function $\hat{o}^i$, $i = 1, 2$. Then, we know from Definition 3.1.2 that this term has spin weight $-N$. The second representation is

$$\begin{aligned}
&\frac{1}{2^n}(1-t)^{n-k+\frac{j-N}{2}}(1+t)^{k-\frac{j-N}{2}}e^{ij\varphi}\\
&=\left(o_\xi^2\right)^{n-k}(-1)^{n-k}\left(\hat{o}_\xi^1\right)^{n-k}\left(o_\xi^2\right)^{j}(-1)^{-N}\left(\hat{o}_\xi^1\right)^{-N}\left(\hat{o}_\xi^2\right)^{k}\left(o_\xi^1\right)^{k}\left(o_\xi^1\right)^{-j}\left(o_\xi^1\right)^{N}\\
&=(-1)^{n-k-N}\left(o_\xi^1\right)^{k+N-j}\left(o_\xi^2\right)^{n-k+j}\left(\hat{o}_\xi^1\right)^{n-k-N}\left(\hat{o}_\xi^2\right)^{k}.
\end{aligned}$$

We see that there are $k+N-j+n-k+j=n+N$ times a function o^i and $n-k-N+k=n-N$ times a function $\hat{o}^i$, $i=1,2$. Then, we know from Definition 3.1.2 that this term has spin weight N. We wanted to represent the spin-weighted spherical harmonics ${}_NY_{n,j}$, from which we know that they have spin weight N. Therefore, the second representation is the desired one and we get the proposition. □

Lemma 3.4.20. *The Jacobi polynomials for $t\in[-1,1]$, $n\in\mathbb{N}_0$, and $\alpha,\beta>-1$ are given by [58]*

$$P_n^{(\alpha,\beta)}(t):=\frac{(-1)^n}{2^n n!}(1-t)^{-\alpha}(1+t)^{-\beta}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\left[(1-t)^{\alpha+n}(1+t)^{\beta+n}\right].$$

The spin-weighted spherical harmonics can be written with help of the Jacobi polynomials for $j+N, j-N\geq 0$ with $\xi=\xi(t,\varphi)\in\Omega$, $N\in\mathbb{Z}$, $n\in\mathbb{N}_0$, $n\geq|N|$, $j=-n,\ldots,n$ by [18, 77, 93]

$${}_NY_{n,j}(\xi)=\frac{(-1)^j}{2^j}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n+j)!(n-j)!}{(n+N)!(n-N)!}}\,e^{ij\varphi}(1-t)^{\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}}P_{n-j}^{(j+N,j-N)}(t).$$

Proof. For $\vartheta\in[0,\pi]$, we know from 2. from Remark 3.4.4 that

$$d_{j,-N}^n(\vartheta)=(-1)^{n-N+n+j}d_{-j,N}^n(\vartheta)=(-1)^{j+N}d_{-j,N}^n(\vartheta).$$

Then, we get with Definition 3.4.1 for $t=\cos\vartheta$

$$\begin{aligned}
&d_{j,-N}^n(\vartheta)\\
&=\frac{(-1)^{j+N+n-j}}{2^n}\sqrt{\frac{(n+j)!}{(n-j)!(n-N)!(n+N)!}}(1-t)^{-\frac{j+N}{2}}(1+t)^{-\frac{j-N}{2}}\\
&\quad\times\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-j}\left[(1-t)^{n+N}(1+t)^{n-N}\right]\\
&=\frac{(-1)^{j+N}}{2^j}\sqrt{\frac{(n+j)!(n-j)!}{(n-N)!(n+N)!}}(1-t)^{\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}}\\
&\quad\times\underbrace{\frac{(-1)^{n-j}}{2^{n-j}(n-j)!}(1-t)^{-(j+N)}(1+t)^{-(j-N)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-j}\left[(1-t)^{n+N}(1+t)^{n-N}\right]}_{=P_{n-j}^{(j+N,j-N)}(t)},
\end{aligned}$$

if $j+N, j-N>-1$. Then, the proposition follows directly with Theorem 3.4.9. □

The orthonormality of the spin-weighted spherical harmonics is easy to show by using the representation with the Wigner D-function from Theorem 3.4.9.

Theorem 3.4.21. *The spin-weighted spherical harmonics are orthonormal for the* $\mathrm{L}^2(\Omega)$*-*

norm [12, 15, 34, 42, 53, 63, 97], this means that

$$\int_{\Omega} {}_{N}Y_{n,j}(\xi)\ \overline{{}_{N}Y_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) = \delta_{n,n'}\delta_{j,j'}.$$

Proof. With $\xi = \xi(t, \varphi) \in \Omega$ and $t = \cos\vartheta$, we get, using 8. from Remark 3.4.4,

$$\begin{aligned}
&\int_{\Omega} {}_{N}Y_{n,j}(\xi)\ \overline{{}_{N}Y_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) \\
&= (-1)^{N+N}\sqrt{\frac{2n+1}{4\pi}}\ \sqrt{\frac{2n'+1}{4\pi}}\ \int_0^{2\pi} e^{ij\varphi}e^{-ij'\varphi}\ \mathrm{d}\varphi\ \int_0^{\pi} d^n_{j,-N}(\vartheta)d^{n'}_{j',-N}(\vartheta)\sin\vartheta\ \mathrm{d}\vartheta \\
&= \sqrt{\frac{2n+1}{4\pi}}\ \sqrt{\frac{2n'+1}{4\pi}}\ \underbrace{\int_0^{2\pi} e^{-i\varphi(j'-j)}\ \mathrm{d}\varphi}_{=2\pi\ \delta_{j,j'}}\ \frac{2}{2n+1}\ \delta_{n,n'} \\
&= \frac{2n+1}{4\pi}\ \frac{2}{2n+1}\ \delta_{n,n'}\cdot 2\pi\ \delta_{j,j'} \\
&= \delta_{n,n'}\delta_{j,j'}.
\end{aligned}$$

□

Now, we obtain the addition theorem for the spin-weighted spherical harmonics.

Theorem 3.4.22 (Addition Theorem for Spin-Weighted Spherical Harmonics)**.** *The spin-weighted spherical harmonics satisfy the addition theorem for $N_1, N_2 \in \mathbb{Z}$ and for $n \in \mathbb{N}_0$, $n \geq \max\{|N_1|, |N_2|\}$,*

$$\sum_{j=-n}^{n} {}_{N_1}Y_{n,j}(\xi_1)\ \overline{{}_{N_2}Y_{n,j}(\xi_2)} = (-1)^{N_1}\sqrt{\frac{2n+1}{4\pi}}\ {}_{N_2}Y_{n,-N1}(\xi)\ e^{-iN_2\gamma},$$

where $\xi_1 = \xi_1(t_1, \varphi_1)$, $\xi_2 = \xi_2(t_2, \varphi_2)$, $\xi = \xi(t, \alpha) \in \Omega$, $t = \cos\beta$, and α, β, and γ are the Euler angles given by

- *for* $\sin(\varphi_1 - \varphi_2) \neq 0$

$$\begin{aligned}
\cot\alpha &= \cos\vartheta_1\cot(\varphi_1 - \varphi_2) - \cot\vartheta_2\frac{\sin\vartheta_1}{\sin(\varphi_1 - \varphi_2)}, \\
\cos\beta &= \cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos(\varphi_1 - \varphi_2), \\
\cot\gamma &= \cos\vartheta_2\cot(\varphi_1 - \varphi_2) - \cot\vartheta_1\frac{\sin\vartheta_2}{\sin(\varphi_1 - \varphi_2)}.
\end{aligned}$$

- *for* $\sin(\varphi_1 - \varphi_2) = 0$*, so* $\varphi_1 - \varphi_2 = k\pi$*,* $k \in \mathbb{Z}$*, then*

$$\left\{\begin{matrix}
\alpha = \pi, \beta = \vartheta_1 - \vartheta_2, \gamma = \pi & , \textit{ if } k \textit{ even}, -\vartheta_1 + \vartheta_2 \in [-\pi, 0) \\
\alpha = 0, \beta = -\vartheta_1 + \vartheta_2, \gamma = 0 & , \textit{ if } k \textit{ even}, -\vartheta_1 + \vartheta_2 \in [0, \pi) \\
\alpha = \pi, \beta = \vartheta_1 + \vartheta_2, \gamma = 0 & , \textit{ if } k \textit{ odd}, \vartheta_1 + \vartheta_2 \in [0, \pi) \\
\alpha = 0, \beta = 2\pi - (\vartheta_1 + \vartheta_2), \gamma = \pi & , \textit{ if } k \textit{ odd}, \vartheta_1 + \vartheta_2 \in [\pi, 2\pi)
\end{matrix}\right\}.$$

Proof. With $\xi_1 = \xi_1(t_1, \varphi_1)$, $\xi_2 = \xi_2(t_2, \varphi_2) \in \Omega$, $t_1 = \cos\vartheta_1$, $t_2 = \cos\vartheta_2$, we get for $N_1, N_2 \in \mathbb{Z}$ and for $n \in \mathbb{N}_0$, $n \geq \max\{|N_1|, |N_2|\}$, using Theorem 3.4.9, 4. from Remark

3.4.4, and Corollary 3.4.6,

$$\begin{aligned}\sum_{j=-n}^{n} {}_{N_1}Y_{n,j}(\xi_1)\,\overline{{}_{N_2}Y_{n,j}(\xi_2)} &= (-1)^{N_1+N_2}\frac{2n+1}{4\pi}\sum_{j=-n}^{n} e^{ij\varphi_1}d^n_{j,-N_1}(\vartheta_1)e^{-ij\varphi_2}d^n_{j,-N_2}(\vartheta_2)\\ &= (-1)^{N_1+N_2}\frac{2n+1}{4\pi}\sum_{j=-n}^{n} d^n_{-N_1,j}(-\vartheta_1)d^n_{j,-N_2}(\vartheta_2)e^{ij(\varphi_1-\varphi_2)}\\ &= (-1)^{N_1+N_2}\frac{2n+1}{4\pi}\; e^{i(-N_1)\alpha}d^n_{-N_1,-N_2}(\beta)e^{i(-N_2)\gamma},\end{aligned}$$

where

- for $\sin(\varphi_1-\varphi_2)\neq 0$, we get

$$\begin{aligned}\cot\alpha &= \cos(-\vartheta_1)\cot(\varphi_1-\varphi_2)+\cot\vartheta_2\frac{\sin(-\vartheta_1)}{\sin(\varphi_1-\varphi_2)}\\ &= \cos\vartheta_1\cot(\varphi_1-\varphi_2)-\cot\vartheta_2\frac{\sin\vartheta_1}{\sin(\varphi_1-\varphi_2)},\\ \cos\beta &= \cos(-\vartheta_1)\cos\vartheta_2-\sin(-\vartheta_1)\sin\vartheta_2\cos(\varphi_1-\varphi_2)\\ &= \cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\cos(\varphi_1-\varphi_2),\\ \cot\gamma &= \cos\vartheta_2\cot(\varphi_1-\varphi_2)+\cot(-\vartheta_1)\frac{\sin\vartheta_2}{\sin(\varphi_1-\varphi_2)}\\ &= \cos\vartheta_2\cot(\varphi_1-\varphi_2)-\cot\vartheta_1\frac{\sin\vartheta_2}{\sin(\varphi_1-\varphi_2)}.\end{aligned}$$

- for $\sin(\varphi_1-\varphi_2)=0$, we have $\varphi_1-\varphi_2=k\pi$, $k\in\mathbb{Z}$. Because $\xi_1,\xi_2\in\Omega$ is $\vartheta_1,\vartheta_2\in[0,\pi]$ and therefore, $-\vartheta_1\in[-\pi,0]$. So, it holds $(-\vartheta_1+\vartheta_2)\in[-\pi,\pi]$, $\vartheta_2\geq-\vartheta_1$ and $\vartheta_1+\vartheta_2\in[0,2\pi]$. Then,

$$\left\{\begin{array}{ll}\alpha=0,\beta=-\vartheta_1+\vartheta_2,\gamma=0 & ,\text{ if } k \text{ even}, -\vartheta_1+\vartheta_2\in[0,\pi)\\ \alpha=\pi,\beta=\vartheta_1-\vartheta_2,\gamma=\pi & ,\text{ if } k \text{ even}, -\vartheta_1+\vartheta_2\in[-\pi,0)\\ \alpha=0,\beta=2\pi-(\vartheta_1+\vartheta_2),\gamma=\pi & ,\text{ if } k \text{ odd}, \vartheta_1+\vartheta_2\in[\pi,2\pi)\\ \alpha=\pi,\beta=\vartheta_1+\vartheta_2,\gamma=0 & ,\text{ if } k \text{ odd}, \vartheta_1+\vartheta_2\in[0,\pi)\end{array}\right\}.$$

  We omit the cases $-\vartheta_1+\vartheta_2=\pi$ and $\vartheta_1+\vartheta_2=2\pi$, because of reasons of continuity. Besides, in this work we do not need those cases.

Continuing, we obtain

$$\sum_{j=-n}^{n} {}_{N_1}Y_{n,j}(\xi_1)\,\overline{{}_{N_2}Y_{n,j}(\xi_2)} = (-1)^{N_1+N_2}\frac{2n+1}{4\pi}\;\overline{D^n_{-N_1,-N_2}(\alpha,\beta,0)}e^{-iN_2\gamma}.$$

With Theorem 3.4.9, we get the proposition

$$\sum_{j=-n}^{n} {}_{N_1}Y_{n,j}(\xi_1)\,\overline{{}_{N_2}Y_{n,j}(\xi_2)} = (-1)^{N_1}\sqrt{\frac{2n+1}{4\pi}}\;{}_{N_2}Y_{n,-N_1}(\xi)\;e^{-iN_2\gamma},$$

where $\xi=\xi(t,\alpha)\in\Omega$ and $t=\cos\beta$. □

Remark 3.4.23. *The addition theorem was previously remarked by [6, 7, 42, 77, 98] without proof and without the special cases. For example, by [42] the addition theorem for* $\xi_1,\xi_2,\xi\in\Omega$

with $\xi = \xi(t, \alpha)$, $t = \cos\beta$, where α, β, γ are the Euler angles, is given by

$$\sum_{j=-n}^{n} \overline{{}_{N_1}Y_{n,j}(\xi_1)}\ {}_{N_2}Y_{n,j}(\xi_2) = \sqrt{\frac{2n+1}{4\pi}}\ {}_{N_2}Y_{n,-N_1}(\xi)e^{-iN_2\gamma}.$$

This is equal to the addition theorem from the previous theorem for an alternative system of the spin-weighted spherical harmonics (see Remark 3.4.10).

Corollary 3.4.24. *With $N_1 = N_2 = 0$, we obtain the well-known addition theorem for the fully normalized spherical harmonics, Theorem 2.4.28, because*

$$\begin{aligned}
&\sum_{j=-n}^{n} {}_0Y_{n,j}(\xi_1)\ \overline{{}_0Y_{n,j}(\xi_2)} = \sqrt{\frac{2n+1}{4\pi}}\ {}_0Y_{n,0}(\xi)\cdot 1\\
\Leftrightarrow &\sum_{j=-n}^{n} Y_{n,j}(\xi_1)\overline{Y_{n,j}(\xi_2)} = \frac{2n+1}{4\pi}\ P_n(\cos\beta)\\
\Leftrightarrow &\sum_{j=-n}^{n} Y_{n,j}(\xi_1)\overline{Y_{n,j}(\xi_2)} = \frac{2n+1}{4\pi}\ P_n(\xi_1\cdot\xi_2)
\end{aligned}$$

and

$$\begin{aligned}
\xi_1\cdot\xi_2 &= \begin{pmatrix}\sin\vartheta_1\cos\varphi_1\\ \sin\vartheta_1\sin\varphi_1\\ \cos\vartheta_1\end{pmatrix}\cdot\begin{pmatrix}\sin\vartheta_2\cos\varphi_2\\ \sin\vartheta_2\sin\varphi_2\\ \cos\vartheta_2\end{pmatrix}\\
&= \sin\vartheta_1\sin\vartheta_2\underbrace{(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)}_{\cos(\varphi_1-\varphi_2)}+\cos\vartheta_1\cos\vartheta_2\\
&= \cos\beta
\end{aligned}$$

for $\xi_1 = \xi_1(\cos\vartheta_1, \varphi_1)$, $\xi_2 = \xi_2(\cos\vartheta_2, \varphi_2)$, $\xi = \xi(\cos\beta, \varphi) \in \Omega$.

Corollary 3.4.25. *With $\xi_1 = \xi_2 = \xi$, $\xi_1 = \xi_1(t_1, \varphi_1), \xi_2 = \xi_2(t_2, \varphi_2), \xi = \xi(t, \varphi) \in \Omega$, we get on the one hand $\varphi_1 = \varphi_2$. So, $\sin(\varphi_1 - \varphi_2) = 0$ and $\varphi_1 - \varphi_2 = 0 = k\pi$, where $k = 0$ is even. On the other hand, $\vartheta_1 = \vartheta_2$, so $-\vartheta_1 + \vartheta_2 = 0$. Then, we obtain for the Euler angles $\alpha = \beta = \gamma = 0$. So, the addition theorem, Theorem 3.4.22, reduces with 7. from Remark 3.4.4 to*

$$\begin{aligned}
\sum_{j=-n}^{n} {}_{N_1}Y_{n,j}(\xi)\ \overline{{}_{N_2}Y_{n,j}(\xi)} &= (-1)^{N_1+N_2}\frac{2n+1}{4\pi}\ \overline{D^n_{-N_1,-N_2}}(0,0,0)\\
&= (-1)^{N_1+N_2}\frac{2n+1}{4\pi}\ \delta_{-N_1,-N_2}\\
&= \frac{2n+1}{4\pi}\ \delta_{N_1,N_2}.
\end{aligned}$$

Additionally, we obtain for $N_1 = N_2 = N$

$$\sum_{j=-n}^{n} {}_NY_{n,j}(\xi)\ \overline{{}_NY_{n,j}(\xi)} = \frac{2n+1}{4\pi}.$$

Lemma 3.4.26. *The complex conjugation of the spin-weighted spherical harmonics yields*

$$\overline{{}_NY_{n,j}} = (-1)^{N+j}\ {}_{-N}Y_{n,-j}$$

for $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, *and* $j = -n, \ldots, n$.

Proof. The proof is straight forward. For $\xi = \xi(t, \varphi) \in \Omega$, $t = \cos\vartheta$, $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, and $j = -n, \ldots, n$, we get with Theorem 3.4.9 and with 2. from Remark 3.4.4

$$\begin{aligned}
\overline{{}_NY_{n,j}}(\xi) &= (-1)^N \sqrt{\frac{2n+1}{4\pi}}\; e^{-ij\varphi} d^n_{j,-N}(\vartheta) \\
&= (-1)^N \sqrt{\frac{2n+1}{4\pi}}\; e^{-ij\varphi} (-1)^{N+j} d^n_{-j,N}(\vartheta) \\
&= (-1)^{N+j} (-1)^N \sqrt{\frac{2n+1}{4\pi}}\; e^{i(-j)\varphi} d^n_{-j,-(-N)}(\vartheta) \\
&= (-1)^{N+j}\; {}_{-N}Y_{n,-j}(\xi).
\end{aligned}$$

□

Now, we formulate the following new theorems, which are Green's second surface identity for the spin-weighted Beltrami operator on the unit sphere and on an arbitrary region.

Theorem 3.4.27. *With Theorem 2.3.8, we can formulate Green's second surface identity on the unit sphere for the operator* $\Delta^{*,N}$ *by*

$$\int_\Omega \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi) \right)\; d\omega(\xi) = 0$$

for $F, G \in \mathrm{X}^2(\Omega_0)$.

Proof. With the definition of the operator $\Delta^{*,N}$, Corollary 3.3.7, we get for $F, G \in \mathrm{X}^2(\Omega_0)$

$$\begin{aligned}
&\int_\Omega \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi) \right)\; d\omega(\xi) \\
&= \int_\Omega \Bigg(F(\xi)\overline{\Delta^*_\xi G(\xi)} - F(\xi)\; \frac{N^2 + 2itN\partial_\varphi}{1-t^2}\; \overline{G(\xi)} - \overline{G(\xi)}\Delta^*_\xi F(\xi) \\
&\quad + \overline{G(\xi)}\; \frac{N^2 - 2itN\partial_\varphi}{1-t^2}\; F(\xi) \Bigg)\; d\omega(\xi) \\
&= \int_\Omega \left(F(\xi)\Delta^*_\xi \overline{G(\xi)} - \overline{G(\xi)}\Delta^*_\xi F(\xi) \right)\; d\omega(\xi) + \int_\Omega \Bigg(-\frac{N^2}{1-t^2}\; F(\xi)\overline{G(\xi)} \\
&\quad - \frac{2itN}{1-t^2}\; F(\xi)\partial_\varphi \overline{G(\xi)} + \frac{N^2}{1-t^2}\; F(\xi)\overline{G(\xi)} - \frac{2itN}{1-t^2}\; \overline{G(\xi)}\partial_\varphi F(\xi) \Bigg)\; d\omega(\xi).
\end{aligned}$$

Here, we can use Green's second surface identity on the unit sphere, Theorem 2.3.8, such that we obtain

$$\begin{aligned}
&\int_\Omega \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi) \right)\; d\omega(\xi) \\
&= -\int_\Omega \frac{2itN}{1-t^2}\; \left(F(\xi)\partial_\varphi \overline{G(\xi)} + \overline{G(\xi)}\partial_\varphi F(\xi) \right)\; d\omega(\xi) \\
&= -\int_{-1}^{1} \frac{2itN}{1-t^2}\; \left(\int_0^{2\pi} F(\xi)\partial_\varphi \overline{G(\xi)}\; d\varphi + \int_0^{2\pi} \overline{G(\xi)}\partial_\varphi F(\xi)\; d\varphi \right)\; dt.
\end{aligned}$$

With integration by parts, we get

$$\int_0^{2\pi} F(\xi)\partial_\varphi \overline{G(\xi)}\ \mathrm{d}\varphi = \underbrace{F(\xi)\overline{G(\xi)}\Big|_{\varphi=0}^{\varphi=2\pi}}_{=0,\text{ because } F \text{ and } G \text{ are } 2\pi\text{-periodic in } \varphi} - \int_0^{2\pi} \overline{G(\xi)}\partial_\varphi F(\xi)\ \mathrm{d}\varphi.$$

So, the whole integral over φ exists and vanishes. Therefore, the integral over Ω exists and vanishes also and the singularities at the poles do not matter. Then, we get on the whole

$$\begin{aligned}
&\int_\Omega \left(F(\xi)\overline{\Delta_\xi^{*,N} G(\xi)} - \overline{G(\xi)}\Delta_\xi^{*,N} F(\xi)\right)\ \mathrm{d}\omega(\xi)\\
&= -\int_\Omega \frac{2itN}{1-t^2}\ \left(-\overline{G(\xi)}\partial_\varphi F(\xi) + \overline{G(\xi)}\partial_\varphi F(\xi)\right)\ \mathrm{d}\omega(\xi)\\
&= 0.
\end{aligned}$$

□

Theorem 3.4.28. *If we formulate Green's second surface identity for the operator $\Delta^{*,N}$ on a subset $\Gamma \subset \Omega$ with sufficiently smooth boundary $\partial\Gamma$, we get an additional term such that*

$$\begin{aligned}
&\int_\Gamma \left(F(\xi)\overline{\Delta_\xi^{*,N} G(\xi)} - \overline{G(\xi)}\Delta_\xi^{*,N} F(\xi)\right)\ \mathrm{d}\omega(\xi)\\
&= \int_{\partial\Gamma} \left(F(\xi)\frac{\partial}{\partial\nu(\xi)}\overline{G(\xi)} - \overline{G(\xi)}\frac{\partial}{\partial\nu(\xi)}F(\xi)\right)\ \mathrm{d}\sigma(\xi) - \int_\Gamma \frac{2iNt}{1-t^2}\ \partial_\varphi \left(F(\xi)\overline{G(\xi)}\right)\ \mathrm{d}\omega(\xi)
\end{aligned}$$

for $F, G \in \mathrm{X}^2\left(\overline{\Gamma}\right)$, if the according integrals exist.

Proof. With the definition of the operator $\Delta^{*,N}$, Corollary 3.3.7, we get for $F, G \in \mathrm{X}^2\left(\overline{\Gamma}\right)$ in analogy to the previous proof

$$\begin{aligned}
&\int_\Gamma \left(F(\xi)\overline{\Delta_\xi^{*,N} G(\xi)} - \overline{G(\xi)}\Delta_\xi^{*,N} F(\xi)\right)\ \mathrm{d}\omega(\xi)\\
&= \int_\Gamma \left(F(\xi)\Delta_\xi^{*}\overline{G(\xi)} - \overline{G(\xi)}\Delta_\xi^{*} F(\xi)\right)\ \mathrm{d}\omega(\xi) + \int_\Gamma \Bigg(-\frac{N^2}{1-t^2}\ F(\xi)\overline{G(\xi)}\\
&\quad -\frac{2itN}{1-t^2}\ F(\xi)\partial_\varphi\overline{G(\xi)} + \frac{N^2}{1-t^2}\ F(\xi)\overline{G(\xi)} - \frac{2itN}{1-t^2}\ \overline{G(\xi)}\partial_\varphi F(\xi)\Bigg)\ \mathrm{d}\omega(\xi).
\end{aligned}$$

Now, with Green's second surface identity, Theorem 2.3.8, we obtain

$$\begin{aligned}
&\int_\Gamma \left(F(\xi)\overline{\Delta_\xi^{*,N} G(\xi)} - \overline{G(\xi)}\Delta_\xi^{*,N} F(\xi)\right)\ \mathrm{d}\omega(\xi)\\
&= \int_{\partial\Gamma} \left(F(\xi)\frac{\partial}{\partial\nu(\xi)}\overline{G(\xi)} - \overline{G(\xi)}\frac{\partial}{\partial\nu(\xi)}F(\xi)\right)\ \mathrm{d}\sigma(\xi)\\
&\quad - \int_\Gamma \frac{2itN}{1-t^2}\ \underbrace{\left(F(\xi)\partial_\varphi\overline{G(\xi)} + \overline{G(\xi)}\partial_\varphi F(\xi)\right)}_{=\partial_\varphi\left(F(\xi)\overline{G(\xi)}\right)}\ \mathrm{d}\omega(\xi).
\end{aligned}$$

□

3.5 Properties of the Operator ð

In this chapter, we look at further details of the operator ð. Most of the properties are previously mentioned in literature without proof.

Lemma 3.5.1. *The operators* ð *and* $\overline{ð}$ *are linear [91], this means that for all functions* $F, G \in \mathrm{C}^{(1)}(\Omega_0)$ *and* $N \in \mathbb{Z}$*, we get*

$$\begin{aligned} ð_N(\lambda F) &= \lambda ð_N\ F, \\ ð_N(F+G) &= ð_N\ F + ð_N\ G \end{aligned}$$

and

$$\begin{aligned} \overline{ð}_N(\lambda F) &= \lambda \overline{ð}_N\ F, \\ \overline{ð}_N(F+G) &= \overline{ð}_N\ F + \overline{ð}_N\ G, \end{aligned}$$

because of the linearity and additivity of the partial derivatives.

Lemma 3.5.2. *The operators* ð *and* $\overline{ð}$ *satisfy the Leibniz rule, this means that for every function* $F, G \in \mathrm{C}^{(1)}(\Omega_0)$ *and for every* $N, M \in \mathbb{Z}$ *the following equations are fulfilled [17]*

$$\begin{aligned} ð_{N+M}(FG) &= (ð_N\ F)\, G + F\, (ð_M\ G)\,, \\ \overline{ð}_{N+M}(FG) &= \left(\overline{ð}_N\ F\right) G + F \left(\overline{ð}_M\ G\right). \end{aligned}$$

Proof. The proof is straight forward by using Definition 3.2.3. For $\xi = \xi(t, \varphi) \in \Omega_0$, we get for $N, M \in \mathbb{Z}$

$$\begin{aligned} ð_{N+M}\left(F(\xi)G(\xi)\right) &= \left(\sqrt{1-t^2}\partial_t + \frac{(N+M)t - i\partial_\varphi}{\sqrt{1-t^2}}\right)\left(F(\xi)G(\xi)\right) \\ &= \sqrt{1-t^2}\left(\partial_t F(\xi)\right) G(\xi) + F(\xi)\sqrt{1-t^2}\left(\partial_t G(\xi)\right) + \frac{Nt}{\sqrt{1-t^2}}\ F(\xi)G(\xi) \\ &\quad + F(\xi)\frac{Mt}{\sqrt{1-t^2}}\ G(\xi) - \frac{i}{\sqrt{1-t^2}}\left(\partial_\varphi F(\xi)\right) G(\xi) - F(\xi)\frac{i}{\sqrt{1-t^2}}\left(\partial_\varphi G(\xi)\right) \\ &= \left(\sqrt{1-t^2}\partial_t F(\xi) + \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\ F(\xi)\right) G(\xi) \\ &\quad + F(\xi)\left(\sqrt{1-t^2}\partial_t G(\xi) + \frac{Mt - i\partial_\varphi}{\sqrt{1-t^2}}\ G(\xi)\right) \\ &= \left(ð_N\ F(\xi)\right) G(\xi) + F(\xi)\left(ð_M\ G(\xi)\right). \end{aligned}$$

The second equation follows analogously.

$$\begin{aligned} \overline{ð}_{N+M}\left(F(\xi)G(\xi)\right) &= \left(\sqrt{1-t^2}\partial_t - \frac{(N+M)t - i\partial_\varphi}{\sqrt{1-t^2}}\right)\left(F(\xi)G(\xi)\right) \\ &= \sqrt{1-t^2}\left(\partial_t F(\xi)\right) G(\xi) + F(\xi)\sqrt{1-t^2}\left(\partial_t G(\xi)\right) - \frac{Nt}{\sqrt{1-t^2}}\ F(\xi)G(\xi) \\ &\quad - F(\xi)\frac{Mt}{\sqrt{1-t^2}}\ G(\xi) + \frac{i}{\sqrt{1-t^2}}\left(\partial_\varphi F(\xi)\right) G(\xi) + F(\xi)\frac{i}{\sqrt{1-t^2}}\left(\partial_\varphi G(\xi)\right) \\ &= \left(\sqrt{1-t^2}\partial_t F(\xi) - \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\ F(\xi)\right) G(\xi) \\ &\quad + F(\xi)\left(\sqrt{1-t^2}\partial_t G(\xi) - \frac{Mt - i\partial_\varphi}{\sqrt{1-t^2}}\ G(\xi)\right) \end{aligned}$$

$$= \left(\overline{\eth}_N\ F(\xi)\right) G(\xi) + F(\xi) \left(\overline{\eth}_M\ G(\xi)\right).$$

□

Lemma 3.5.3. *If we apply the operators* $\eth$ *and* $\overline{\eth}$ *to the functions* o^k *and* $\hat{o}^k$ *for* $k = 1, 2$, *we get [91]*

$$\eth_{\frac{1}{2}}\ o^k = 0, \qquad \eth_{-\frac{1}{2}}\ \hat{o}^k = o^k, \qquad \overline{\eth}_{\frac{1}{2}}\ o^k = -\hat{o}^k, \qquad \overline{\eth}_{-\frac{1}{2}}\ \hat{o}^k = 0.$$

Proof. The proof is straight forward. Let $\xi = \xi(t, \varphi) \in \Omega_0$.

- The first equation.

$$\begin{aligned}
\eth_{\frac{1}{2}}\ o^1_\xi &= \left(\sqrt{1-t^2}\partial_t + \frac{\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}}\right) e^{-i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}} \\
&= e^{-i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \frac{1}{2}\left(\sqrt{1+t}\right)^{-1} + \frac{\frac{t}{2}}{\sqrt{1-t}} - \frac{i(-i)\frac{1}{2}}{\sqrt{1-t}}\right) \\
&= e^{-i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1-t)}} (1 - t + t - 1) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\eth_{\frac{1}{2}}\ o^2_\xi &= \left(\sqrt{1-t^2}\partial_t + \frac{\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}}\right) e^{i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}} \\
&= e^{i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \left(-\frac{1}{2}\right)\left(\sqrt{1-t}\right)^{-1} + \frac{\frac{t}{2}}{\sqrt{1+t}} - \frac{i\frac{i}{2}}{\sqrt{1+t}}\right) \\
&= e^{i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1+t)}} (-1 - t + t + 1) \\
&= 0.
\end{aligned}$$

- The second equation.

$$\begin{aligned}
\eth_{-\frac{1}{2}}\ \hat{o}^1_\xi &= \left(\sqrt{1-t^2}\partial_t + \frac{-\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}}\right) \left(-e^{-i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}}\right) \\
&= -e^{-i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \left(-\frac{1}{2}\right)\left(\sqrt{1-t}\right)^{-1} - \frac{\frac{t}{2}}{\sqrt{1+t}} - \frac{i(-i)\frac{1}{2}}{\sqrt{1+t}}\right) \\
&= -e^{-i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1+t)}} (-1 - t - t - 1) \\
&= e^{-i\frac{\varphi}{2}} \frac{1}{\sqrt{2(1+t)}} (1+t) \\
&= e^{-i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}} \\
&= o^1_\xi
\end{aligned}$$

and

$$\eth_{-\frac{1}{2}}\ \hat{o}^2_\xi = \left(\sqrt{1-t^2}\partial_t + \frac{-\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}}\right) e^{i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}}$$

$$
\begin{aligned}
&= e^{i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \frac{1}{2}\left(\sqrt{1+t}\right)^{-1} - \frac{\frac{t}{2}}{\sqrt{1-t}} - \frac{i\frac{i}{2}}{\sqrt{1-t}} \right) \\
&= e^{i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1-t)}} (1-t-t+1) \\
&= e^{i\frac{\varphi}{2}} \frac{1}{\sqrt{2(1-t)}} (1-t) \\
&= e^{i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}} \\
&= o_\xi^2.
\end{aligned}
$$

- The third equation.

$$
\begin{aligned}
\overline{ð}_{\frac{1}{2}}\ o_\xi^1 &= \left(\sqrt{1-t^2}\partial_t - \frac{\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}} \right) e^{-i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}} \\
&= e^{-i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \frac{1}{2}\left(\sqrt{1+t}\right)^{-1} - \frac{\frac{t}{2}}{\sqrt{1-t}} + \frac{i(-i)\frac{1}{2}}{\sqrt{1-t}} \right) \\
&= e^{-i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1-t)}} (1-t-t+1) \\
&= e^{-i\frac{\varphi}{2}} \frac{1}{\sqrt{2(1-t)}} (1-t) \\
&= e^{-i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}} \\
&= -\hat{o}_\xi^1
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{ð}_{\frac{1}{2}}\ o_\xi^2 &= \left(\sqrt{1-t^2}\partial_t - \frac{\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}} \right) e^{i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}} \\
&= e^{i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \left(-\frac{1}{2}\right)\left(\sqrt{1-t}\right)^{-1} - \frac{\frac{t}{2}}{\sqrt{1+t}} + \frac{i\frac{i}{2}}{\sqrt{1+t}} \right) \\
&= e^{i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1+t)}} (-1-t-t-1) \\
&= -e^{i\frac{\varphi}{2}} \frac{1}{\sqrt{2(1+t)}} (1+t) \\
&= -e^{i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}} \\
&= -\hat{o}_\xi^2.
\end{aligned}
$$

- The fourth equation.

$$
\begin{aligned}
\overline{ð}_{-\frac{1}{2}}\ \hat{o}_\xi^1 &= \left(\sqrt{1-t^2}\partial_t - \frac{-\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}} \right) \left(-e^{-i\frac{\varphi}{2}} \sqrt{\frac{1-t}{2}} \right) \\
&= -e^{-i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \left(-\frac{1}{2}\right)\left(\sqrt{1-t}\right)^{-1} + \frac{\frac{t}{2}}{\sqrt{1+t}} + \frac{i(-i)\frac{1}{2}}{\sqrt{1+t}} \right)
\end{aligned}
$$

$$= -e^{-i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1+t)}} (-1 - t + t + 1)$$
$$= 0$$

and

$$\begin{aligned}
\overline{\eth}_{-\frac{1}{2}}\ \hat{o}_\xi^2 &= \left(\sqrt{1-t^2}\partial_t - \frac{-\frac{t}{2} - i\partial_\varphi}{\sqrt{1-t^2}}\right) e^{i\frac{\varphi}{2}} \sqrt{\frac{1+t}{2}} \\
&= e^{i\frac{\varphi}{2}} \frac{1}{\sqrt{2}} \left(\sqrt{1-t}\sqrt{1+t}\ \frac{1}{2}\left(\sqrt{1+t}\right)^{-1} + \frac{\frac{t}{2}}{\sqrt{1-t}} + \frac{i\frac{i}{2}}{\sqrt{1-t}}\right) \\
&= e^{i\frac{\varphi}{2}} \frac{1}{2\sqrt{2(1-t)}} (1 - t + t - 1) \\
&= 0.
\end{aligned}$$

□

The operator $\eth$ is called spin raising and the operator $\overline{\eth}$ is called spin lowering [97]. We see this in the following theorem.

Theorem 3.5.4. *Let ${}_NF_n \in \mathrm{C}^{(1)}(\Omega_0)$ be a function of spin weight $N \in \mathbb{Z}$ and degree $n \in \mathbb{N}_0$, $n \geq |N|$. Then, from [15] the operator $\eth$ raises the spin weight by one*

$$\mathrm{sw}\left(\eth_N\ {}_NF_n\right) = N + 1$$

and the operator $\overline{\eth}$ lowers it by one

$$\mathrm{sw}\left(\overline{\eth}_N\ {}_NF_n\right) = N - 1.$$

Proof. With Definition 3.1.2, with Lemma 3.5.2, and with the previous lemma, we get for $N \in \mathbb{Z}$ and for $n \in \mathbb{N}_0$, $n \geq |N|$, on the one hand

$$\begin{aligned}
\eth_N\ {}_NF_n = &\sum_{i_1,\dots,i_{2n}=1}^{2} d_{i_1 i_2 \dots i_{2n}} \left(\sum_{j=1}^{n+N} o^{i_1} \dots o^{i_{j-1}} o^{i_{j+1}} \dots o^{i_{n+N}} \underbrace{\left(\eth_{\frac{1}{2}} o^{i_j}\right)}_{=0} \hat{o}^{i_{n+N+1}} \dots \hat{o}^{i_{2n}} \right. \\
&\left. + \sum_{j=n+N+1}^{2n} o^{i_1} \dots o^{i_{n+N}} \hat{o}^{i_{n+N+1}} \dots \hat{o}^{i_{j-1}} \hat{o}^{i_{j+1}} \dots \hat{o}^{i_{2n}} \underbrace{\left(\eth_{-\frac{1}{2}} \hat{o}^{i_j}\right)}_{=o^{i_j}} \right) \\
= &\sum_{i_1,\dots,i_{2n}=1}^{2} \tilde{d}_{i_1 i_2 \dots i_{2n}} \underbrace{o^{i_1} \dots o^{i_{n+N+1}}}_{n+(N+1)} \underbrace{\hat{o}^{i_{n+N+2}} \dots \hat{o}^{i_{2n}}}_{n-(N+1)},
\end{aligned}$$

where we use the symmetry of the coefficients, such that

$$\mathrm{sw}\left(\eth_N\ {}_NF_n\right) = \frac{1}{2}(n + (N+1)) + \left(-\frac{1}{2}\right)(n - (N+1)) = N + 1.$$

On the other hand, we obtain

$$\begin{aligned}\overline{\eth}_N\ {}_NF_n =& \sum_{i_1,\dots,i_{2n}=1}^{2} d_{i_1 i_2 \dots i_{2n}} \left(\sum_{j=1}^{n+N} o^{i_1}\dots o^{i_{j-1}} o^{i_{j+1}} \dots o^{i_{n+N}} \underbrace{\left(\overline{\eth}_{\frac{1}{2}} o^{i_j}\right)}_{=-\hat{o}^{i_j}} \hat{o}^{i_{n+N+1}} \dots \hat{o}^{i_{2n}} \right. \\ &\left. + \sum_{j=n+N+1}^{2n} o^{i_1} \dots o^{i_{n+N}} \hat{o}^{i_{n+N+1}} \dots \hat{o}^{i_{j-1}} \hat{o}^{i_{j+1}} \dots \hat{o}^{i_{2n}} \underbrace{\left(\overline{\eth}_{-\frac{1}{2}} \hat{o}^{i_j}\right)}_{=0} \right) \\ =& - \sum_{i_1,\dots,i_{2n}=1}^{2} d_{i_1 i_2 \dots i_{2n}} \underbrace{o^{i_1}\dots o^{i_{n+N-1}}}_{n+(N-1)} \underbrace{\hat{o}^{i_{n+N}} \dots \hat{o}^{i_{2n}}}_{n-(N-1)},\end{aligned}$$

such that

$$\mathrm{sw}\left(\overline{\eth}_N\ {}_NF_n\right) = \frac{1}{2}(n+(N-1)) + \left(-\frac{1}{2}\right)(n-(N-1)) = N-1.$$

□

Corollary 3.5.5. *From the previous theorem, for every function ${}_NF_n \in \mathrm{C}^{(M)}(\Omega_0)$ of spin weight $N \in \mathbb{Z}$ and degree $n \in \mathbb{N}_0$, $n \geq |N|$, and for $M \in \mathbb{N}_0$, we conclude that*

$$\mathrm{sw}\left(\eth_N^M\ {}_NF_n\right) = N+M$$

and

$$\mathrm{sw}\left(\overline{\eth}_N^M\ {}_NF_n\right) = N-M.$$

Remark 3.5.6. *Now, we can show that the spin-weighted spherical harmonics ${}_NY_{n,j}$ of spin weight $N \in \mathbb{Z}$, degree $n \in \mathbb{N}_0$, $n \geq |N|$, and order $j=-n,\dots,n$ do have spin weight N [15].*

If we take a look at the spherical harmonics from Definition 2.4.37

$$Y_{n,j}(\xi(t,\varphi)) = \begin{cases} (-1)^j \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-j)!}{(n+j)!}} P_{n,j}(t) e^{ij\varphi}, & j \geq 0 \\ (-1)^j \overline{Y_{n,-j}(\xi(t,\varphi))}, & j<0 \end{cases}$$

for $\xi \in \Omega$ and use the expression from Lemma 2.4.15

$$P_{n,j}(t) = \left(1-t^2\right)^{\frac{j}{2}} \sum_{k=0}^{\left\lfloor \frac{n-j}{2} \right\rfloor} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-j-2k)!} t^{n-j-2k},$$

we can prove that the spherical harmonics have spin weight zero. Because

$$\begin{aligned}\left(1-t^2\right)^{\frac{j}{2}} e^{ij\varphi} &= 2^j \left(\sqrt{\frac{1-t}{2}}\right)^j e^{ij\frac{\varphi}{2}} \left(\sqrt{\frac{1+t}{2}}\right)^j e^{ij\frac{\varphi}{2}} \\ &= 2^j \left(\sqrt{\frac{1-t}{2}} e^{i\frac{\varphi}{2}}\right)^j \left(\sqrt{\frac{1+t}{2}} e^{i\frac{\varphi}{2}}\right)^j \\ &= 2^j \left(o_\xi^2\right)^j \left(\hat{o}_\xi^2\right)^j,\end{aligned}$$

we get

$$\operatorname{sw}\left(\left(1-t^2\right)^{\frac{j}{2}} e^{ij\varphi}\right) = \frac{1}{2}\, j + \left(-\frac{1}{2}\right) j = 0.$$

It is obvious that

$$\begin{aligned} t &= \frac{1}{2}\left((1+t)-(1-t)\right) \\ &= \frac{1}{2}\left(\sqrt{1+t}\sqrt{1+t} - \sqrt{1-t}\sqrt{1-t}\right) \\ &= \frac{1}{2}\left(2\left(\sqrt{\frac{1+t}{2}}e^{-i\frac{\varphi}{2}}\right)\left(\sqrt{\frac{1+t}{2}}e^{i\frac{\varphi}{2}}\right) + 2\left(\sqrt{\frac{1-t}{2}}e^{i\frac{\varphi}{2}}\right)\left(-\sqrt{\frac{1-t}{2}}e^{-i\frac{\varphi}{2}}\right)\right) \\ &= o_\xi^1 \hat{o}_\xi^2 + o_\xi^2 \hat{o}_\xi^1. \end{aligned} \tag{3.19}$$

Because

$$\operatorname{sw}\left(o_\xi^1 \hat{o}_\xi^2\right) = \operatorname{sw}\left(o_\xi^2 \hat{o}_\xi^1\right) = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0,$$

we get

$$\operatorname{sw}(t) = 0$$

and therefore, with Lemma 3.1.7 for all $k = 0, \ldots, \left\lfloor \frac{n-j}{2} \right\rfloor$,

$$\operatorname{sw}\left(t^{n-j-2k}\right) = (n-j-2k)\operatorname{sw}(t) = 0.$$

Then, we obtain for $j \geq 0$

$$\operatorname{sw}(Y_{n,j}) = \operatorname{sw}\left(\left(1-t^2\right)^{\frac{j}{2}} e^{ij\varphi}\right) = 0$$

and for $j < 0$

$$\operatorname{sw}(Y_{n,j}) = \operatorname{sw}\left(\overline{Y_{n,-j}}\right) = -\operatorname{sw}(Y_{n,-j}) = 0.$$

Altogether, we get

$$\operatorname{sw}(Y_{n,j}) = 0.$$

With Definition 3.2.6 and with the previous corollary, we see that for $N \geq 0$

$$\operatorname{sw}\left({}_N Y_{n,j}\right) = \operatorname{sw}\left(\eth_0^N\ Y_{n,j}\right) = 0 + N = N$$

and for $N \leq 0$

$$\operatorname{sw}\left({}_N Y_{n,j}\right) = \operatorname{sw}\left(\bar{\eth}_0^{-N}\ Y_{n,j}\right) = 0 - (-N) = N.$$

Remark 3.5.7. *We cannot conclude uniquely from a function* F_n *of degree* $n \in \mathbb{N}_0$ *to its spin weight* $N \in \mathbb{Z}$ *with* $n \geq |N|$*. This means that arbitrary functions can have different spin weights. For example, from Remark 3.2.12, we know already that for* $\xi = \xi(t,\varphi) \in \Omega_0$

$$\sqrt{1-t^2} = \sqrt{\frac{8\pi}{3}}\ {}_1Y_{1,0}(\xi).$$

With Lemma 3.2.9, we also get

$$\begin{aligned} {}_{-1}Y_{1,0}(\xi) &= \frac{-1}{\sqrt{1(1+1)-0(0-1)}}\ \bar{\eth}_0\ {}_0Y_{1,0}(\xi) \\ &= \frac{-1}{\sqrt{2}}\left(\sqrt{1-t^2}\partial_t - \frac{0\cdot t - i\partial_\varphi}{\sqrt{1-t^2}}\right) Y_{1,0}(\xi) \end{aligned}$$

$$= \frac{-1}{\sqrt{2}}\left(\sqrt{1-t^2}\partial_t + \frac{i\partial_\varphi}{\sqrt{1-t^2}}\right)\sqrt{\frac{3}{4\pi}}t$$
$$= -\sqrt{\frac{3}{8\pi}}\sqrt{1-t^2}.$$

Now, we use Definition 3.1.2 and see that on the one hand

$$\sqrt{1-t^2} = 2\left(\sqrt{\frac{1+t}{2}}e^{-i\frac{\varphi}{2}}\right)\left(\sqrt{\frac{1-t}{2}}e^{i\frac{\varphi}{2}}\right)$$
$$= 2o_\xi^1 o_\xi^2$$

and on the other hand

$$\sqrt{1-t^2} = -2\left(-\sqrt{\frac{1-t}{2}}e^{-i\frac{\varphi}{2}}\right)\left(\sqrt{\frac{1+t}{2}}e^{i\frac{\varphi}{2}}\right)$$
$$= -2\hat{o}_\xi^1 \hat{o}_\xi^2.$$

So,

$$\sqrt{1-t^2} = \sqrt{\frac{8\pi}{3}}\ {}_1Y_{1,0}$$
$$= 2o_\xi^1 o_\xi^2$$

and

$$\sqrt{1-t^2} = -\sqrt{\frac{8\pi}{3}}\ {}_{-1}Y_{1,0}$$
$$= -2\hat{o}_\xi^1 \hat{o}_\xi^2$$

have spin weight 1 *and spin weight* -1 *at the same time. As another example, we get from Definition 2.4.37*

$$Y_{2,0}(\xi(t,\varphi)) = (-1)^0\sqrt{\frac{2\cdot 2+1}{4\pi}}\sqrt{\frac{(2-0)!}{(2+0)!}}\ \frac{1}{2^2 2!}\ \left(1-t^2\right)^0\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2+0}\left(t^2-1\right)^2$$
$$= \sqrt{\frac{5}{4\pi}}\ \frac{1}{8}\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^4-2t^2+1\right)$$
$$= \sqrt{\frac{5}{4\pi}}\ \frac{1}{8}\ \left(12t^2-4\right)$$
$$= \sqrt{\frac{5}{16\pi}}\ \left(3t^2-1\right).$$

Then, with Lemma 3.2.9, we obtain

$${}_1Y_{2,0}(\xi(t,\varphi)) = \frac{1}{\sqrt{2(2+1)-0(0+1)}}\ ð_0\ {}_0Y_{2,0}(\xi)$$
$$= \frac{1}{\sqrt{6}}\left(\sqrt{1-t^2}\partial_t + \frac{0\cdot t - i\partial_\varphi}{\sqrt{1-t^2}}\right)\sqrt{\frac{5}{16\pi}}\ \left(3t^2-1\right)$$
$$= \sqrt{\frac{5}{96\pi}}\sqrt{1-t^2}\cdot 6t$$

$$= \sqrt{\frac{15}{8\pi}} \sqrt{1-t^2}\, t$$

and

$$\begin{aligned} {}_{-1}Y_{2,0}(\xi(t,\varphi)) &= \frac{-1}{\sqrt{2(2+1)-0(0-1)}}\ \overline{\eth}_0\ {}_0Y_{2,0}(\xi) \\ &= \frac{-1}{\sqrt{6}} \left(\sqrt{1-t^2}\partial_t - \frac{0 \cdot t - i\partial_\varphi}{\sqrt{1-t^2}} \right) \sqrt{\frac{5}{16\pi}}\ (3t^2-1) \\ &= -\sqrt{\frac{15}{8\pi}} \sqrt{1-t^2}\, t. \end{aligned}$$

Furthermore, again with (3.19)

$$t = o^1_\xi \hat{o}^2_\xi + o^2_\xi \hat{o}^1_\xi$$

and therefore,

$$\begin{aligned} \sqrt{1-t^2}\ t &= \sqrt{\frac{8\pi}{15}}\ {}_1Y_{2,0}(\xi) \\ &= 2\left(o^1_\xi o^1_\xi o^2_\xi \hat{o}^2_\xi + o^1_\xi o^2_\xi o^2_\xi \hat{o}^1_\xi \right) \end{aligned}$$

and

$$\begin{aligned} \sqrt{1-t^2}\ t &= -\sqrt{\frac{8\pi}{15}}\ {}_{-1}Y_{2,0}(\xi) \\ &= -2\left(o^1_\xi \hat{o}^1_\xi \hat{o}^2_\xi \hat{o}^2_\xi + o^2_\xi \hat{o}^1_\xi \hat{o}^1_\xi \hat{o}^2_\xi \right) \end{aligned}$$

also have spin weight 1 *and spin weight* -1 *at the same time.*

More generally, the spin-weighted spherical harmonics of spin weight $N \in \mathbb{Z}$*, degree* $n \in \mathbb{N}_0$*, and order zero have two possible spin weights. This means that*

$${}_NY_{n,0} = (-1)^N\ {}_{-N}Y_{n,0},$$

because we know from Lemma 3.4.26

$$\overline{{}_NY_{n,0}} = (-1)^N\ {}_{-N}Y_{n,0}$$

and from Theorem 3.4.9

$$\overline{{}_NY_{n,0}}(\xi) = (-1)^N \sqrt{\frac{2n+1}{4\pi}}\ d^n_{0,-N}(\vartheta) = {}_NY_{n,0}(\xi)$$

for $\xi = \xi(\cos\vartheta, \varphi) \in \Omega_0$*. So, they also have the same possibilities of representation in the functions* o^k *and* $\hat{o}^k$*,* $k = 1, 2$*.*

Next, we show the needed properties borrowed from [63], where they are given without proof.

Lemma 3.5.8. *For all functions* $A, B \in \mathrm{C}^{(1)}(\Omega_0)$ *and for all* $N \in \mathbb{Z}$*, we get [63]*

$$\int_\Omega \left(\eth_N\ A(\xi)\right) B(\xi)\ \mathrm{d}\omega(\xi) = -\int_\Omega A(\xi)\ \left(\eth_{-(N+1)}\ B(\xi)\right)\ \mathrm{d}\omega(\xi)$$

and

$$\int_\Omega \left(\overline{\eth}_N\ A(\xi)\right) B(\xi)\ \mathrm{d}\omega(\xi) = -\int_\Omega A(\xi)\ \left(\overline{\eth}_{-(N-1)}\ B(\xi)\right)\ \mathrm{d}\omega(\xi)$$

as far as the left- or the right-hand integral exists.

Proof. We prove only the first identity. The second one follows analogously. Let $A, B \in \mathrm{C}^{(1)}(\Omega_0)$. Then, for all $N \in \mathbb{Z}$, we get with the definition of the operator ð from Definition 3.2.3

$$\begin{aligned}
&\int_\Omega (\eth_N\ A(\xi))\, B(\xi)\ \mathrm{d}\omega(\xi)\\
&= \int_\Omega \left(\sqrt{1-t^2}\,(\partial_t A(\xi))\, B(\xi) + B(\xi)\ \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\ A(\xi)\right)\ \mathrm{d}\omega(\xi)\\
&= \int_0^{2\pi}\int_{-1}^1 (\partial_t A(\xi))\,\sqrt{1-t^2}\ B(\xi)\ \mathrm{d}t\ \mathrm{d}\varphi - \int_\Omega \frac{t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\ \mathrm{d}\omega(\xi)\\
&\quad - \int_\Omega \frac{(-1-N)t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\ \mathrm{d}\omega(\xi) - \int_{-1}^1 \frac{i}{\sqrt{1-t^2}}\int_0^{2\pi} (\partial_\varphi A(\xi))\, B(\xi)\ \mathrm{d}\varphi\ \mathrm{d}t.
\end{aligned}$$

Now, we integrate by parts such that

$$\begin{aligned}
&\int_\Omega (\eth_N\ A(\xi))\, B(\xi)\ \mathrm{d}\omega(\xi)\\
&= \int_0^{2\pi}\left(\underbrace{\sqrt{1-t^2}A(\xi)B(\xi)\Big|_{t=-1}^{t=1}}_{=0} - \int_{-1}^1 A(\xi)\,\underbrace{\partial_t\left(\sqrt{1-t^2}\ B(\xi)\right)}_{=\frac{-t}{\sqrt{1-t^2}}B(\xi)+\sqrt{1-t^2}\partial_t B(\xi)}\ \mathrm{d}t\right)\ \mathrm{d}\varphi\\
&\quad - \int_\Omega \frac{t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\ \mathrm{d}\omega(\xi) - \int_\Omega \frac{-(N+1)t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\ \mathrm{d}\omega(\xi)\\
&\quad - \int_{-1}^1 \frac{i}{\sqrt{1-t^2}}\left(\underbrace{A(\xi)B(\xi)|_{\varphi=0}^{\varphi=2\pi}}_{=0} - \int_0^{2\pi} A(\xi)\,(\partial_\varphi B(\xi))\ \mathrm{d}\varphi\right)\ \mathrm{d}t\\
&= \int_\Omega \frac{t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\ \mathrm{d}\omega(\xi) - \int_\Omega \sqrt{1-t^2}\ A(\xi)\,(\partial_t B(\xi))\ \mathrm{d}\omega(\xi)\\
&\quad - \int_\Omega \frac{t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\ \mathrm{d}\omega(\xi) - \int_\Omega \frac{-(N+1)t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\ \mathrm{d}\omega(\xi)\\
&\quad + \int_\Omega \frac{i}{\sqrt{1-t^2}}\ A(\xi)\,(\partial_\varphi B(\xi))\ \mathrm{d}\omega(\xi)\\
&= -\int_\Omega \left(\sqrt{1-t^2}\ A(\xi)\,(\partial_t B(\xi)) + \frac{-(N+1)t}{\sqrt{1-t^2}}\ A(\xi)B(\xi)\right.\\
&\quad \left. - \frac{i}{\sqrt{1-t^2}}\ A(\xi)\,(\partial_\varphi B(\xi))\right)\ \mathrm{d}\omega(\xi)\\
&= -\int_\Omega A(\xi)\ \left(\eth_{-(N+1)}\ B(\xi)\right)\ \mathrm{d}\omega(\xi).
\end{aligned}$$

□

Now, we show more properties of these operators.

Lemma 3.5.9. *The Beltrami operator can be described by the operators* ð *and* $\overline{\eth}$. *For every* *function $F \in \mathrm{C}^{(2)}(\Omega_0)$, we get [91]*

$$\overline{\eth}_1 \eth_0 \; F = \Delta^* F = \eth_{-1} \overline{\eth}_0 \; F.$$

Proof. The first equation is proved for $\xi = \xi(t, \varphi) \in \Omega_0$ by

$$\begin{aligned}
&\overline{\eth}_1 \eth_0 \; F(\xi) \\
&= \left(\sqrt{1-t^2}\partial_t - \frac{t - i\partial_\varphi}{\sqrt{1-t^2}} \right) \left(\sqrt{1-t^2}\partial_t F(\xi) - \frac{i\partial_\varphi}{\sqrt{1-t^2}} \; F(\xi) \right) \\
&= \sqrt{1-t^2}\frac{1}{2}(-2t)\frac{1}{\sqrt{1-t^2}}\partial_t F(\xi) + \left(1-t^2\right)\partial_t^2 F(\xi) - i\sqrt{1-t^2}\left(-\frac{1}{2}\right)(-2t)\frac{1}{(1-t^2)^{\frac{3}{2}}}\partial_\varphi F(\xi) \\
&\quad - i\partial_t \partial_\varphi F(\xi) - t\partial_t F(\xi) + i\partial_\varphi \partial_t F(\xi) + \frac{it}{1-t^2}\partial_\varphi F(\xi) - \frac{i \cdot i}{1-t^2}\partial_\varphi^2 F(\xi) \\
&= -t\partial_t F(\xi) + \left(1-t^2\right)\partial_t^2 F(\xi) - \frac{it}{1-t^2}\partial_\varphi F(\xi) - t\partial_t F(\xi) + \frac{it}{1-t^2}\partial_\varphi F(\xi) + \frac{1}{1-t^2}\partial_\varphi^2 F(\xi) \\
&= \left(1-t^2\right)\partial_t^2 F(\xi) - 2t\partial_t F(\xi) + \frac{1}{1-t^2}\partial_\varphi^2 F(\xi) \\
&= \partial_t \left(\left(1-t^2\right)\partial_t F(\xi) \right) + \frac{1}{1-t^2}\partial_\varphi^2 F(\xi) \\
&= \Delta_\xi^* F(\xi).
\end{aligned}$$

For the second equation, we get for $\xi = \xi(t, \varphi) \in \Omega_0$

$$\begin{aligned}
&\eth_{-1} \overline{\eth}_0 \; F(\xi) \\
&= \left(\sqrt{1-t^2}\partial_t + \frac{-t - i\partial_\varphi}{\sqrt{1-t^2}} \right) \left(\sqrt{1-t^2}\partial_t F(\xi) + \frac{i\partial_\varphi}{\sqrt{1-t^2}} \; F(\xi) \right) \\
&= -t\partial_t F(\xi) + \left(1-t^2\right)\partial_t^2 F(\xi) + \frac{it}{1-t^2}\partial_\varphi F(\xi) + i\partial_t \partial_\varphi F(\xi) - t\partial_t F(\xi) - i\partial_\varphi \partial_t F(\xi) \\
&\quad - \frac{it}{1-t^2}\partial_\varphi F(\xi) - \frac{i \cdot i}{1-t^2}\partial_\varphi^2 F(\xi) \\
&= \left(1-t^2\right)\partial_t^2 F(\xi) - 2t\partial_t F(\xi) + \frac{1}{1-t^2}\partial_\varphi^2 F(\xi) \\
&= \Delta_\xi^* F(\xi).
\end{aligned}$$

□

Lemma 3.5.10. *Furthermore, we see that for every function $F \in \mathrm{C}^{(2)}(\Omega_0)$ and $N \in \mathbb{Z}$ [53, 63]*

$$\left(\overline{\eth}_{N+1} \eth_N - \eth_{N-1} \overline{\eth}_N \right) F = 2NF.$$

Proof. Let $\xi = \xi(t, \varphi) \in \Omega_0$. Here, we first look at

$$\begin{aligned}
&\overline{\eth}_{N+1} \eth_N \; F(\xi) \\
&= \left(\sqrt{1-t^2}\partial_t - \frac{Nt + t - i\partial_\varphi}{\sqrt{1-t^2}} \right) \left(\sqrt{1-t^2}\partial_t F(\xi) + \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}} F(\xi) \right) \\
&= \sqrt{1-t^2}\partial_t \left(\sqrt{1-t^2}\partial_t F(\xi) \right) + NF(\xi) + Nt\left(-\frac{1}{2}\right)(-2t)\frac{1}{1-t^2}F(\xi) - \frac{it}{1-t^2}\partial_\varphi F(\xi) \\
&\quad - i\partial_t \partial_\varphi F(\xi) + Nt\partial_t F(\xi) - Nt\partial_t F(\xi) - t\partial_t F(\xi) + i\partial_\varphi \partial_t F(\xi) - \frac{N^2 t^2}{1-t^2} F(\xi)
\end{aligned}$$

$$- \frac{Nt^2}{1-t^2}F(\xi) + \frac{iNt}{1-t^2}\partial_\varphi F(\xi) + \frac{iNt}{1-t^2}\partial_\varphi F(\xi) + \frac{it}{1-t^2}\partial_\varphi F(\xi) - \frac{i \cdot i}{1-t^2}\partial_\varphi^2 F(\xi)$$
$$= \sqrt{1-t^2}\partial_t\left(\sqrt{1-t^2}\partial_t F(\xi)\right) + NF(\xi) + \frac{Nt^2}{1-t^2}F(\xi) - t\partial_t F(\xi) - \frac{N^2t^2}{1-t^2}F(\xi)$$
$$- \frac{Nt^2}{1-t^2}F(\xi) + \frac{2iNt}{1-t^2}\partial_\varphi F(\xi) + \frac{1}{1-t^2}\partial_\varphi^2 F(\xi)$$
$$= \sqrt{1-t^2}\partial_t\left(\sqrt{1-t^2}\partial_t F(\xi)\right) + NF(\xi) - t\partial_t F(\xi) - \frac{N^2t^2}{1-t^2}F(\xi) + \frac{2iNt}{1-t^2}\partial_\varphi F(\xi)$$
$$+ \frac{1}{1-t^2}\partial_\varphi^2 F(\xi). \tag{3.20}$$

Next, we see that

$$\eth_{N-1}\overline{\eth}_N\ F(\xi)$$
$$= \left(\sqrt{1-t^2}\partial_t + \frac{Nt - t - i\partial_\varphi}{\sqrt{1-t^2}}\right)\left(\sqrt{1-t^2}\partial_t F(\xi) - \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}F(\xi)\right)$$
$$= \sqrt{1-t^2}\partial_t\left(\sqrt{1-t^2}\partial_t F(\xi)\right) - NF(\xi) - \frac{Nt^2}{1-t^2}F(\xi) - Nt\partial_t F(\xi) + \frac{it}{1-t^2}\partial_\varphi F(\xi)$$
$$+ i\partial_t\partial_\varphi F(\xi) + Nt\partial_t F(\xi) - t\partial_t F(\xi) - i\partial_\varphi\partial_t F(\xi) - \frac{N^2t^2}{1-t^2}F(\xi) + \frac{Nt^2}{1-t^2}F(\xi)$$
$$+ \frac{iNt}{1-t^2}\partial_\varphi F(\xi) + \frac{iNt}{1-t^2}\partial_\varphi F(\xi) - \frac{it}{1-t^2}\partial_\varphi F(\xi) - \frac{i \cdot i}{1-t^2}\partial_\varphi^2 F(\xi)$$
$$= \sqrt{1-t^2}\partial_t\left(\sqrt{1-t^2}\partial_t F(\xi)\right) - NF(\xi) - t\partial_t F(\xi) - \frac{N^2t^2}{1-t^2}F(\xi) + \frac{2iNt}{1-t^2}\partial_\varphi F(\xi)$$
$$+ \frac{1}{1-t^2}\partial_\varphi^2 F(\xi).$$

If we compare these two results, we see directly that

$$\overline{\eth}_{N+1}\eth_N\ F - \eth_{N-1}\overline{\eth}_N\ F = NF + NF = 2NF.$$

□

Corollary 3.5.11. *From the previous lemma, we can conclude that for every function* $F \in \mathrm{C}^{(2)}(\Omega_0)$ *and* $N \in \mathbb{Z}$

$$\overline{\eth}_{N+1}\eth_N\ F = \Delta^{*,N}F + N(N+1)F,$$
$$\eth_{N-1}\overline{\eth}_N\ F = \Delta^{*,N}F + N(N-1)F.$$

To the knowledge of the author, the first identity is new. The second identity was previously remarked in [53] without proof.

Proof. For $\xi = \xi(t,\varphi) \in \Omega_0$, we get the first equation with (3.20) by

$$\overline{\eth}_{N+1}\eth_N\ F(\xi)$$
$$= \sqrt{1-t^2}\partial_t\left(\sqrt{1-t^2}\partial_t F(\xi)\right) + NF(\xi) - t\partial_t F(\xi) - \frac{N^2t^2}{1-t^2}F(\xi) + \frac{2iNt}{1-t^2}\partial_\varphi F(\xi)$$
$$+ \frac{1}{1-t^2}\partial_\varphi^2 F(\xi)$$
$$= -t\partial_t F(\xi) + \left(1-t^2\right)\partial_t^2 F(\xi) + NF(\xi) - t\partial_t F(\xi) - \frac{N^2t^2}{1-t^2}F(\xi) + \frac{2iNt}{1-t^2}\partial_\varphi F(\xi)$$

$$
\begin{aligned}
&+ \frac{1}{1-t^2}\partial_\varphi^2 F(\xi) \\
&= \left(1-t^2\right)\partial_t^2 F(\xi) - 2t\partial_t F(\xi) + \frac{1}{1-t^2}\partial_\varphi^2 F(\xi) + NF(\xi) - \frac{N^2t^2 - 2iNt\partial_\varphi}{1-t^2}F(\xi) \\
&= \Delta_\xi^* F(\xi) - \frac{N^2 - 2iNt\partial_\varphi}{1-t^2}F(\xi) + \frac{N^2 - N^2t^2}{1-t^2}F(\xi) + NF(\xi) \\
&= \Delta_\xi^{*,N} F(\xi) + N(N+1)F(\xi).
\end{aligned}
$$

Then, we obtain directly the second equation with Lemma 3.5.10 by

$$
\begin{aligned}
\eth_{N-1}\overline{\eth}_N\ F &= \overline{\eth}_{N+1}\eth_N\ F - 2NF \\
&= \Delta^{*,N}F + \left(N^2 + N - 2N\right)F \\
&= \Delta^{*,N}F + N(N-1)F.
\end{aligned}
$$

□

Now, we look at properties borrowed from [53], where they are given without proof.

Lemma 3.5.12. *Let $F \in \mathrm{C}^{(4)}(\Omega_0)$ be a function of spin weight zero. Then, we get [53]*

$$\overline{\eth}_1\overline{\eth}_2\eth_1\eth_0\ F = \eth_{-1}\eth_{-2}\overline{\eth}_{-1}\overline{\eth}_0\ F = \left(\Delta^* + 2\right)\Delta^* F.$$

Proof. We get the first proposition with Lemma 3.5.10 and with Lemma 3.5.9

$$
\begin{aligned}
\overline{\eth}_1\overline{\eth}_2\eth_1\eth_0\ F &= \overline{\eth}_1\big(\eth_0\overline{\eth}_1\eth_0\ F + 2\cdot 1\underbrace{\eth_0\ F}_{\mathrm{sw}=1}\big) \\
&= \underbrace{\overline{\eth}_1\eth_0}_{=\Delta^*}\underbrace{\overline{\eth}_1\eth_0}_{=\Delta^*}\ F + 2\underbrace{\overline{\eth}_1\eth_0}_{=\Delta^*}\ F \\
&= \left(\Delta^* + 2\right)\Delta^* F.
\end{aligned}
$$

The second proposition follows analogously. □

Lemma 3.5.13. *For $N \in \mathbb{Z}$ and $F \in \mathrm{C}^{(N+1)}(\Omega_0)$, we obtain [53]*

$$\left(\overline{\eth}_N\eth_0^N - \eth_{-1}^N\overline{\eth}_0\right)F = N(N-1)\eth_0^{N-1}\ F.$$

Lemma 3.5.14. *Let $P, Q \in \mathrm{C}^{(2)}(\Omega_0)$ and $N, M \in \mathbb{Z}$. Then, we get [53]*

$$
\begin{aligned}
&P\ \eth_{M-1}\overline{\eth}_M\ Q - Q\ \overline{\eth}_{N+1}\eth_N\ P = \eth_{N+M-1}\left(P\ \overline{\eth}_M\ Q\right) - \overline{\eth}_{N+M+1}\left(Q\ \eth_N\ P\right), \\
&P\ \eth_{M+1}\eth_M\ Q - Q\ \eth_{N+1}\eth_N\ P = \eth_{N+M+1}\left(P\ \eth_M\ Q - Q\ \eth_N\ P\right).
\end{aligned}
$$

Proof. Let $P, Q \in \mathrm{C}^{(2)}(\Omega_0)$ and $N, M \in \mathbb{Z}$.

- We get the first property with Lemma 3.5.2 by

$$
\begin{aligned}
&\eth_{N+M-1}\left(P\ \overline{\eth}_M\ Q\right) - \overline{\eth}_{N+M+1}\left(Q\eth_N\ P\right) \\
&= \left(\eth_N\ P\right)\left(\overline{\eth}_M\ Q\right) + P\left(\eth_{M-1}\overline{\eth}_M\ Q\right) - \left(\overline{\eth}_M\ Q\right)\left(\eth_N\ P\right) - Q\left(\overline{\eth}_{N+1}\eth_N\ P\right) \\
&= P\ \eth_{M-1}\overline{\eth}_M\ Q - Q\ \overline{\eth}_{N+1}\eth_N\ P.
\end{aligned}
$$

- We get the second property with Lemma 3.5.1 and Lemma 3.5.2 by

$$\eth_{N+M+1}\left(P\ \eth_M\ Q - Q\ \eth_N\ P\right)$$

$$= \eth_{N+M+1}(P\,\eth_M\,Q) - \eth_{N+M+1}(Q\,\eth_N\,P)$$
$$= (\eth_N\,P)(\eth_M\,Q) + P(\eth_{M+1}\eth_M\,Q) - (\eth_M\,Q)(\eth_N\,P) - Q(\eth_{N+1}\eth_N\,P)$$
$$= P\,\eth_{M+1}\eth_M\,Q - Q\,\eth_{N+1}\eth_N\,P.$$

□

Lemma 3.5.15. *Let $P, Q \in \mathrm{C}^{(2)}(\Omega_0)$ and $N, M \in \mathbb{Z}$. Then, for $\mathrm{sw}(PQ) = 0$, we get*

$$P\,\eth_{M-1}\overline{\eth}_M\,Q - Q\,\overline{\eth}_{N+1}\eth_N\,P = P\,\overline{\Delta^{*,N}}Q - Q\,\Delta^{*,N}P.$$

Proof. Let $P, Q \in \mathrm{C}^{(2)}(\Omega_0)$ and $N, M \in \mathbb{Z}$. Then, for $\mathrm{sw}(PQ) = 0$, we get with Lemma 3.1.7 that $\mathrm{sw}(PQ) = N + M$ and so, $M = -N$. Therefore, we get with Corollary 3.5.11

$$\begin{aligned} P\,\eth_{M-1}\overline{\eth}_M\,Q - Q\,\overline{\eth}_{N+1}\eth_N\,P &= P\,\eth_{-N-1}\overline{\eth}_{-N}\,Q - Q\,\overline{\eth}_{N+1}\eth_N\,P \\ &= P\,\Delta^{*,-N}Q + N(N+1)PQ - Q\,\Delta^{*,N}P - N(N+1)QP \\ &= P\,\Delta^{*,-N}Q - Q\,\Delta^{*,N}P. \end{aligned}$$

From Corollary 3.3.7, it is obvious that $\Delta^{*,-N} = \overline{\Delta^{*,N}}$. So, we get the proposition. □

Lemma 3.5.16. *We borrow from [17, 63] that for every function $F \in \mathrm{C}^{(p+q)}(\Omega_0)$ and for $N \in \mathbb{Z}$*

$$\eth^p_{N-q}\overline{\eth}^q_N\,F = \overline{\eth}^q_{N+p}\eth^p_N\,F, \qquad \text{if } q - p = 2N$$

for $p, q \in \mathbb{N}_0$.

3.6 The Uniqueness of the Eigenfunctions of $\Delta^{*,N}$

In this chapter, we point out further properties of the spin-weighted spherical harmonics and formulate the space, which is spanned by the spin-weighted spherical harmonics. So, we can show that the spin-weighted spherical harmonics are the only eigenvalues of the operator $\Delta^{*,N}$.

Now, we start with a property that follows from the definition of the spin-weighted spherical harmonics [63].

Lemma 3.6.1. *We see that*

$$\overline{\eth}_{N+1}\eth_N\;{}_NY_{n,j} = -(n(n+1) - N(N+1))\;{}_NY_{n,j}$$

and

$$\eth_{N-1}\overline{\eth}_N\;{}_NY_{n,j} = -(n(n+1) - N(N-1))\;{}_NY_{n,j}$$

for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, $n \geq |N|$, and all $j = -n, \ldots, n$.

The first equation was previously mentioned in [34, 53, 63, 96, 97] without proof.

Proof. The proof is straight forward. With Corollary 3.5.11 and with Corollary 3.3.7, we get directly

$$\begin{aligned} \overline{\eth}_{N+1}\eth_N\;{}_NY_{n,j} &= \Delta^{*,N}\;{}_NY_{n,j} + N(N+1)\;{}_NY_{n,j} \\ &= -n(n+1)\;{}_NY_{n,j} + N(N+1)\;{}_NY_{n,j} \\ &= -(n(n+1) - N(N+1))\;{}_NY_{n,j}. \end{aligned}$$

Analogously, we get the second equation with Corollary 3.5.11 and with Corollary 3.3.7 by

$$\begin{aligned}\eth_{N-1}\overline{\eth}_N\ {}_NY_{n,j} &= \Delta^{*,N}\ {}_NY_{n,j} + N(N-1)\ {}_NY_{n,j}\\ &= -n(n+1)\ {}_NY_{n,j} + N(N-1)\ {}_NY_{n,j}\\ &= -(n(n+1) - N(N-1))\ {}_NY_{n,j}.\end{aligned}$$

□

Now, we can show that Lemma 3.2.8 is satisfied for all $N \in \mathbb{Z}$. This was previously mentioned in [15, 34, 53, 63, 91, 96, 97] without proof.

Lemma 3.6.2. *The spin-weighted spherical harmonics fulfill for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, and all $j = -n, \ldots, n$ the following properties [15, 34, 53, 63, 91, 96, 97]*

$$\eth_N\ {}_NY_{n,j} = \sqrt{n(n+1) - N(N+1)}\ {}_{N+1}Y_{n,j} \tag{3.21}$$

and

$$\overline{\eth}_N\ {}_NY_{n,j} = -\sqrt{n(n+1) - N(N-1)}\ {}_{N-1}Y_{n,j}. \tag{3.22}$$

Proof. Let $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, and $j = -n, \ldots, n$. We start with the first property. To show it, we have to look at four different cases.

- For $n < |N|$ and $n < |N+1|$, we get from Definition 3.2.6 that ${}_NY_{n,j} = {}_{N+1}Y_{n,j} = 0$ and equation (3.21) is trivially satisfied by $0 = 0$.

- For $n \geq |N|$ and $n \geq |N+1|$, we have to distinguish between $N \geq 0$ and $N < 0$. We have already proven the case $N \geq 0$ in Lemma 3.2.8. For $N < 0$, we get with Definition 3.2.6, with (3.1), and with Lemma 3.6.1

$$\begin{aligned}\eth_N\ {}_NY_{n,j} &= (-1)^N\sqrt{\frac{(n+N)!}{(n-N)!}}\ \eth_N\overline{\eth}_0^{-N}\ Y_{n,j}\\ &= \frac{-1}{\sqrt{(n+N+1)(n-N)}}\ (-1)^{N+1}\sqrt{\frac{(n+N+1)!}{(n-N-1)!}}\ \eth_N\overline{\eth}_{N+1}\overline{\eth}_0^{-(N+1)}\ Y_{n,j}\\ &= \frac{-1}{\sqrt{n(n+1)-N(N+1)}}\ \eth_N\overline{\eth}_{N+1}\ {}_{N+1}Y_{n,j}\\ &= \frac{-1}{\sqrt{n(n+1)-N(N+1)}}\ (-(n(n+1)-N(N+1)))\ {}_{N+1}Y_{n,j}\\ &= \sqrt{n(n+1)-N(N+1)}\ {}_{N+1}Y_{n,j}.\end{aligned}$$

- For $n < |N|$ and $n \geq |N+1|$, we get that $N < 0$ and $n = -(N+1)$. Then, we know from Definition 3.2.6 that ${}_NY_{n,j} = 0$. So, the left-hand side of equation (3.21) is zero. For the right-hand side, we also get zero, because

$$\begin{aligned}&\sqrt{-(N+1)(-N-1+1) - N(N+1)}\ {}_{N+1}Y_{-(N+1),j}\\ &= \sqrt{N(N+1) - N(N+1)}\ {}_{N+1}Y_{-(N+1),j}\\ &= 0.\end{aligned}$$

- For $n \geq |N|$ and $n < |N+1|$, we get that $N \geq 0$ and $n = N$. Then, we know again from Definition 3.2.6 that ${}_{N+1}Y_{n,j} = 0$. So, the right-hand side of equation (3.21) is zero and we have to show that $\eth_N \; {}_N Y_{N,j} = 0$. With (3.12), we obtain for $\xi = \xi(t,\varphi) \in \Omega_0$ and $t = \cos\vartheta$

$$\begin{aligned}
{}_N Y_{N,j}(\xi) &= (-1)^N \sqrt{\frac{2N+1}{4\pi}}\; e^{ij\varphi} d^N_{j,-N}(\vartheta)\\
&= (-1)^N \sqrt{\frac{2N+1}{4\pi}}\; e^{ij\varphi}\; \frac{(-1)^{N+j}}{2^N} \sqrt{\frac{(N+N)!}{(N+j)!(N-j)!(N-N)!}}\; (1-t)^{-\frac{j+N}{2}}\\
&\quad \times (1+t)^{\frac{j-N}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{N-N} \left[(1-t)^{N+j}(1+t)^{N-j}\right]\\
&= \frac{(-1)^j}{2^N} \sqrt{\frac{2N+1}{4\pi}} \sqrt{\frac{(2N)!}{(N+j)!(N-j)!}}\; e^{ij\varphi}(1-t)^{\frac{j+N}{2}}(1+t)^{-\frac{j-N}{2}}.
\end{aligned}$$

Then, we get with Definition 3.2.3

$$\begin{aligned}
&\eth_N \; {}_N Y_{N,j}(\xi)\\
&= \left(\sqrt{1-t^2}\,\partial_t + \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}}\right) {}_N Y_{N,j}(\xi)\\
&= \frac{(-1)^j}{2^N} \sqrt{\frac{2N+1}{4\pi}} \sqrt{\frac{(2N)!}{(N+j)!(N-j)!}}\; e^{ij\varphi} \left(-\frac{j+N}{2}\;(1-t)^{\frac{j+N}{2}-1+\frac{1}{2}}(1+t)^{-\frac{j-N}{2}+\frac{1}{2}}\right.\\
&\quad \left. -\frac{j-N}{2}\;(1-t)^{\frac{j+N}{2}+\frac{1}{2}}(1+t)^{-\frac{j-N}{2}-1+\frac{1}{2}} + (Nt+j)(1-t)^{\frac{j+N}{2}-\frac{1}{2}}(1+t)^{-\frac{j-N}{2}-\frac{1}{2}}\right)\\
&= \frac{(-1)^j}{2^N} \sqrt{\frac{2N+1}{4\pi}} \sqrt{\frac{(2N)!}{(N+j)!(N-j)!}}\; e^{ij\varphi}(1-t)^{\frac{j+N}{2}-\frac{1}{2}}(1+t)^{-\frac{j-N}{2}-\frac{1}{2}}\;\frac{1}{2}\\
&\quad \times \underbrace{\left(-(j+N)(1+t) - (j-N)(1-t) + 2(Nt+j)\right)}_{=-j-jt-N-Nt-j+jt+N-Nt+2Nt+2j=0}\\
&= 0.
\end{aligned}$$

The second property can be shown analogously.

- For $n < |N|$ and $n < |N-1|$, we get from Definition 3.2.6 that ${}_N Y_{n,j} = {}_{N-1}Y_{n,j} = 0$. Then, equation (3.22) is trivially satisfied by $0 = 0$.
- For $n \geq |N|$ and $n \geq |N-1|$, we have to distinguish between $N \leq 0$ and $N > 0$. The case $N \leq 0$ has previously been proven in Lemma 3.2.8. For $N > 0$, we get with Definition 3.2.6, with (3.1), and with Lemma 3.6.1

$$\begin{aligned}
\bar{\eth}_N \; {}_N Y_{n,j} &= \sqrt{\frac{(n-N)!}{(n+N)!}}\; \bar{\eth}_N \eth_0^N \; Y_{n,j}\\
&= \frac{1}{\sqrt{(n+N)(n-N+1)}} \sqrt{\frac{(n-(N-1))!}{(n+N-1)!}}\; \bar{\eth}_N \eth_{N-1} \eth_0^{N-1}\; Y_{n,j}\\
&= \frac{1}{\sqrt{n(n+1) - N(N-1)}}\; \bar{\eth}_N \eth_{N-1}\; {}_{N-1}Y_{n,j}\\
&= \frac{1}{\sqrt{n(n+1) - N(N-1)}}\; \left(-(n(n+1) - N(N-1))\right)\; {}_{N-1}Y_{n,j}\\
&= -\sqrt{n(n+1) - N(N-1)}\; {}_{N-1}Y_{n,j}.
\end{aligned}$$

- For $n < |N|$ and $n \geq |N-1|$, we get that $N > 0$ and $n = N-1$. Then, we know from Definition 3.2.6 that ${}_NY_{n,j} = 0$. So, the left-hand side of equation (3.22) is zero. For the right-hand side, we get also zero because

$$\begin{aligned}&-\sqrt{(N-1)(N-1+1)-N(N-1)}\ {}_{N-1}Y_{N-1,j}\\&=-\sqrt{N(N-1)-N(N-1)}\ {}_{N-1}Y_{N-1,j}\\&=0.\end{aligned}$$

- For $n \geq |N|$ and $n < |N-1|$, we get that $N \leq 0$ and $n = -N$. Then, we know again from Definition 3.2.6 that ${}_{N-1}Y_{n,j} = 0$. So, the right-hand side of equation (3.22) is zero and we have to show that $\overline{\eth}_N\ {}_NY_{-N,j} = 0$. With (3.11), we obtain for $\xi = \xi(t,\varphi) \in \Omega_0$ and $t = \cos\vartheta$

$$\begin{aligned}{}_NY_{-N,j}(\xi) &= (-1)^N\sqrt{\frac{-2N+1}{4\pi}}\ e^{ij\varphi}d^{-N}_{j,-N}(\vartheta)\\&=(-1)^N\sqrt{\frac{-2N+1}{4\pi}}\ e^{ij\varphi}\ \frac{(-1)^{-N+N}}{2^{-N}}\ \sqrt{\frac{(-N-N)!}{(-N+j)!(-N-j)!(-N+N)!}}\\&\quad\times(1-t)^{\frac{j+N}{2}}(1+t)^{-\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-N+N}\left[(1-t)^{-N-j}(1+t)^{-N+j}\right]\\&=\frac{(-1)^N}{2^{-N}}\ \sqrt{\frac{-2N+1}{4\pi}}\ \sqrt{\frac{(-2N)!}{(-N+j)!(-N-j)!}}\ e^{ij\varphi}(1-t)^{-\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}}.\end{aligned}$$

Then, we get with Definition 3.2.3

$$\begin{aligned}&\overline{\eth}_N\ {}_NY_{-N,j}(\xi)\\&=\left(\sqrt{1-t^2}\partial_t-\frac{Nt-i\partial_\varphi}{\sqrt{1-t^2}}\right)\ {}_NY_{-N,j}(\xi)\\&=\frac{(-1)^N}{2^{-N}}\ \sqrt{\frac{-2N+1}{4\pi}}\ \sqrt{\frac{(-2N)!}{(-N+j)!(-N-j)!}}\ e^{ij\varphi}\\&\quad\times\left(\frac{j+N}{2}(1-t)^{-\frac{j+N}{2}-1+\frac{1}{2}}(1+t)^{\frac{j-N}{2}+\frac{1}{2}}+\frac{j-N}{2}\ (1-t)^{-\frac{j+N}{2}+\frac{1}{2}}(1+t)^{\frac{j-N}{2}-1+\frac{1}{2}}\right.\\&\quad\left.-(Nt+j)(1-t)^{-\frac{j+N}{2}-\frac{1}{2}}(1+t)^{\frac{j-N}{2}-\frac{1}{2}}\right)\\&=\frac{(-1)^N}{2^{-N}}\ \sqrt{\frac{-2N+1}{4\pi}}\ \sqrt{\frac{(-2N)!}{(-N+j)!(-N-j)!}}\ e^{ij\varphi}(1-t)^{-\frac{j+N}{2}-\frac{1}{2}}(1+t)^{\frac{j-N}{2}-\frac{1}{2}}\ \frac{1}{2}\\&\quad\times\underbrace{((j+N)(1+t)+(j-N)(1-t)-2(Nt+j))}_{=j+jt+N+Nt+j-jt-N+Nt-2Nt-2j=0}\\&=0.\end{aligned}$$

□

Corollary 3.6.3. *For the spin raising and lowering operator, we get*

$$\begin{aligned}\eth_N\ {}_NY_{n,j}(\xi)&=\mathcal{O}(1),\\\overline{\eth}_N\ {}_NY_{n,j}(\xi)&=\mathcal{O}(1)\end{aligned}$$

for all $\xi = \xi(t, \varphi) \in \Omega_0$*, all* $n \in \mathbb{N}_0$*,* $N \in \mathbb{Z}$*,* $n \geq |N|$*, and* $j = -n, \ldots, n$*.*

Proof. We conclude the corollary directly from the previous lemma and from Corollary 3.4.13. □

Next, we can conclude that the operator $\overline{\eth}_{N+1}\eth_N$ is self-adjoint.

Lemma 3.6.4. *The operator* $\overline{\eth}_{N+1}\eth_N$ *is self-adjoint. This means that for every function* $F, G \in \mathrm{X}^2(\Omega_0)$ *and for* $N \in \mathbb{Z}$*, we obtain [91]*

$$\left\langle \overline{\eth}_{N+1}\eth_N\ F, G \right\rangle_{\mathrm{L}^2(\Omega)} = \left\langle F, \overline{\eth}_{N+1}\eth_N\ G \right\rangle_{\mathrm{L}^2(\Omega)}$$

provided that $\eth_N\ F, \overline{\eth}_{-N}\ G \in \mathrm{L}^2(\Omega)$*.*

Proof. Let $F, G \in \mathrm{X}^2(\Omega_0)$. Then, from the previous lemma, we conclude with Corollary 3.4.15 that $\overline{\eth}_{N+1}\eth_N\ F$ respectively $\overline{\eth}_{N+1}\eth_N\ G$ is bounded and with $F, G \in \mathrm{X}^2(\Omega_0)$, we integrate bounded functions. Therefore, the left- and the right-hand integral exist.

With Lemma 3.5.8, we get for $N \in \mathbb{Z}$

$$\begin{aligned}\left\langle \overline{\eth}_{N+1}\eth_N\ F, G \right\rangle_{\mathrm{L}^2(\Omega)} &= \int_\Omega \left(\overline{\eth}_{N+1}\eth_N\ F(\xi)\right)\ \overline{G(\xi)}\ \mathrm{d}\omega(\xi) \\ &= -\int_\Omega \left(\eth_N\ F(\xi)\right)\left(\overline{\eth}_{-N}\ \overline{G(\xi)}\right)\ \mathrm{d}\omega(\xi).\end{aligned}$$

The last integral exists, because $\eth_N\ F, \overline{\eth}_{-N}\ G \in \mathrm{L}^2(\Omega)$. Further, we obtain again with Lemma 3.5.8

$$\left\langle \overline{\eth}_{N+1}\eth_N\ F, G \right\rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega F(\xi)\left(\eth_{-(N+1)}\overline{\eth}_{-N}\ \overline{G(\xi)}\right)\ \mathrm{d}\omega(\xi).$$

Then, with Remark 3.2.4, this leads us to

$$\begin{aligned}\left\langle \overline{\eth}_{N+1}\eth_N\ F, G \right\rangle_{\mathrm{L}^2(\Omega)} &= \int_\Omega F(\xi)\left(\overline{\overline{\eth}_{N+1}}\ \overline{\eth_N}\ \overline{G(\xi)}\right)\ \mathrm{d}\omega(\xi) \\ &= \int_\Omega F(\xi)\left(\overline{\overline{\eth}_{N+1}\eth_N\ G(\xi)}\right)\ \mathrm{d}\omega(\xi) \\ &= \left\langle {}_N F, \overline{\eth}_{N+1}\eth_N\ G \right\rangle_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

□

Due to Corollary 3.6.3, the previous lemma holds true for all ${}_N Y_{n,j}$.

Now, we can show a more general version of Lemma 3.6.1.

Lemma 3.6.5. *Let* $N \in \mathbb{Z}$*,* $n \in \mathbb{N}_0$*,* $n \geq |N|$*,* $j = -n, \ldots, n$*, and* $p \in \mathbb{N}_0$*. We get*

$$\overline{\eth}^p_{N+p}\eth^p_N\ {}_N Y_{n,j} = (-1)^p\ \frac{(n-N)!}{(n-N-p)!}\ \frac{(n+N+p)!}{(n+N)!}\ {}_N Y_{n,j}$$

for $p \leq n - N$ *and*

$$\eth^p_{N-p}\overline{\eth}^p_N\ {}_N Y_{n,j} = (-1)^p\ \frac{(n+N)!}{(n+N-p)!}\ \frac{(n-N+p)!}{(n-N)!}\ {}_N Y_{n,j}$$

for $p \leq n + N$*.*

Proof. From Lemma 3.6.2, we know that for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, and all $j = -n, \ldots, n$

$$\eth_N \; {}_NY_{n,j} = \sqrt{n(n+1) - N(N+1)} \; {}_{N+1}Y_{n,j}$$

and

$$\overline{\eth}_N \; {}_NY_{n,j} = -\sqrt{n(n+1) - N(N-1)} \; {}_{N-1}Y_{n,j}.$$

Let $p \in \mathbb{N}_0$. With (3.1), we get for $p \leq n - N$

$$\begin{aligned}
\eth_N^p \; {}_NY_{n,j} &= \sqrt{(n-N)(n+N+1)} \; \eth^{p-1} {}_{N+1}Y_{n,j} \\
&= \sqrt{(n-N)\ldots(n-N-p+1)(n+N+1)\ldots(n+N+p)} \; {}_{N+p}Y_{n,j} \\
&= \sqrt{\frac{(n-N)!(n+N+p)!}{(n-N-p)!(n+N)!}} \; {}_{N+p}Y_{n,j}
\end{aligned}$$

and for $p \leq n + N$

$$\begin{aligned}
\overline{\eth}_N^p \; {}_NY_{n,j} &= -\sqrt{(n+N)(n-N+1)} \; \overline{\eth}^{p-1} {}_{N-1}Y_{n,j} \\
&= (-1)^p \sqrt{(n+N)\ldots(n+N-p+1)(n-N+1)\ldots(n-N+p)} \; {}_{N-p}Y_{n,j} \\
&= (-1)^p \sqrt{\frac{(n+N)!(n-N+p)!}{(n+N-p)!(n-N)!}} \; {}_{N-p}Y_{n,j}.
\end{aligned}$$

Then, we obtain for $p \leq n - N$

$$\overline{\eth}_{N+p}^p \; {}_{N+p}Y_{n,j} = (-1)^p \sqrt{\frac{(n+N+p)!(n-(N+p)+p)!}{(n+N+p-p)!(n-(N+p))!}} \; {}_NY_{n,j}$$

and for $p \leq n + N$

$$\eth_{N-p}^p \; {}_{N-p}Y_{n,j} = \sqrt{\frac{(n-N+p)!(n+N)!}{(n-N)!(n+N-p)!}} \; {}_NY_{n,j}.$$

Altogether, we get for $p \leq n - N$

$$\begin{aligned}
\overline{\eth}_{N+p}^p \eth_N^p \; {}_NY_{n,j} &= \sqrt{\frac{(n-N)!(n+N+p)!}{(n-N-p)!(n+N)!}} \; \overline{\eth}_{N+p}^p \; {}_{N+p}Y_{n,j} \\
&= \sqrt{\frac{(n-N)!(n+N+p)!}{(n-N-p)!(n+N)!}} \; (-1)^p \sqrt{\frac{(n+N+p)!(n-N-p+p)!}{(n+N+p-p)!(n-N-p)!}} \; {}_NY_{n,j} \\
&= (-1)^p \; \frac{(n-N)!(n+N+p)!}{(n-N-p)!(n+N)!} \; {}_NY_{n,j}
\end{aligned}$$

and for $p \leq n + N$

$$\begin{aligned}
\overline{\eth}_{N+p}^p \eth_N^p \; {}_NY_{n,j} &= (-1)^p \; \sqrt{\frac{(n+N)!(n-N+p)!}{(n+N-p)!(n-N)!}} \; \sqrt{\frac{(n-N+p)!(n+N)!}{(n-N)!(n+N-p)!}} \; {}_NY_{n,j} \\
&= (-1)^p \; \frac{(n+N)!(n-N+p)!}{(n-N)!(n+N-p)!} \; {}_NY_{n,j}
\end{aligned}$$

□

Lemma 3.6.6. *Let $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, and $j = -n, \ldots, n$. For $n < N$, we obtain the property [63]*

$$\overline{\eth}_0^N \; Y_{n,j} = 0.$$

Proof. From the definition of the spin-weighted spherical harmonics, Lemma 3.2.9, we know that

$$\overline{\eth}_N \; {}_N Y_{n,j} = -\sqrt{n(n+1) - N(N-1)} \; {}_{N-1} Y_{n,j}$$

for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, and all $j = -n, \ldots, n$. Then, we get

$$\begin{aligned}
\overline{\eth}_0^N \; Y_{n,j} &= \overline{\eth}_{-(N-1)} \ldots \overline{\eth}_0 \; Y_{n,j} \\
&= (-1)^N \left(\prod_{k=-(N-1)}^{0} (n(n+1) - k(k-1)) \right)^{\frac{1}{2}} {}_{-N} Y_{n,j} \\
&= (-1)^N \left(\prod_{k=0}^{N-1} (n(n+1) - k(k+1)) \right)^{\frac{1}{2}} {}_{-N} Y_{n,j}.
\end{aligned}$$

The product is zero, if $k = n$. Because $k = 0, \ldots, N-1$ and $n < N$, so $n \in \{0, \ldots, N-1\}$, the value $k = n$ occurs. Therefore, for $n < N$,

$$\overline{\eth}_0^N \; Y_{n,j} = 0.$$

□

Now, we take a look at the kernel of the operators $\eth$ and $\overline{\eth}$.

Theorem 3.6.7. *The kernel of the operator $\eth_N$ is given for $N \in \mathbb{Z}$ by [17]*

$$\ker(\eth_N) = \begin{cases} \left\{ F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega) \;\middle|\; F = \sum_{j=-N}^{N} c_j \; {}_N Y_{N,j}, \; c_j \in \mathbb{C} \right\}, & N \geq 0 \\ \{0\}, & \textit{else} \end{cases},$$

where for $\xi = \xi(t, \varphi) \in \Omega_0$, $j = -N, \ldots, N$, and

$${}_N Y_{N,j}(\xi) = \underbrace{\sqrt{\frac{2N+1}{4\pi}} \, (-1)^j \frac{1}{2^N} \sqrt{\frac{(2N)!}{(N-j)!(N+j)!}}}_{=: c_{N,j}} \; e^{ij\varphi} (1-t)^{\frac{N+j}{2}} (1+t)^{\frac{N-j}{2}}.$$

Proof. To get the kernel of the operator $\eth_N$, we have to solve the equation

$$0 = \eth_N \; F(\xi) = \left(\sqrt{1-t^2} \partial_t + \frac{Nt - i\partial_\varphi}{\sqrt{1-t^2}} \right) F(\xi),$$

where $F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega)$ and $N \in \mathbb{Z}$. Therefore, we use separation of variables and set $F(\xi(t, \varphi)) := f(t) g(\varphi)$, $\xi \in \Omega_0$. Then, we obtain

$$\begin{aligned}
& 0 = \sqrt{1-t^2} f'(t) g(\varphi) + \frac{Nt}{\sqrt{1-t^2}} \; f(t) g(\varphi) - \frac{i}{\sqrt{1-t^2}} \; f(t) g'(\varphi) \\
\Leftrightarrow \qquad & 0 = \left(1 - t^2\right) f'(t) g(\varphi) + Nt f(t) g(\varphi) - i f(t) g'(\varphi)
\end{aligned}$$

and consequently,

$$\left(1-t^2\right)\frac{f'(t)}{f(t)}+Nt=i\,\frac{g'(\varphi)}{g(\varphi)}.$$

Now, we see that the left-hand side depends only on t and the right-hand side only on φ. So, both sides must be equal to a constant c. Then, we get on the one hand

$$c=i\,\frac{g'}{g} \qquad \Leftrightarrow g'=-icg$$

and consequently, we have the solution for g given by

$$g(\varphi)=a\cdot e^{-ic\varphi},$$

where $g(0)=g(2\pi)$. So,

$$a=ae^{-2i\pi c} \qquad \Leftrightarrow \qquad e^{-2i\pi c}=1$$

and therefore,

$$-2i\pi c=2i\pi j \qquad \Leftrightarrow \qquad c=-j, \qquad j\in\mathbb{Z}.$$

Then, we obtain on the other hand for all $j\in\mathbb{Z}$

$$\left(1-t^2\right)\frac{f'}{f}+Nt=-j \qquad \Leftrightarrow \qquad \frac{1}{f}\,\frac{\mathrm{d}f}{\mathrm{d}t}=-\frac{j+Nt}{1-t^2}.$$

So, we get by integration

$$\begin{aligned}
\ln\left(|f(t)|\right)&=-j\underbrace{\int\frac{1}{1-t^2}\,\mathrm{d}t}_{=\frac{1}{2}(\ln(1+t)-\ln(1-t))}-N\underbrace{\int\frac{t}{1-t^2}\,\mathrm{d}t}_{=-\frac{1}{2}\ln(1-t^2)}\\
&=-\frac{j}{2}\ln\left(\frac{1+t}{1-t}\right)+\frac{N}{2}\ln\left(1-t^2\right)+b\\
&=\ln\left((1+t)^{-\frac{j}{2}}(1-t)^{\frac{j}{2}}\right)+\ln\left((1+t)^{\frac{N}{2}}(1-t)^{\frac{N}{2}}\right)+\ln\left(\tilde{b}\right)\\
&=\ln\Bigg(\tilde{b}\underbrace{(1+t)^{\frac{N-j}{2}}(1-t)^{\frac{N+j}{2}}}_{\geq 0}\Bigg)
\end{aligned}$$

and so, we obtain the solution for f (without loss of generality) by

$$f(t)=\tilde{b}(1-t)^{\frac{N+j}{2}}(1+t)^{\frac{N-j}{2}}.$$

Altogether, we get

$$\begin{aligned}
F(\xi)&=f(t)g(\varphi)\\
&=\tilde{c}e^{ij\varphi}\underbrace{(1-t)^{\frac{N+j}{2}}(1+t)^{\frac{N-j}{2}}}_{=:I}.
\end{aligned}$$

For $N\geq 0$ and $N\geq|j|$, the exponents of I are non-negative. Therefore, $I\in\mathrm{L}^2(\Omega)$. For $N<0$, at least one of the exponents is negative. Therefore, $I\notin\mathrm{L}^2(\Omega)$ and the only solution is zero.

So, we only have to regard the case $N\geq 0$ further. From Definition 3.4.1 and Theorem

3.4.9, we know that

$$
\begin{aligned}
{}_NY_{N,j}(\xi) &= (-1)^j \frac{1}{2^N}\sqrt{\frac{(N-j)!}{(N+j)!(2N)!}}\sqrt{\frac{2N+1}{4\pi}}\ e^{ij\varphi}(1-t)^{\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}} \\
&\qquad \times \underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{N+j}(1+t)^{2N}}_{\substack{=\frac{(2N)!}{(2N-N-j)!}(1+t)^{2N-N-j} \\ =\frac{(2N)!}{(N-j)!}(1+t)^{N-j}}} \\
&= \underbrace{(-1)^j \frac{1}{2^N}\sqrt{\frac{(2N)!}{(N+j)!(N-j)!}}\sqrt{\frac{2N+1}{4\pi}}}_{=c_{N,j}}\ e^{ij\varphi}(1-t)^{\frac{j+N}{2}}(1+t)^{\frac{N-j}{2}}.
\end{aligned}
$$

So, we obtain that

$$
F(\xi) = \frac{\tilde{c}}{c_{N,j}}\ {}_NY_{N,j}(\xi)
$$

satisfies $0 = \eth_N\ F(\xi)$ for all $j = -N, \ldots, N$ and therefore, all linear combinations of the spin-weighted spherical harmonics ${}_NY_{N,j}$ of spin weight and degree N. So, F is a function of spin weight and degree N. □

Theorem 3.6.8. *In the same way, the kernel of the operator $\overline{\eth}_N$ is given for $N \in \mathbb{Z}$ by [17]*

$$
\begin{aligned}
\ker\left(\overline{\eth}_N\right) &= \begin{cases} \left\{F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega) \ \middle|\ F = \sum_{j=-N}^{N} c_j\ \overline{{}_{-N}Y_{-N,j}},\ c_j \in \mathbb{C}\right\}, & N \leq 0 \\ \{0\}, & \textit{else} \end{cases} \\
&= \begin{cases} \left\{F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega) \ \middle|\ F = \sum_{j=-N}^{N} c_j\ {}_NY_{-N,j},\ c_j \in \mathbb{C}\right\}, & N \leq 0 \\ \{0\}, & \textit{else} \end{cases}.
\end{aligned}
$$

Proof. With the previous theorem, we see directly that for $N \in \mathbb{Z}$

$$
\ker\left(\eth_{-N}\right) = \begin{cases} \left\{F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega) \ \middle|\ F = \sum_{j=N}^{-N} c_j\ {}_{-N}Y_{-N,j},\ c_j \in \mathbb{C}\right\}, & -N \geq 0 \Leftrightarrow N \leq 0 \\ \{0\}, & \text{else} \end{cases}.
$$

With Remark 3.2.4, this is equal to

$$
\ker\left(\overline{\eth}_N\right) = \begin{cases} \left\{F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega) \ \middle|\ F = \sum_{j=N}^{-N} c_j\ \overline{{}_{-N}Y_{-N,j}},\ c_j \in \mathbb{C}\right\}, & N \leq 0 \\ \{0\}, & \text{else} \end{cases}.
$$

From Definition 3.4.1 and Theorem 3.4.9, we know for $\xi = \xi(t,\varphi) \in \Omega_0$ that

$$
\begin{aligned}
{}_NY_{-N,-j}(\xi) &= \sqrt{\frac{1-2N}{4\pi}}\ (-1)^{-j}2^N\sqrt{\frac{(-N+j)!}{(-N-j)!(-2N)!}}\ (1-t)^{-\frac{j-N}{2}}(1+t)^{-\frac{j+N}{2}}e^{-ij\varphi} \\
&\qquad \times \underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{-j-N}(1-t)^{-2N}}_{\substack{=\frac{(-2N)!}{(-2N+N+j)!}(1-t)^{-2N+N+j}(-1)^{-j-N} \\ =\frac{(-2N)!}{(j-N)!}(1-t)^{j-N}(-1)^{-j-N}}}
\end{aligned}
$$

$$= (-1)^N \sqrt{\frac{1-2N}{4\pi}}\ 2^N \sqrt{\frac{(-2N)!}{(-N+j)!(-j-N)!}}\ (1-t)^{-\frac{N-j}{2}}(1+t)^{-\frac{N+j}{2}} e^{-ij\varphi}$$

and therefore, from the previous proof, we see that

$$\begin{aligned}\overline{{}_{-N}Y_{-N,j}}(\xi) &= \sqrt{\frac{1-2N}{4\pi}}\ (-1)^j 2^N \sqrt{\frac{(-2N)!}{(j-N)!(-j-N)!}}\ (1-t)^{-\frac{N-j}{2}}(1+t)^{-\frac{N+j}{2}} e^{-ij\varphi}\\ &= (-1)^j\ {}_NY_{-N,-j}(\xi).\end{aligned}$$

Then, we obtain

$$\ker\left(\overline{\eth}_N\right) = \begin{cases}\left\{F \in \mathrm{C}^{(1)}(\Omega_0)\cap \mathrm{L}^2(\Omega)\ \middle|\ F = \sum_{j=-N}^{N} c_j\ {}_NY_{-N,-j},\ c_j \in \mathbb{C}\right\}, & N \leq 0\\ \{0\}, & \text{else}\end{cases}$$

and consequently, we get

$$\ker\left(\overline{\eth}_N\right) = \begin{cases}\left\{F \in \mathrm{C}^{(1)}(\Omega_0)\cap \mathrm{L}^2(\Omega)\ \middle|\ F = \sum_{j=-N}^{N} c_j\ {}_NY_{-N,j},\ c_j \in \mathbb{C}\right\}, & N \leq 0\\ \{0\}, & \text{else}\end{cases}.$$

□

Corollary 3.6.9. *So, the operator $\eth_N$ is injective for $N < 0$ and the operator $\overline{\eth}_N$ is injective for $N > 0$.*

Lemma 3.6.10. *Let $N \in \mathbb{Z}$. So, we can recapitulate that for all functions $F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega)$ and for $N < 0$ the identity [63]*

$$\eth_N\ F = 0$$

implies that

$$F = 0.$$

Furthermore, we obtain that for all functions $F \in \mathrm{C}^{(1)}(\Omega_0) \cap \mathrm{L}^2(\Omega)$ and for $N > 0$

$$\overline{\eth}_N\ F = 0$$

implies that

$$F = 0.$$

Lemma 3.6.11. *For every function $F \in \mathrm{L}^2(\Omega)$ and for every $N \in \mathbb{Z}$, we get that*

$$\langle F, {}_NY_{n,j}\rangle_{\mathrm{L}^2(\Omega)} = 0$$

for all $n \in \mathbb{N}_0$, $n \geq |N|$, and all $j = -n, \ldots, n$ implies that

$$F = 0.$$

Proof. Let $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, and $j = -n, \ldots, n$. Furthermore, let $F \in \mathrm{C}^{(\infty)}(\Omega)$. This is enough, because $\mathrm{C}^{(\infty)}(\Omega)$ is dense in $\mathrm{L}^2(\Omega)$ [94].

- For $N = 0$, we get

$$0 = \langle F, Y_{n,j}\rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega F(\xi)\overline{Y_{n,j}(\xi)}\ \mathrm{d}\omega(\xi)$$

and know directly, for example from [58], that $F = 0$.

- For $N > 0$, we obtain with Definition 3.2.6 that

$$\begin{aligned} 0 &= \langle F, {}_N Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} \\ &= \sqrt{\frac{(n-N)!}{(n+N)!}} \, \langle F, \eth_0^N \, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} \\ &= \sqrt{\frac{(n-N)!}{(n+N)!}} \int_\Omega F(\xi) \overline{\eth_0^N \, Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= \sqrt{\frac{(n-N)!}{(n+N)!}} \int_\Omega F(\xi) \overline{\eth_{N-1} \dots \eth_1 \eth_0 \, Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= \sqrt{\frac{(n-N)!}{(n+N)!}} \int_\Omega F(\xi) \overline{\eth}_{-N+1} \dots \overline{\eth}_{-1} \overline{\eth}_0 \, \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi). \end{aligned}$$

These integrals exist, because we know from Corollary 3.4.13 that the integrands are bounded. Now, we apply N times Lemma 3.5.8. So,

$$\begin{aligned} 0 &= \sqrt{\frac{(n-N)!}{(n+N)!}} \, (-1)^N \int_\Omega \left(\overline{\eth}_1 \overline{\eth}_2 \dots \overline{\eth}_N \, F(\xi) \right) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= \sqrt{\frac{(n-N)!}{(n+N)!}} \, (-1)^N \int_\Omega \left(\overline{\eth}_N^N \, F(\xi) \right) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= \sqrt{\frac{(n-N)!}{(n+N)!}} \, (-1)^N \left\langle \overline{\eth}_N^N \, F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)}. \end{aligned}$$

Therefore, we see that

$$\left\langle \overline{\eth}_N^N \, F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)} = 0$$

for $n \geq N$. Now, we look at $\left\langle \overline{\eth}_N^N \, F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)}$ for $n < N$. Here, we get the other way around to above

$$\begin{aligned} \left\langle \overline{\eth}_N^N \, F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)} &= \int_\Omega \left(\overline{\eth}_1 \overline{\eth}_2 \dots \overline{\eth}_N \, F(\xi) \right) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= (-1)^N \int_\Omega F(\xi) \overline{\eth_{N-1} \dots \eth_1 \eth_0 \, Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi). \end{aligned}$$

We obtain that for $\xi \in \Omega$

$$\eth_{N-1} \dots \eth_1 \eth_0 \, Y_{n,j}(\xi) = 0$$

for all $n = 0, \dots, N-1$. This is obvious, because for $n = 0$, we receive from Definition 2.4.37 and from Definition 3.2.3

$$\eth_0 \, Y_{0,0}(\xi) = \eth_0 \, \frac{1}{\sqrt{4\pi}} = 0.$$

For $n = 1$, we can use Definition 3.2.6 such that

$$\eth_0 \, Y_{1,j}(\xi) = \sqrt{2} \, {}_1 Y_{1,j}(\xi),$$

where we know from Theorem 3.6.7 that ${}_1Y_{1,j} \in \ker(\eth_1)$. This means that

$$\eth_1\eth_0 \; Y_{1,j}(\xi) = \sqrt{2}\; \eth_1 \; {}_1Y_{1,j}(\xi) = 0.$$

Inductively, we get the proposition for all $n \in \mathbb{N}_0$ up to $n = N-1$, where we get from Definition 3.2.6 and from Theorem 3.6.7

$$\eth_{N-1}\ldots\eth_1\eth_0 \; Y_{N-1,j}(\xi) = \sqrt{(2(N-1))!}\; \eth_{N-1} \; \underbrace{{}_{N-1}Y_{N-1,j}(\xi)}_{\in\ker(\eth_{N-1})} = 0.$$

All in all, we obtain

$$\left\langle \overline{\eth}_N^N \; F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)} = 0$$

for all $n \in \mathbb{N}_0$ and $j = -n, \ldots, n$. Then, we have the same case like for $N = 0$ and we get from [58] that

$$\overline{\eth}_N^N \; F = 0.$$

Now, we apply N times Lemma 3.6.10. So, we get with $N > 0$ the proposition that $F = 0$.

- For $N < 0$, we get with Definition 3.2.6 that

$$\begin{aligned}
0 &= \langle F, {}_NY_{n,j}\rangle_{\mathrm{L}^2(\Omega)} \\
&= (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}} \; \left\langle F, \overline{\eth}_0^{-N} \; Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)} \\
&= (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}} \int_\Omega F(\xi)\overline{\overline{\eth}_0^{-N} \; Y_{n,j}(\xi)} \; \mathrm{d}\omega(\xi) \\
&= (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}} \int_\Omega F(\xi)\overline{\overline{\eth}_{N+1}\ldots\overline{\eth}_{-1}\overline{\eth}_0 \; Y_{n,j}(\xi)} \; \mathrm{d}\omega(\xi) \\
&= (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}} \int_\Omega F(\xi)\eth_{-(N+1)}\ldots\eth_1\eth_0 \; \overline{Y_{n,j}(\xi)} \; \mathrm{d}\omega(\xi).
\end{aligned}$$

Like before, these integrals exist, because we know from Corollary 3.4.13 that the integrands are bounded. Then, we apply N times Lemma 3.5.8.

$$\begin{aligned}
0 &= (-1)^N \sqrt{\frac{(n+N)!}{(n-N)!}} \; (-1)^N \int_\Omega \left(\eth_{-1}\eth_{-2}\ldots\eth_N \; F(\xi)\right) \overline{Y_{n,j}(\xi)} \; \mathrm{d}\omega(\xi) \\
&= \sqrt{\frac{(n+N)!}{(n-N)!}} \int_\Omega \left(\eth_N^{-N} \; F(\xi)\right) \overline{Y_{n,j}(\xi)} \; \mathrm{d}\omega(\xi) \\
&= \sqrt{\frac{(n+N)!}{(n-N)!}} \; \left\langle \eth_N^{-N} \; F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)}.
\end{aligned}$$

So, we see that

$$\left\langle \eth_N^{-N} \; F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)} = 0$$

for $n \geq -N$. Now, we look at $\left\langle \eth_N^{-N} \; F, Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)}$ for $n < -N$. Here, we get the other way around to above

$$\begin{aligned}\left\langle \eth_N^{-N} F, Y_{n,j} \right\rangle_{L^2(\Omega)} &= \int_\Omega \left(\eth_{-1}\eth_{-2}\ldots\eth_N\ F(\xi)\right) \overline{Y_{n,j}(\xi)}\ \mathrm{d}\omega(\xi)\\ &= (-1)^N \int_\Omega F(\xi) \overline{\overline{\eth}_{N+1}\ldots\overline{\eth}_{-1}\overline{\eth}_0\ Y_{n,j}(\xi)}\ \mathrm{d}\omega(\xi).\end{aligned}$$

We obtain that for $\xi \in \Omega$

$$\overline{\eth}_{N+1}\ldots\overline{\eth}_{-1}\overline{\eth}_0\ Y_{n,j}(\xi) = 0$$

for all $n = 0, \ldots, -N-1$ (see also Lemma 3.6.6). This is obvious, because for $n = 0$, we receive from Definition 2.4.37 and from Definition 3.2.3

$$\overline{\eth}_0\ Y_{0,0}(\xi) = \overline{\eth}_0\ \frac{1}{\sqrt{4\pi}} = 0.$$

For $n = 1$, we can use Definition 3.2.6 such that

$$\overline{\eth}_0\ Y_{1,j}(\xi) = -\sqrt{2}\ {}_{-1}Y_{1,j}(\xi),$$

where we know from Theorem 3.6.8 that ${}_{-1}Y_{1,j} \in \ker\left(\overline{\eth}_{-1}\right)$. This means that

$$\overline{\eth}_{-1}\overline{\eth}_0\ Y_{1,j}(\xi) = -\sqrt{2}\ \overline{\eth}_{-1}\ {}_{-1}Y_{1,j}(\xi) = 0.$$

Inductively, we get the proposition for all $n \in \mathbb{N}_0$ up to $n = -N-1$, where we get from Definition 3.2.6 and from Theorem 3.6.8

$$\overline{\eth}_{N+1}\ldots\overline{\eth}_{-1}\overline{\eth}_0\ Y_{-(N+1),j}(\xi) = (-1)^{-(N+1)}\sqrt{(2(-N-1))!}\ \overline{\eth}_{N+1}\ \underbrace{{}_{N+1}Y_{-(N+1),j}(\xi)}_{\in\ker\left(\overline{\eth}_{-(N+1)}\right)} = 0.$$

All in all, we obtain

$$\left\langle \eth_N^{-N} F, Y_{n,j} \right\rangle_{L^2(\Omega)} = 0$$

for all $n \in \mathbb{N}_0$ and $j = -n, \ldots, n$. So, we get from [58] that

$$\eth_N^{-N}\ F = 0.$$

Now, we apply N times Lemma 3.6.10. Then, we get with $N < 0$ the proposition that $F = 0$.

□

From the previous lemma, we can directly conclude the following theorem.

Theorem 3.6.12. *The system* $\{{}_NY_{n,j}\}_{n\geq|N|,j=-n,\ldots,n}$ *is complete in* $\left(L^2(\Omega), \langle .,. \rangle_{L^2(\Omega)}\right)$. *Consequently, for* $F \in L^2(\Omega)$*, we obtain*

$$\lim_{L\to\infty}\left\| F - \sum_{n=|N|}^{L}\sum_{j=-n}^{n} \langle F, {}_NY_{n,j}\rangle_{L^2(\Omega)}\ {}_NY_{n,j} \right\|_{L^2(\Omega)} = 0.$$

This means that, for every function $F \in L^2(\Omega)$ *and for every* $N \in \mathbb{Z}$*, there exist coefficients* ${}_NF_{n,j}$ *such that*

$$F = \sum_{n=|N|}^{\infty}\sum_{j=-n}^{n} {}_NF_{n,j}\ {}_NY_{n,j}.$$

So, in combination with the orthonormality of the spin-weighted spherical harmonics from Theorem 3.4.21, we obtain that $\{{}_N Y_{n,j}\}_{n\geq|N|,j=-n,\ldots,n}$ *is an orthonormal basis of* $\mathrm{L}^2(\Omega)$. *Therefore, the coefficients* ${}_N F_{n,j}$ *are unique.*

Remark 3.6.13. *There is an alternative proof of the previous theorem given by [63]. However, the proposition is shown for a "suitably regular" [63] function* ${}_N F$ *of spin weight* N. *If we want to perform the proof for functions in* $\mathrm{L}^2(\Omega)$, *we have to show that* $\left(\overline{\eth}_N^N\ {}_N F\right) \in \mathrm{L}^2(\Omega)$ *at least for all* ${}_N F$ *in a dense subset of* $\mathrm{L}^2(\Omega)$. *Then, we can go on like in [63] and write* $\left(\overline{\eth}_N^N\ {}_N F\right)$ *in the basis of the fully normalized spherical harmonics in the* $\mathrm{L}^2(\Omega)$*-sense as*

$$\overline{\eth}_N^N\ {}_N F = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} F_{n,j} Y_{n,j}.$$

From Lemma 3.6.5, we conclude

$$\overline{\eth}_N^N \eth_0^N\ Y_{n,j} = (-1)^N\ \frac{n!}{(n-N)!}\ \frac{(n+N)!}{n!}\ Y_{n,j} = (-1)^N\ \frac{(n+N)!}{(n-N)!}\ Y_{n,j}.$$

Then,

$$\begin{aligned}
\overline{\eth}_N^N\ {}_N F &= \sum_{n=0}^{N-1}\sum_{j=-n}^{n} F_{n,j} Y_{n,j} + \sum_{n=N}^{\infty}\sum_{j=-n}^{n} F_{n,j} Y_{n,j} \\
&= \sum_{n=0}^{N-1}\sum_{j=-n}^{n} F_{n,j} Y_{n,j} + \sum_{n=N}^{\infty}\sum_{j=-n}^{n} \underbrace{(-1)^N\ \frac{(n-N)!}{(n+N)!}\ F_{n,j}}_{=:c_{n,j}} \overline{\eth}_N^N \eth_0^N\ Y_{n,j}
\end{aligned}$$

such that

$$\overline{\eth}_N^N \left({}_N F - \sum_{n=N}^{\infty}\sum_{j=-n}^{n} c_{n,j} \eth_0^N\ Y_{n,j} \right) = \sum_{n=0}^{N-1}\sum_{j=-n}^{n} F_{n,j} Y_{n,j}. \tag{3.23}$$

The right-hand side is zero, because if we multiply (3.23) by $\overline{Y_{n',j}(\xi)}$ *with* $n' < N$ *and then integrate over* Ω *with Theorem 3.4.21, we get*

$$\begin{aligned}
&\int_\Omega \overline{Y_{n',j}(\xi)}\ \overline{\eth}_N^N \left({}_N F(\xi) - \sum_{n=N}^{\infty}\sum_{j=-n}^{n} c_{n,j} \eth_0^N\ Y_{n,j}(\xi) \right)\ \mathrm{d}\omega(\xi) \\
&= \sum_{n=0}^{N-1}\sum_{j=-n}^{n} F_{n,j} \underbrace{\int_\Omega \overline{Y_{n',j}(\xi)} Y_{n,j}(\xi)\ \mathrm{d}\omega(\xi)}_{=\delta_{n,n'}} \\
&= F_{n',j}.
\end{aligned}$$

Now, we apply N *times Lemma 3.5.8, so*

$$\int_\Omega (-1)^N \left(\eth_0^N\ \overline{Y_{n',j}(\xi)} \right) \left({}_N F(\xi) - \sum_{n=N}^{\infty}\sum_{j=-n}^{n} c_{n,j} \eth_0^N\ Y_{n,j}(\xi) \right)\ \mathrm{d}\omega(\xi) = F_{n',j}.$$

These integrals exists, because we know from Lemma 3.4.16 that the integrand is bounded.

With Lemma 3.6.6 and $n' < N$, the left-hand side is zero, so

$$F_{n,j} = 0 \qquad \text{for all } n < N$$

and therefore, the right-hand side of (3.23) also vanishes. Then, (3.23) reduces to

$$\bar{\eth}_0^N \left({}_N F - \sum_{n=N}^{\infty} \sum_{j=-n}^{n} c_{n,j} \eth_0^N \; Y_{n,j} \right) = 0.$$

Now, we apply N times Lemma 3.6.10 such that

$${}_N F - \sum_{n=N}^{\infty} \sum_{j=-n}^{n} c_{n,j} \eth_0^N \; Y_{n,j} = 0$$

and therefore,

$${}_N F = \sum_{n=N}^{\infty} \sum_{j=-n}^{n} c_{n,j} \eth_0^N \; Y_{n,j}.$$

With Definition 3.2.6, this leads to the proposition

$$\begin{aligned} {}_N F &= \sum_{n=N}^{\infty} \sum_{j=-n}^{n} (-1)^N \, \frac{(n-N)!}{(n+N)!} \; F_{n,j} \, \sqrt{\frac{(n+N)!}{(n-N)!}} \; {}_N Y_{n,j} \\ &= \sum_{n=N}^{\infty} \sum_{j=-n}^{n} \underbrace{(-1)^N \, \sqrt{\frac{(n-N)!}{(n+N)!}} \; F_{n,j}}_{=:{}_N F_{n,j}} \; {}_N Y_{n,j}. \end{aligned}$$

Corollary 3.6.14 (Parseval Identity for the Spin-Weighted Spherical Harmonics)**.** *For every function $F, G \in \mathrm{L}^2(\Omega)$ and every $N \in \mathbb{Z}$, the Parseval identity for the spin-weighted spherical is fulfilled such that*

$$\langle F, G \rangle_{\mathrm{L}^2(\Omega)} = \sum_{n=|N|}^{\infty} \sum_{j=-n}^{n} \langle F, {}_N Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} \, \overline{\langle G, {}_N Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)}}$$

and consequently,

$$\|F\|^2_{\mathrm{L}^2(\Omega)} = \sum_{n=|N|}^{\infty} \sum_{j=-n}^{n} \left| \langle F, {}_N Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} \right|^2.$$

In the following, we define the space, which is spanned by the spin-weighted spherical harmonics.

Lemma 3.6.15. *The definition of a homogeneous polynomial from Definition 2.4.17 is equivalent to*

$$P(x) = |x|^n \sum_{i_1,\ldots,i_{2n}=1}^{2} d_{i_1,\ldots,i_{2n}} \underbrace{o^{i_1}_{\frac{x}{|x|}} \ldots o^{i_n}_{\frac{x}{|x|}}}_{n} \underbrace{\hat{o}^{i_{n+1}}_{\frac{x}{|x|}} \ldots \hat{o}^{i_{2n}}_{\frac{x}{|x|}}}_{n}$$

for $x \in \mathbb{R}^3$. So, all homogeneous polynomials have spin weight zero [91].

Proof. From Definition 2.4.17, we know that for $x \in \mathbb{R}^3$

$$P(x) = \sum_{|\alpha|=n} C_\alpha x^\alpha$$

$$
\begin{aligned}
&= \sum_{|\alpha|=n} C_\alpha x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \\
&= \sum_{i_1,\ldots,i_n=1}^{3} \tilde{C}_i x_{i_1} \ldots x_{i_n},
\end{aligned}
$$

where $i = (i_1, \ldots, i_n)$ and without loss of generality $\tilde{C}_i$ is totally symmetric. With $x = |x|\xi$ and $\xi \in \Omega$, we get

$$
\begin{aligned}
P(x) &= |x|^n \sum_{|\alpha|=n} C_\alpha \xi^\alpha \\
&= |x|^n \sum_{i_1,\ldots,i_n=1}^{3} \tilde{C}_i \xi_{i_1} \ldots \xi_{i_n}.
\end{aligned}
$$

Because

$$
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \sqrt{1-t^2}\cos\varphi \\ \sqrt{1-t^2}\sin\varphi \\ t \end{pmatrix},
$$

$\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right)$, and $\sin\varphi = \frac{1}{2i}\left(e^{i\varphi} - e^{-i\varphi}\right)$, we can formulate the components of the vector ξ with Definition 3.1.2 and with (3.19) by

$$
\begin{aligned}
\xi_1 &= \sqrt{1-t^2}\cos\varphi \\
&= 2\sqrt{\frac{1-t}{2}}\sqrt{\frac{1+t}{2}}\frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) \\
&= \left(\sqrt{\frac{1-t}{2}}e^{i\frac{\varphi}{2}}\right)\left(\sqrt{\frac{1+t}{2}}e^{i\frac{\varphi}{2}}\right) - \left(\sqrt{\frac{1+t}{2}}e^{-i\frac{\varphi}{2}}\right)\left(-\sqrt{\frac{1-t}{2}}e^{-i\frac{\varphi}{2}}\right) \\
&= o_\xi^2 \hat{o}_\xi^2 - o_\xi^1 \hat{o}_\xi^1 \\
&= -o_\xi^1 \hat{o}_\xi^1 + o_\xi^2 \hat{o}_\xi^2, \\
\xi_2 &= \sqrt{1-t^2}\sin\varphi \\
&= 2\sqrt{\frac{1-t}{2}}\sqrt{\frac{1+t}{2}}\frac{1}{2i}\left(e^{i\varphi} - e^{-i\varphi}\right) \\
&= -i\left(\sqrt{\frac{1-t}{2}}e^{i\frac{\varphi}{2}}\right)\left(\sqrt{\frac{1+t}{2}}e^{i\frac{\varphi}{2}}\right) + \left(\sqrt{\frac{1+t}{2}}e^{-i\frac{\varphi}{2}}\right)\left(-\sqrt{\frac{1-t}{2}}e^{-i\frac{\varphi}{2}}\right) \\
&= -i\left(o_\xi^2 \hat{o}_\xi^2 + o_\xi^1 \hat{o}_\xi^1\right) \\
&= -i\left(o_\xi^1 \hat{o}_\xi^1 + o_\xi^2 \hat{o}_\xi^2\right), \\
\xi_3 &= t = o_\xi^1 \hat{o}_\xi^2 + o_\xi^2 \hat{o}_\xi^1.
\end{aligned}
$$

So, we obtain that

$$
\xi_i = \sum_{A,B=1}^{2} \sigma^i_{A,B} o_\xi^A \hat{o}_\xi^B
$$

with

$$
\sigma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^2 = -i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Furthermore, we see that the spin weight satisfies $\mathrm{sw}(\xi_i) = 0$ for $i = 1, 2, 3$ and therefore,

$$\mathrm{sw}(\xi_{i_1} \cdot \ldots \cdot \xi_{i_n}) = 0 + \ldots + 0 = 0.$$

Altogether, we obtain

$$\begin{aligned}
P(x) &= |x|^n \sum_{i_1,\ldots,i_n=1}^{3} \tilde{C}_i \left(\sum_{A_1,B_1=1}^{2} \sigma^{i_1}_{A_1,B_1} o_\xi^{A_1} \hat{o}_\xi^{B_1} \right) \ldots \left(\sum_{A_n,B_n=1}^{2} \sigma^{i_n}_{A_n,B_n} o_\xi^{A_n} \hat{o}_\xi^{B_n} \right) \\
&= |x|^n \sum_{A_1,\ldots,A_n,B_1,\ldots,B_n=1}^{2} \underbrace{\left(\sum_{i_1,\ldots,i_n=1}^{3} \tilde{C}_i \sigma^{i_1}_{A_1,B_1} \ldots \sigma^{i_n}_{A_n,B_n} \right)}_{=: d_{A_1,B_1,\ldots,A_n,B_n} \text{ totally symmetric}} o_\xi^{A_1} o_\xi^{A_2} \ldots o_\xi^{A_n} \hat{o}_\xi^{B_1} \hat{o}_\xi^{B_2} \ldots \hat{o}_\xi^{B_n} \\
&= |x|^n \sum_{A_1,\ldots,A_n,B_1,\ldots,B_n=1}^{2} d_{A_1,\ldots,A_n,B_1,\ldots,B_n} o_\xi^{A_1} o_\xi^{A_2} \ldots o_\xi^{A_n} \hat{o}_\xi^{B_1} \hat{o}_\xi^{B_2} \ldots \hat{o}_\xi^{B_n} \\
&= |x|^n \sum_{i_1,\ldots,i_{2n}=1}^{2} d_{i_1,\ldots,i_{2n}} \underbrace{o_\xi^{i_1} \ldots o_\xi^{i_n}}_{n} \underbrace{\hat{o}_\xi^{i_{n+1}} \ldots \hat{o}_\xi^{i_{2n}}}_{n} .
\end{aligned}$$

□

With the previous lemma, we can show that the spin-weighted spherical harmonics are continuous differentiable.

Remark 3.6.16. *For $N = 0$, we get from the previous lemma that the spin-weighted spherical harmonics are homogeneous polynomials. So, they are infinitely differentiable on the unit sphere Ω. This means that they are in $\mathrm{C}^{(\infty)}(\Omega)$. For $N \neq 0$, this is not the case. Here, we know already from Corollary 3.2.11 that the spin-weighted spherical harmonics are infinitely differentiable on Ω_0. So, there are problems for $t = \pm 1$. We see this in the following examples.*

Let $\xi = \xi(t, \varphi) \in \Omega$. Then, we get from Definition 2.4.37

$$\begin{aligned}
Y_{2,2}(\xi) &= \sqrt{\frac{5}{4\pi} \frac{(-1)^4}{8}} \sqrt{\frac{1}{24}} \, e^{2i\varphi} \left(1 - t^2\right) \underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^4 \left(1 - t^2\right)^2}_{=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^4 (1-2t^2+t^4) = 24} \\
&= \sqrt{\frac{15}{32\pi}} \left(1 - t^2\right) e^{2i\varphi}
\end{aligned}$$

and

$$Y_{2,-2}(\xi) = (-1)^2 \, \overline{Y_{2,2}(\xi)} = \sqrt{\frac{15}{32\pi}} \left(1 - t^2\right) e^{-2i\varphi}.$$

So, we can calculate the according spin-weighted spherical harmonics of spin weight 2 with help of (3.3)

$$\begin{aligned}
&{}_2Y_{2,\pm 2}(\xi) \\
&= \sqrt{\frac{5}{256\pi}} \, e^{\pm 2i\varphi} \Bigg(\left(1 - t^2\right)(-2) - \frac{1}{1 - t^2} \left(1 - t^2\right)(\pm 2i)^2 - 2i \Bigg((\pm 2i)(-2t) \\
&\quad + \frac{t}{1 - t^2} \left(1 - t^2\right)(\pm 2i) \Bigg) \Bigg)
\end{aligned}$$

$$= \sqrt{\frac{5}{256\pi}}\ e^{\pm 2i\varphi}\left(-2+2t^2+4\pm 4\left(-2t+t\right)\right)$$
$$= \sqrt{\frac{5}{256\pi}}\ e^{\pm 2i\varphi}\left(2+2t^2\mp 4t\right)$$
$$= \sqrt{\frac{5}{64\pi}}\ \left(1\mp t\right)^2 e^{\pm 2i\varphi}.$$

Then, we see that for $t \to -1$

$${}_2Y_{2,2}(\xi) \to \sqrt{\frac{5}{4\pi}}\ e^{2i\varphi}$$

and for $t \to 1$

$${}_2Y_{2,-2}(\xi) \to \sqrt{\frac{5}{4\pi}}\ e^{-2i\varphi}$$

depend on φ *and therefore, the spin-weighted spherical harmonics are* not *in* $\mathrm{C}^{(\infty)}(\Omega)$*.*

Now, we can formulate a set for the spin-weighted spherical harmonics.

Remark 3.6.17. *We know from Definition 2.4.16 that a function* $F \in \mathrm{C}^{(2)}(\mathbb{R}^3)$ *is called harmonic, if for all* $x \in \mathbb{R}^3$

$$\Delta_x F(x) = 0.$$

This is equivalent to the proposition that the polynomial of degree $n \in \mathbb{N}_0$ *given by* $F(r\xi) = r^n Y_n(\xi)$*,* $r \in \mathbb{R}$*,* $\xi \in \Omega$*, is harmonic, if*

$$\Delta^*_\xi Y_n(\xi) = -n(n+1)Y_n(\xi).$$

Therefore, we formulate the following definition.

Definition 3.6.18. *A function* $Y_n \in \mathrm{C}^{(2)}(\Omega_0)$ *of degree* $n \in \mathbb{N}_0$ *is called* $(*, N)$*-harmonic for* $N \in \mathbb{Z}$*, if for all* $\xi \in \Omega_0$ *the equation*

$$\Delta^{*,N}_\xi\ Y_n(\xi) = -n(n+1)Y_n(\xi)$$

for the spin-weighted Beltrami operator from Corollary 3.3.7 is fulfilled.

Definition 3.6.19. *With* $\mathrm{Harm}_n^N(\Omega_0)$*, we denote the set of the* $(*, N)$*-harmonic polynomials of degree* $n \in \mathbb{N}_0$ *and spin weight* $N \in \mathbb{Z}$*. This means that*

$$\mathrm{Harm}_n^N(\Omega_0) := \left\{ {}_NP_n \;\middle|\; {}_NP_n \in \mathrm{C}^{(2)}(\Omega_0) \text{ is a } (*, N)\text{-harmonic function of spin weight } N \text{ and degree } n \right\}.$$

Definition 3.6.20. *Analogously to Definition 2.4.18, we define the spaces*

$$\mathrm{Harm}_{0\ldots n}^N(\Omega_0) := \bigoplus_{i=0}^{n} \mathrm{Harm}_i^N(\Omega_0)$$

and

$$\mathrm{Harm}_{0\ldots\infty}^N(\Omega_0) := \bigcup_{i=0}^{\infty} \mathrm{Harm}_{0\ldots i}^N(\Omega_0).$$

Note that a $(*, N)$-harmonic function gets, in general, *not* harmonic for $N \neq 0$ by multiplication with r^n, $r \in \mathbb{R}$, $n \in \mathbb{N}_0$. Similarly a function of spin weight $N \neq 0$ is, in general, *not* homogeneous. Only for the case of spin weight zero, we get by $\mathrm{Harm}_n^N(\Omega_0) = \mathrm{Harm}_n(\Omega_0)$ the set of the on the unit sphere restricted harmonic and homogeneous polynomials of degree $n \in \mathbb{N}_0$.

Corollary 3.6.21. *The spin-weighted spherical harmonics ${}_NY_n$ of spin weight $N \in \mathbb{Z}$ and degree $n \in \mathbb{N}_0$, $n \geq |N|$, form the set $\mathrm{Harm}_n^N(\Omega_0)$. The system $\{{}_NY_{n,j}\}_{j=-n,\ldots,n}$ forms an orthonormal system on* $\left(\mathrm{Harm}_n^N(\Omega_0), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)}\right)$.

Proof. From Theorem 3.6.12, we know that the system $\{{}_NY_{n,j}\}_{n\geq|N|,j=-n,\ldots,n}$ is complete for all $N \in \mathbb{Z}$, so $\{{}_NY_{n,j}\}_{j=-n,\ldots,n}$ is also complete in $\mathrm{Harm}_n^N(\Omega_0)$ for all $N \in \mathbb{Z}$ and all $n \in \mathbb{N}_0$, $n \geq |N|$. Furthermore, from Theorem 3.4.21, we can conclude that

$$\langle {}_NY_{n,j}, {}_NY_{n,j'} \rangle_{\mathrm{L}^2(\Omega)} = \delta_{j,j'}$$

for all $j, j' \in \{-n, \ldots, n\}$. Finally, from Lemma 3.6.11, we obtain that

$$\langle F, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = 0$$

for all $j = -n, \ldots, n$ and $F \in \mathrm{Harm}_n^N(\Omega_0)$ induces that $F = 0$. □

Remark 3.6.22. *It is obvious that for $0 \leq p \leq q \leq \infty$*

$$\mathrm{Harm}_{p\ldots q}^N(\Omega_0) = \mathrm{span}\,\{{}_NY_{n,j}\}_{n=p,\ldots,q,j=-n,\ldots,n}\,.$$

So, we can conclude the following lemma for the $\mathrm{C}(\Omega_0)$-norm of the spin-weighted spherical harmonics.

Lemma 3.6.23. *For every ${}_NY_n \in \mathrm{Harm}_n^N(\Omega_0)$, we get for $n \in \mathbb{N}_0$ and $N \in \mathbb{Z}$ with $n \geq |N|$*

$$\|{}_NY_n\|_{\mathrm{C}(\Omega_0)} \leq \sqrt{\frac{2n+1}{4\pi}}\ \|{}_NY_n\|_{\mathrm{L}^2(\Omega)}\,.$$

In particular, we obtain

$$\|{}_NY_{n,j}\|_{\mathrm{C}(\Omega_0)} \leq \sqrt{\frac{2n+1}{4\pi}}.$$

Proof. For every ${}_NY_n \in \mathrm{Harm}_n^N(\Omega_0)$, $n \in \mathbb{N}_0$, $N \in \mathbb{Z}$, $n \geq |N|$, we know from Corollary 3.6.21 that for $\xi \in \Omega_0$

$${}_NY_n(\xi) = \sum_{j=-n}^{n} \langle {}_NY_n, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)}\ {}_NY_{n,j}(\xi).$$

Then, we get with the Cauchy-Schwarz inequality

$$\begin{aligned} |{}_NY_n(\xi)| &= \left| \sum_{j=-n}^{n} \langle {}_NY_n, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)}\ {}_NY_{n,j}(\xi) \right| \\ &= \left(\sum_{j=-n}^{n} \left| \langle {}_NY_n, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{j=-n}^{n} |{}_NY_{n,j}(\xi)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

With Corollary 3.4.25, this leads directly to the proposition

$$\|{}_NY_n\|_{\mathrm{C}(\Omega_0)} \leq \sqrt{\frac{2n+1}{4\pi}}\ \|{}_NY_n\|_{\mathrm{L}^2(\Omega)}$$

and particularly to

$$\|{}_NY_{n,j}\|_{\mathrm{C}(\Omega_0)} \leq \sqrt{\frac{2n+1}{4\pi}}.$$

□

Now, we can show that the spin-weighted spherical harmonics are the only eigenfunctions of the operator $\Delta^{*,N}$.

Theorem 3.6.24. *The spin-weighted spherical harmonics ${}_NY_{n,j}$ and their linear combinations are the only eigenfunctions of the differential operator $\Delta^{*,N}$ to the eigenvalues $-n(n+1)$ for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, $n \geq |N|$, and all $j = -n, \dots, n$ in $\mathrm{X}^2(\Omega_0)$.*

Proof. Let $\xi \in \Omega_0$. From Corollary 3.3.7, we know already that for all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, $n \geq |N|$, and all $j = -n, \dots, n$

$$\Delta_\xi^{*,N}\ {}_NY_{n,j}(\xi) = -n(n+1)\ {}_NY_{n,j}(\xi).$$

(1) Determination of the eigenvalues
We assume that there exists a $\lambda \neq -n(n+1)$ for all $n \in \mathbb{N}_0$, $n \geq |N|$, and a $K \in \mathrm{X}^2(\Omega_0)$ such that $\Delta^{*,N} K = \lambda K$. We apply Green's second surface identity on the unit sphere for the operator $\Delta^{*,N}$, Theorem 3.4.27, so that for all $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, and $j = -n, \dots, n$

$$\begin{aligned}
\int_\Omega \Bigg(K(\xi) \underbrace{\overline{\Delta_\xi^{*,N}\ {}_NY_{n,j}(\xi)}}_{=-n(n+1)\ \overline{{}_NY_{n,j}}} - \overline{{}_NY_{n,j}(\xi)}\ \underbrace{\Delta_\xi^{*,N} K(\xi)}_{=\lambda K(\xi)} \Bigg)\, \mathrm{d}\omega(\xi) &= 0 \\
\Leftrightarrow \underbrace{(-n(n+1) - \lambda)}_{\neq 0} \int_\Omega K(\xi)\ \overline{{}_NY_{n,j}(\xi)}\, \mathrm{d}\omega(\xi) &= 0 \\
\Leftrightarrow \langle K, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)} &= 0.
\end{aligned}$$

From Theorem 3.6.12, we know that $\{{}_NY_{n,j}\}_{n,j}$ is an orthonormal basis. So, we obtain $K = 0$ in $\mathrm{L}^2(\Omega)$, and therefore, λ is not an eigenvalue.

(2) Determination of the eigenfunctions
We assume that there exists a $\lambda = -k(k+1)$ for $k \in \mathbb{N}_0$, $k \geq |N|$, and a $K \in \mathrm{X}^2(\Omega_0)$, $K \neq 0$, such that $\Delta^{*,N} K = \lambda K$. In analogy to the previous considerations, we get for all $N \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, $n \geq |N|$, and $j = -n, \dots, n$ the equation

$$(-n(n+1) - \lambda) \langle K, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = 0,$$

where $-n(n+1) - \lambda \neq 0$ for $n \neq k$. Then, for all $N \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $n \geq |N|$, $n \neq k$, and $j = -n, \dots, n$

$$\langle K, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = 0.$$

From Theorem 3.6.12, we know again that $\{{}_NY_{n,j}\}_{n,j}$ is an orthonormal basis. So, we obtain

$$\begin{aligned}
K &= \sum_{n=|N|}^{\infty} \sum_{j=-n}^{n} \langle K, {}_NY_{n,j} \rangle_{\mathrm{L}^2(\Omega)}\ {}_NY_{n,j} \\
&= \sum_{j=-k}^{k} \langle K, {}_NY_{k,j} \rangle_{\mathrm{L}^2(\Omega)}\ {}_NY_{k,j} \in \operatorname{span} \{{}_NY_{k,j}\}_{j=-k,\dots,k}.
\end{aligned}$$

□

3.7 Spin-Weighted Legendre Polynomials

In this section, we look at the spin-weighted Legendre polynomials and their properties.

Definition 3.7.1. *We define the spin-weighted Legendre polynomials of spin weight $N \in \mathbb{Z}$ and degree $n \in \mathbb{N}_0$, $n \geq |N|$, by*

$$_NP_n(t) := \frac{(-1)^n}{2^n\sqrt{(n+N)!(n-N)!}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\left[(1-t)^{n-N}(1+t)^{n+N}\right]$$

for $t \in [-1,1]$.

The spin-weighted Legendre polynomials are polynomials of degree n. So, they are infinitely differentiable on $[-1,1]$. Furthermore, it is obvious that for spin weight zero we get the Legendre polynomials $_0P_n = P_n$.

Definition 3.7.2. *Furthermore, we define the associated spin-weighted Legendre functions of spin weight $N \in \mathbb{Z}$, degree $n \in \mathbb{N}_0$, $n \geq |N|$, and order $j = -n, \ldots, n$ by*

$$_NP_{n,j}(t) := \begin{cases} (1-t)^{\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \ _NP_n(t), & j \geq 0 \\ (-1)^{N+j} \ _{-N}P_{n,-j}(t), & j < 0 \end{cases}$$

for $t \in [-1,1]$. So, for $t = \cos\vartheta$, we conclude from (3.9) and (3.10) that

$$d^n_{j,-N}(\vartheta) = (-1)^j\sqrt{\frac{(n-|j|)!}{(n+|j|)!}} \ _NP_{n,j}(t).$$

Furthermore, for spin weight zero, we get the associated Legendre functions $_0P_{n,j} = P_{n,j}$.

Note that in [12] the associated generalized Legendre functions are defined by

$$P^N_{n,j} = d^n_{j,N},$$

which do not yield the associated Legendre functions for spin weight zero.

Corollary 3.7.3. *With Theorem 3.4.9 and with Definition 3.4.1, we see directly that the spin-weighted spherical harmonics can be written by the associated spin-weighted Legendre functions and by the spin-weighted Legendre polynomials for $\xi = \xi(t,\varphi) \in \Omega$ by*

$$\begin{aligned} _NY_{n,j}(\xi) &= {}_NX_{n,j}(t)e^{ij\varphi} \\ &= (-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|j|)!}{(n+|j|)!}} \ e^{ij\varphi} \ _NP_{n,j}(t) \\ &= (-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|j|)!}{(n+|j|)!}} \ e^{ij\varphi}(1-t)^{\frac{j+N}{2}}(1+t)^{\frac{j-N}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \ _NP_n(t), \end{aligned}$$

where for $t = \cos\vartheta$

$$_NX_{n,j}(t) := (-1)^N\sqrt{\frac{2n+1}{4\pi}} \ d^n_{j,-N}(\vartheta)$$

denote the fully normalized associated spin-weighted Legendre functions (compare Corollary 3.2.10). Note that

$$_NY_{n,j}(\xi) = (-1)^{N+j} \ \overline{_{-N}Y_{n,-j}(\xi)}.$$

Particularly, we get for $j = 0$

$$_NY_{n,0}(\xi) = (-1)^N \sqrt{\frac{2n+1}{4\pi}} \left(\frac{1-t}{1+t}\right)^{\frac{N}{2}} {}_NP_n(t)$$

and

$$_NP_{n,0}(t) = \left(\frac{1-t}{1+t}\right)^{\frac{N}{2}} {}_NP_n(t).$$

Remark 3.7.4. *We get directly from Definition 3.7.1 and Definition 3.7.2 with 7. from Remark 3.4.4 that the associated spin-weighted Legendre functions satisfy*

$$_NP_{n,j}(1) = (-1)^j \sqrt{\frac{(n+|j|)!}{(n-|j|)!}}\ \delta_{j,-N}$$

and consequently,

$$_NP_{n,0}(1) = \delta_{N,0}.$$

Furthermore, we get directly from Definition 3.7.1 and Definition 3.7.2 for $t \in [-1,1]$

$$_NP_n(-t) = (-1)^n\ {}_{-N}P_n(t),$$
$$_NP_{n,j}(-t) = \begin{cases} (-1)^{n+j}\ {}_{-N}P_{n,j}(t) & j \geq 0 \\ (-1)^{n+N}\ {}_NP_{n,-j}(t) & g < 0 \end{cases}.$$

The last equation was mentioned in [12].

Definition 3.7.5. *For* ${}_NF, {}_NG \in \mathrm{L}^2([-1,1])$*, functions of spin weight* $N \in \mathbb{Z}$*, we define with*

$$\langle {}_NF, {}_NG \rangle_{w_N} := \int_{-1}^{1} {}_NF(t)\, {}_NG(t) w_N(t)\ \mathrm{d}t$$

a weighted scalar product. Here, we use as a weight function

$$w_N(t) := \left(\frac{1-t}{1+t}\right)^N > 0$$

for $t \in [-1,1]$*. Note that this weight function is not integrable for* $N \neq 0$*. In the following, we use only functions such that the weighted integrals exist.*

Lemma 3.7.6. *The associated spin-weighted Legendre functions are orthogonal on* $[-1,1]$ *such that*

$$\int_{-1}^{1} {}_NP_{n,j}(t)\ {}_NP_{n',j'}(t)\ \mathrm{d}t = \frac{2}{2n+1} \frac{(n+|j|)!}{(n-|j|)!}\ \delta_{n,n'}\delta_{j,j'}$$

for all $N \in \mathbb{Z}$*, all* $n, n' \in \mathbb{N}_0$*,* $n, n' \geq |N|$*, all* $j = -n, \ldots, n$*, and all* $j' = -n', \ldots, n'$*.*

Proof. For the spin-weighted spherical harmonics, we know from Theorem 3.4.21 that

$$\int_\Omega {}_NY_{n,j}(\xi)\ \overline{{}_NY_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) = \delta_{n,n'}\delta_{j,j'}$$

for all $N \in \mathbb{Z}$, all $n, n' \in \mathbb{N}_0$, $n, n' \geq |N|$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$. Then, we get for the associated spin-weighted Legendre functions with Definition 3.7.2

$$\begin{aligned}&(-1)^{N+j+N+j'} \underbrace{\int_0^{2\pi} e^{i(j-j')\varphi} \, \mathrm{d}\varphi}_{=2\pi\delta_{j,j'}} \frac{\sqrt{(2n+1)(2n'+1)}}{4\pi} \sqrt{\frac{(n-|j|)!(n-|j'|)!}{(n+|j|)!(n+|j'|)!}} \\ &\times \int_{-1}^{1} {}_NP_{n,j}(t) \ {}_NP_{n',j'}(t) \ \mathrm{d}t \\ &= \delta_{n,n'}\delta_{j,j'},\end{aligned}$$

and therefore,

$$\int_{-1}^{1} {}_NP_{n,j}(t) \ {}_NP_{n',j'}(t) \ \mathrm{d}t = \frac{2}{2n+1} \frac{(n+|j|)!}{(n-|j|)!} \ \delta_{n,n'}\delta_{j,j'}.$$

□

Lemma 3.7.7. *We get for the spin-weighted Legendre polynomials*

$$\int_{-1}^{1} {}_NP_n(t) \ {}_NP_{n'}(t) w_N(t) \ \mathrm{d}t = \frac{2}{2n+1} \ \delta_{n,n'}$$

for all $N \in \mathbb{Z}$ and all $n, n' \in \mathbb{N}_0$, $n, n' \geq |N|$. In particular, we define

$${}_N\gamma_n := \int_{-1}^{1} {}_NP_n^2(t) w_N(t) \ \mathrm{d}t = \frac{2}{2n+1}.$$

So, the spin-weighted Legendre polynomials are orthogonal polynomials with respect to the weight function $w_N(x) = \left(\frac{1-t}{1+t}\right)^N$.

Proof. For the associated spin-weighted Legendre functions, we know from the previous lemma that

$$\int_{-1}^{1} {}_NP_{n,j}(t) \ {}_NP_{n',j'}(t) \ \mathrm{d}t = \frac{2}{2n+1} \frac{(n+|j|)!}{(n-|j|)!} \ \delta_{n,n'}\delta_{j,j'}$$

for all $N \in \mathbb{Z}$, all $n, n' \in \mathbb{N}_0$, $n, n' \geq |N|$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$ This also holds true for $j = 0$, where we get with

$${}_NP_{n,0}(t) = w_N^{\frac{1}{2}}(t) \ {}_NP_n(t)$$

from Definition 3.7.2 that

$$\int_{-1}^{1} {}_NP_n(t) \ {}_NP_{n'}(t) w_N(t) \ \mathrm{d}t = \frac{2}{2n+1} \ \delta_{n,n'}.$$

Then, we obtain directly the proposition. □

Lemma 3.7.8. *The associated spin-weighted Legendre functions fulfill the following recursion relations for $t \in [-1, 1]$*

$$\begin{aligned}(t^2-1) \ {}_NP'_{n,j}(t) &= \left(nt + \frac{Nj}{n}\right) \ {}_NP_{n,j}(t) - \beta^N_{n,|j|} \ {}_NP_{n-1,j}(t), \\ &= -\left((n+1)t + \frac{Nj}{n+1}\right) \ {}_NP_{n,j}(t) + \beta^N_{n+1,-|j|} \ {}_NP_{n+1,j}(t),\end{aligned}$$

$$(2n+1)\left(t + \frac{Nj}{n(n+1)}\right) \ {}_NP_{n,j}(t) = \beta^N_{n,|j|} \ {}_NP_{n-1,j}(t) + \beta^N_{n+1,-|j|} \ {}_NP_{n+1,j}(t),$$

where

$$\beta_{n,j}^N = \sqrt{(n-N)(n+N)}\ \frac{n+j}{n}$$

for all $N \in \mathbb{Z}$*, all* $n \in \mathbb{N}_0$*,* $n \geq |N|$*, and all* $j = -n, \dots, n$*. Furthermore, we denote* ${}_N P_{n,j}(t) := 0$ *for* $n < |j|$*.*

Proof. For the spin-weighted spherical harmonics, we know the following recursion relations from Theorem 3.3.1 for $\xi = \xi(t, \varphi) \in \Omega$, all $N \in \mathbb{Z}$, all $n \in \mathbb{N}_0$, $n \geq |N+1|$, and all $j = -n, \dots, n$

$$\begin{aligned}
\left(t^2-1\right)\partial_t\ {}_N Y_{n,j}(\xi) &= \left(nt + \frac{Nj}{n}\right)\ {}_N Y_{n,j}(\xi) - (2n+1)\alpha_{n,j}^N\ {}_N Y_{n-1,j}(\xi), \\
&= -\left((n+1)t + \frac{Nj}{n+1}\right)\ {}_N Y_{n,j}(\xi) + (2n+1)\alpha_{n+1,j}^N\ {}_N Y_{n+1,j}(\xi), \\
\left(t + \frac{Nj}{n(n+1)}\right)\ {}_N Y_{n,j}(\xi) &= \alpha_{n,j}^N\ {}_N Y_{n-1,j}(\xi) + \alpha_{n+1,j}^N\ {}_N Y_{n+1,j}(\xi),
\end{aligned}$$

where

$$\alpha_{n,j}^N := \frac{\sqrt{(n-N)(n+N)}}{n}\ \sqrt{\frac{(n-j)(n+j)}{(2n-1)(2n+1)}}.$$

From Corollary 3.7.3, we know that

$${}_N Y_{n,j}(\xi) = (-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|j|)!}{(n+|j|)!}}e^{ij\varphi}\ {}_N P_{n,j}(t).$$

- The first recursion leads to

$$\begin{aligned}
&\left(t^2-1\right)(-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}e^{ij\varphi}\sqrt{\frac{(n-|j|)!}{(n+|j|)!}}\ {}_N P'_{n,j}(t) \\
&= \left(nt + \frac{Nj}{n}\right)(-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|j|)!}{(n+|j|)!}}e^{ij\varphi}\ {}_N P_{n,j}(t) \\
&\quad - (2n+1)\alpha_{n,j}^N(-1)^{N+j}\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|j|)!}{(n-1+|j|)!}}e^{ij\varphi}\ {}_N P_{n-1,j}(t),
\end{aligned}$$

  and therefore,

$$\left(t^2-1\right)\ {}_N P'_{n,j}(t) = \left(nt + \frac{Nj}{n}\right)\ {}_N P_{n,j}(t) - \underbrace{(2n+1)\alpha_{n,j}^N\sqrt{\frac{2n-1}{2n+1}}\sqrt{\frac{n+|j|}{n-|j|}}}_{=\sqrt{(n-N)(n+N)}\frac{n+|j|}{n}=\beta_{n,|j|}^N}\ {}_N P_{n-1,j}(t).$$

- From the second recursion relation of the spin-weighted spherical harmonics, we get analogous

$$\begin{aligned}
&\left(t^2-1\right)\ {}_N P'_{n,j}(t) \\
&= -\left((n+1)t + \frac{Nj}{n+1}\right)\ {}_N P_{n,j}(t) + \underbrace{(2n+1)\alpha_{n+1,j}^N\sqrt{\frac{2n+3}{2n+1}}\sqrt{\frac{n+1-|j|}{n+1+|j|}}}_{=\beta_{n+1,-|j|}^N}\ {}_N P_{n+1,j}(t).
\end{aligned}$$

- Furthermore, the third recursion relation of the spin-weighted spherical harmonics leads to

$$\left(t + \frac{Nj}{n(n+1)}\right)(-1)^{N+j}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|j|)!}{(n+|j|)!}}e^{ij\varphi}\ {}_N P_{n,j}(t)$$
$$= \alpha_{n,j}^N (-1)^{N+j}\sqrt{\frac{2n-1}{4\pi}}\sqrt{\frac{(n-1-|j|)!}{(n-1+|j|)!}}e^{ij\varphi}\ {}_N P_{n-1,j}(t)$$
$$+ \alpha_{n+1,j}^N (-1)^{N+j}\sqrt{\frac{2n+3}{4\pi}}\sqrt{\frac{(n+1-|j|)!}{(n+1+|j|)!}}e^{ij\varphi}\ {}_N P_{n+1,j}(t)$$

and therefore,

$$\left(t + \frac{Nj}{n(n+1)}\right)\ {}_N P_{n,j}(t)$$
$$= \underbrace{\alpha_{n,j}^N \sqrt{\frac{2n-1}{2n+1}}\sqrt{\frac{n+|j|}{n-|j|}}}_{=\frac{1}{2n+1}\beta_{n,|j|}^N}\ {}_N P_{n-1,j}(t) + \underbrace{\alpha_{n+1,j}^N \sqrt{\frac{2n+3}{2n+1}}\sqrt{\frac{n+1-|j|}{n+1+|j|}}}_{=\frac{1}{2n+1}\beta_{n+1,-|j|}^N}\ {}_N P_{n+1,j}(t).$$

Then, we get for the associated spin-weighted Legendre functions

$$(2n+1)\left(t + \frac{Nj}{n(n+1)}\right)\ {}_N P_{n,j}(t) = \beta_{n,|j|}^N\ {}_N P_{n-1,j}(t) + \beta_{n+1,-|j|}^N\ {}_N P_{n+1,j}(t).$$

□

Lemma 3.7.9. *The spin-weighted Legendre polynomials fulfill the following recursion relations for $t \in [-1,1]$, for all $N \in \mathbb{Z}$, and all $n \in \mathbb{N}_0$, $n \geq |N|$,*

$$\left(t^2 - 1\right)\ {}_N P_n'(t) = (nt - N)\ {}_N P_n(t) - \sqrt{(n-N)(n+N)}\ {}_N P_{n-1}(t), \tag{3.24}$$
$$= -((n+1)t + N)\ {}_N P_n(t) + \sqrt{(n+1-N)(n+1+N)}\ {}_N P_{n+1}(t), \tag{3.25}$$
$$(2n+1)t\ {}_N P_n(t) = \sqrt{(n-N)(n+N)}\ {}_N P_{n-1}(t) + \sqrt{(n+1-N)(n+1+N)}\ {}_N P_{n+1}(t) \tag{3.26}$$

Proof. Let $t \in [-1,1]$, $N \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, $n \geq |N|$. With the previous lemma, the recursion relations for the associated spin-weighted Legendre functions, and with

$${}_N P_{n,0}(t) = w_N^{\frac{1}{2}}(t)\ {}_N P_n(t)$$

from Definition 3.7.2, we get the properties for $j = 0$.

- The first recursion relation leads to

$$\left(t^2 - 1\right)\ {}_N P_{n,0}'(t) = nt\ {}_N P_{n,0}(t) - \beta_{n,0}^N\ {}_N P_{n-1,0}(t)$$

such that

$$\left(t^2 - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(w_N^{\frac{1}{2}}(t)\ {}_N P_n(t)\right) = ntw_N^{\frac{1}{2}}(t)\ {}_N P_n(t) - \sqrt{(n-N)(n+N)}\ w_N^{\frac{1}{2}}(t)\ {}_N P_{n-1}(t).$$

With

$$\begin{aligned}\left(t^2-1\right)\left(w_N^{\frac{1}{2}}(t)\right)' &= -(1-t)(1+t)\left(-\frac{N}{2}(1-t)^{\frac{N}{2}-1}(1+t)^{-\frac{N}{2}} - \frac{N}{2}(1-t)^{\frac{N}{2}}(1+t)^{-\frac{N}{2}-1}\right) \\ &= \frac{N}{2}\left(w_N^{\frac{1}{2}}(t)(1+t) + (1-t)w_N^{\frac{1}{2}}(t)\right) \\ &= N w_N^{\frac{1}{2}}(t), \qquad (3.27)\end{aligned}$$

we get for the spin-weighted Legendre polynomials

$$N\ {}_NP_n(t) + \left(t^2-1\right)\ {}_NP_n'(t) = nt\ {}_NP_n(t) - \sqrt{(n-N)(n+N)}\ {}_NP_{n-1}(t).$$

This yields (3.24).

- From the second recursion relation of the associated spin-weighted Legendre functions, we get
$$\left(t^2-1\right)\ {}_NP_{n,0}'(t) = -(n+1)t\ {}_NP_{n,0}(t) + \beta_{n+1,0}^N\ {}_NP_{n+1,0}(t)$$
and therefore,
$$\begin{aligned}&\left(t^2-1\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(w_N^{\frac{1}{2}}(t)\ {}_NP_n(t)\right) \\ &= -(n+1)tw_N^{\frac{1}{2}}(t)\ {}_NP_n(t) + \sqrt{(n+1-N)(n+1+N)}\ w_N^{\frac{1}{2}}(t)\ {}_NP_{n+1}(t)\end{aligned}$$
With (3.27), this leads to
$$N\ {}_NP_n(t) + \left(t^2-1\right)\ {}_NP_n'(t) = -(n+1)t\ {}_NP_n(t) + \sqrt{(n+1-N)(n+1+N)}\ {}_NP_{n+1}(t).$$
Then, we obtain (3.25).

- Furthermore, the third recursion relation of the associated spin-weighted Legendre functions leads to
$$(2n+1)t\ {}_NP_{n,0}(t) = \beta_{n,0}^N\ {}_NP_{n-1,0}(t) + \beta_{n+1,0}^N\ {}_NP_{n+1,0}(t)$$
such that
$$\begin{aligned}&(2n+1)tw_N^{\frac{1}{2}}(t)\ {}_NP_n(t) \\ &= \sqrt{(n-N)(n+N)}\ w_N^{\frac{1}{2}}(t)\ {}_NP_{n-1}(t) + \sqrt{(n+1-N)(n+1+N)}\ w_N^{\frac{1}{2}}(t)\ {}_NP_{n+1}(t).\end{aligned}$$
Then, we get for the spin-weighted Legendre polynomials
$$(2n+1)t\ {}_NP_n(t) = \sqrt{(n-N)(n+N)}\ {}_NP_{n-1}(t) + \sqrt{(n+1-N)(n+1+N)}\ {}_NP_{n+1}(t).$$

□

Corollary 3.7.10. *If we know ${}_NP_{|N|}(t)$, ${}_NP_{|N|}'(t)$ and ${}_NP_{|N|+1}(t)$, ${}_NP_{|N|+1}'(t)$, then we can calculate with*

$$\begin{aligned}{}_NP_n(t) &= \frac{(2n-1)t\ {}_NP_{n-1}(t) - \sqrt{(n-1-N)(n-1+N)}\ {}_NP_{n-2}(t)}{\sqrt{(n-N)(n+N)}}, \\ {}_NP_n'(t) &= \frac{\sqrt{(n-N)(n+N)}\ {}_NP_{n-1}(t) - (nt-N)\ {}_NP_n(t)}{1-t^2}\end{aligned}$$

the roots of the spin-weighted Legendre polynomials ${}_NP_n$ *by Newton's method for* $t \in (-1,1)$*, for all* $N \in \mathbb{Z}$ *and all* $n \in \mathbb{N}_0$*,* $n \geq |N+1|$*.*

In this thesis we only need spin weight $N = 0, \pm 1, \pm 2$. Therefore, we calculate the accordingly spin-weighted spherical harmonics.

Lemma 3.7.11. *We get for the needed spin weights for* $t \in [-1,1]$ *and for* $n \in \mathbb{N}_0$ *and greater or equal the absolut value of the spin weight*

$$\begin{aligned}
{}_0P_n(t) &= P_n(t),\\
{}_1P_n(t) &= -\frac{1+t}{\sqrt{n(n+1)}}P_n'(t),\\
{}_{-1}P_n(t) &= \frac{1-t}{\sqrt{n(n+1)}}P_n'(t),\\
{}_2P_n(t) &= \frac{(1+t)^2}{\sqrt{n(n+1)(n(n+1)-2)}}P_n''(t),\\
{}_{-2}P_n(t) &= \frac{(1-t)^2}{\sqrt{n(n+1)(n(n+1)-2)}}P_n''(t).
\end{aligned}$$

Proof. With Corollary 3.7.3, we know for $\xi = \xi(t,\varphi) \in \Omega$ that

$${}_NP_n(t) = (-1)^N\sqrt{\frac{4\pi}{2n+1}}\left(\frac{1+t}{1-t}\right)^{\frac{N}{2}} {}_NY_{n,0}(\xi).$$

From Defition 2.4.37, we get

$$Y_{n,0}(\xi) = (-1)^0\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{n!}{n!}}P_{n,0}(t)e^0 = \sqrt{\frac{2n+1}{4\pi}}P_n(t)$$

such that $Y_{n,0}$ is independent form φ. With Definition 3.2.6, we obtain

$${}_NP_n(t) = (-1)^N\left(\frac{1+t}{1-t}\right)^{\frac{N}{2}}\begin{cases}\sqrt{\frac{(n-N)!}{(n+N)!}}\eth_0^N P_n(t), & 0 \leq N \leq n\\ (-1)^N\sqrt{\frac{(n+N)!}{(n-N)!}}\bar{\eth}_0^{-N} P_n(t), & -n \leq N \leq 0\end{cases}.$$

Because of the independence from φ, the partial differentiation after φ is zero and we can for this case reduce the differentiation operators to

$$\begin{aligned}
\eth_N &= \sqrt{1-t^2}\partial_t + \frac{Nt}{\sqrt{1-t^2}},\\
\bar{\eth}_N &= \sqrt{1-t^2}\partial_t - \frac{Nt}{\sqrt{1-t^2}}.
\end{aligned}$$

Then, we get for the needed spin weights

$$\begin{aligned}
{}_0P_n(t) &= P_n(t),\\
{}_1P_n(t) &= -\sqrt{\frac{1+t}{1-t}}\frac{1}{\sqrt{n(n+1)}}\sqrt{1-t^2}P_n'(t)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1+t}{\sqrt{n(n+1)}} P_n'(t), \\
{}_{-1}P_n(t) &= -\sqrt{\frac{1-t}{1+t}} \frac{-1}{\sqrt{n(n+1)}} \sqrt{1-t^2} P_n'(t) \\
&= \frac{1-t}{\sqrt{n(n+1)}} P_n'(t), \\
{}_{2}P_n(t) &= \frac{1+t}{1-t} \frac{1}{\sqrt{(n-1)n(n+1)(n+2)}} \left(\sqrt{1-t^2} \frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{\sqrt{1-t^2}} \right) \sqrt{1-t^2} P_n'(t) \\
&= \frac{1+t}{1-t} \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}} \left(\sqrt{1-t^2} \frac{\frac{1}{2}(-2t)}{\sqrt{1-t^2}} P_n'(t) + (1-t^2) P_n''(t) + t P_n'(t) \right) \\
&= \frac{(1+t)^2}{\sqrt{n(n+1)(n(n+1)-2)}} P_n''(t), \\
{}_{-2}P_n(t) &= \frac{1-t}{1+t} \frac{(-1)^2}{\sqrt{(n-1)n(n+1)(n+2)}} \left(\sqrt{1-t^2} \frac{\mathrm{d}}{\mathrm{d}t} - \frac{-t}{\sqrt{1-t^2}} \right) \sqrt{1-t^2} P_n'(t) \\
&= \frac{1-t}{1+t} \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}} \left(\sqrt{1-t^2} \frac{\frac{1}{2}(-2t)}{\sqrt{1-t^2}} P_n'(t) + (1-t^2) P_n''(t) + t P_n'(t) \right) \\
&= \frac{(1-t)^2}{\sqrt{n(n+1)(n(n+1)-2)}} P_n''(t).
\end{aligned}
$$

□

Corollary 3.7.12. *For the needed spin weights, we get the following initial conditions for* $t \in [-1, 1]$

$$
\begin{aligned}
{}_0P_0(t) &= 1, \\
{}_0P_0'(t) &= 0, \\
{}_0P_1(t) &= t, \\
{}_0P_1'(t) &= 1 \\
{}_1P_1(t) &= -\frac{1}{\sqrt{2}}(1+t), \\
{}_1P_1'(t) &= -\frac{1}{\sqrt{2}}, \\
{}_1P_2(t) &= -\sqrt{\frac{3}{8}} \left(t + t^2 \right), \\
{}_1P_2'(t) &= -\sqrt{\frac{3}{8}}(1+2t), \\
{}_{-1}P_1(t) &= \frac{1}{\sqrt{2}}(1-t), \\
{}_{-1}P_1'(t) &= -\frac{1}{\sqrt{2}}, \\
{}_{-1}P_2(t) &= \sqrt{\frac{3}{8}} \left(t - t^2 \right), \\
{}_{-1}P_2'(t) &= \sqrt{\frac{3}{8}}(1-2t),
\end{aligned}
$$

$$
\begin{aligned}
{}_2P_2(t) &= \sqrt{\frac{3}{8}}(1+t)^2, \\
{}_2P_2'(t) &= \sqrt{\frac{3}{2}}(1+t), \\
{}_2P_3(t) &= \sqrt{\frac{15}{8}}t(1+t)^2, \\
{}_2P_3'(t) &= \sqrt{\frac{15}{8}}\left(1+4t+3t^2\right), \\
{}_{-2}P_2(t) &= \sqrt{\frac{3}{8}}(1-t)^2, \\
{}_{-2}P_2'(t) &= -\sqrt{\frac{3}{2}}(1-t), \\
{}_{-2}P_3(t) &= \sqrt{\frac{15}{8}}t(1-t)^2, \\
{}_{-2}P_3'(t) &= \sqrt{\frac{15}{8}}\left(1-4t+3t^2\right).
\end{aligned}
$$

Proof. From Definition 2.4.1, we get the corresponding Legendre polynomials for $t \in [-1, 1]$ by

$$
\begin{aligned}
P_0(t) &= 1, \\
P_1(t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2-1\right) \\
&= t, \\
P_2(t) &= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^2-1\right)^2 \\
&= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^4-2t^2+1\right) \\
&= \frac{1}{2}\left(3t^2-1\right), \\
P_3(t) &= \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(t^2-1\right)^3 \\
&= \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(t^6-3t^4+3t^2-1\right) \\
&= \frac{1}{2}\left(5t^3-3t\right),
\end{aligned}
$$

and its derivatives by

$$
\begin{aligned}
P_1'(t) &= 1, \\
P_2'(t) &= 3t, \\
P_2''(t) &= 3, \\
P_3'(t) &= \frac{1}{2}\left(15t^2-3\right), \\
P_3''(t) &= 15t.
\end{aligned}
$$

Then, we can calculate the needed spin-weighted Legendre polynomials with help of the previous lemma. For spin weight zero, we get

$$
\begin{aligned}
{}_0P_0(t) &= P_0(t) = 1,\\
{}_0P_0'(t) &= 0,\\
{}_0P_1(t) &= P_1(t) = t,\\
{}_0P_1'(t) &= 1.
\end{aligned}
$$

For spin weight 1, we gain

$$
\begin{aligned}
{}_1P_1(t) &= -\frac{1+t}{\sqrt{2}}P_1'(t) = -\frac{1}{\sqrt{2}}(1+t),\\
{}_1P_1'(t) &= -\frac{1}{\sqrt{2}},\\
{}_1P_2(t) &= -\frac{1+t}{\sqrt{6}}P_2'(t) = -\sqrt{\frac{3}{2}}t(1+t),\\
{}_1P_2'(t) &= -\sqrt{\frac{3}{2}}(1+2t).
\end{aligned}
$$

For spin weight -1, we obtain

$$
\begin{aligned}
{}_{-1}P_1(t) &= \frac{1-t}{\sqrt{2}}P_1'(t) = \frac{1}{\sqrt{2}}(1-t),\\
{}_{-1}P_1'(t) &= -\frac{1}{\sqrt{2}},\\
{}_{-1}P_2(t) &= \frac{1-t}{\sqrt{6}}P_2'(t) = \sqrt{\frac{3}{2}}t(1-t),\\
{}_{-1}P_2'(t) &= \sqrt{\frac{3}{2}}(1-2t).
\end{aligned}
$$

For spin weight 2, we get

$$
\begin{aligned}
{}_2P_2(t) &= \frac{(1+t)^2}{\sqrt{24}}P_2''(t) = \sqrt{\frac{3}{8}}(1+t)^2,\\
{}_2P_2'(t) &= \sqrt{\frac{3}{2}}(1+t),\\
{}_2P_3(t) &= \frac{(1+t)^2}{\sqrt{120}}P_3''(t) = \sqrt{\frac{15}{8}}t(1+t)^2,\\
{}_2P_3'(t) &= \sqrt{\frac{15}{8}}\left(1+4t+3t^2\right).
\end{aligned}
$$

For spin weight -2, we calculate

$$
\begin{aligned}
{}_{-2}P_2(t) &= \frac{(1-t)^2}{\sqrt{24}}P_2''(t) = \sqrt{\frac{3}{8}}(1-t)^2,\\
{}_{-2}P_2'(t) &= -\sqrt{\frac{3}{2}}(1-t),\\
{}_{-2}P_3(t) &= \frac{(1-t)^2}{\sqrt{120}}P_3''(t) = \sqrt{\frac{15}{8}}t(1-t)^2,
\end{aligned}
$$

$$_{-2}P_3'(t) = \sqrt{\frac{15}{8}}\left(1 - 4t + 3t^2\right).$$

□

Lemma 3.7.13. *The leading coefficient of the spin-weighted Legendre polynomials of spin weight $N \in \mathbb{Z}$ and degree $n \in \mathbb{N}_0$, $n \geq |N|$, is given by*

$$_NA_n = \frac{(-1)^N (2n)!}{2^n n! \sqrt{(n+N)!(n-N)!}}.$$

Proof. With the binomial theorem and the Cauchy product of two sums, we get for $t \in [-1, 1]$

$$\begin{aligned}
_NP_n(t) &= \underbrace{\frac{(-1)^n}{2^n\sqrt{(n+N)!(n-N)!}}}_{=:_Nc_n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left[(1-t)^{n-N}(1+t)^{n+N}\right] \\
&= {_Nc_n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left[\left(\sum_{k=0}^{n-N}\binom{n-N}{k}(-1)^{n-N-k}t^{n-N-k}\right)\left(\sum_{l=0}^{n+N}\binom{n+N}{l}t^{n+N-l}\right)\right] \\
&= {_Nc_n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left[\sum_{i=0}^{2n}\underbrace{\left(\sum_{j=\max\{0,i-(n+N)\}}^{\min\{n-N,i\}}\binom{n-N}{j}\binom{n+N}{i-j}(-1)^{n-N-j}\right)}_{=:_Na_n^i} t^{n-N-j+n+N-i+j}\right] \\
&= {_Nc_n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \sum_{i=0}^{2n} {_Na_n^i}\, t^{2n-i} \\
&= \sum_{i=0}^{n} {_Nc_n}\ {_Na_n^i}\frac{(2n-i)!}{(n-i)!}t^{n-i}
\end{aligned}$$

for all $N \in \mathbb{Z}$ and all $n \in \mathbb{N}_0$, $n \geq |N|$. So, the leading coefficient of the spin-weighted Legendre polynomials of degree n is given by

$$\begin{aligned}
_NA_n &= {_Nc_n}{_Na_n^0}\frac{(2n)!}{n!} \\
&= \frac{(-1)^n}{2^n\sqrt{(n+N)!(n-N)!}}\left(\sum_{j=0}^{0}\binom{n-N}{j}\binom{n+N}{-j}(-1)^{n-N-j}\right)\frac{(2n)!}{n!} \\
&= \frac{(-1)^N(2n)!}{2^n n!\sqrt{(n+N)!(n-N)!}}.
\end{aligned}$$

□

Lemma 3.7.14. *The Christoffel-Darboux formula for the associated spin-weighted Legendre functions is given for $t_1, t_2 \in [-1, 1]$ by*

$$\begin{aligned}
&(t_1 - t_2)\sum_{n=|N|}^{L-1}(2n+1)\frac{(n-j)!}{(n+j)!}\ {_NP_{n,j}(t_1)}\ {_NP_{n,j}(t_2)} \\
&= \beta_{L,j}^N\frac{(L-j)!}{(L+j)!}\left({_NP_{L,j}(t_1)}\ {_NP_{L-1,j}(t_2)} - {_NP_{L-1,j}(t_1)}\ {_NP_{L,j}(t_2)}\right)
\end{aligned}$$

for all $N \in \mathbb{Z}$ and all $j = -L, \dots, L$.

Proof. For $t_1, t_2 \in [-1,1]$, $N \in \mathbb{Z}$, $j = -L, \dots, L$, and $n_j = \max\{|N|, |j|\}$ with

$$(t_1 - t_2) \sum_{n=n_j}^{L-1} \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta) = \alpha_{L,j}^N \left(\overline{{}_NY_{L,j}(\xi)}\ {}_NY_{L-1,j}(\eta) - \overline{{}_NY_{L-1,j}(\xi)}\ {}_NY_{L,j}(\eta) \right)$$

from Theorem 3.3.5, we get

$$\begin{aligned}
&(t_1 - t_2) \sum_{n=|N|}^{L-1} (-1)^{N+j+N+j} \frac{2n+1}{4\pi} \frac{(n-|j|)!}{(n+|j|)!}\ {}_NP_{n,j}(t_1)\ {}_NP_{n,j}(t_2) \\
&= (-1)^{N+j+N+j} \frac{\sqrt{(L-N)(L+N)}}{L} \sqrt{\frac{(L-j)(L+j)}{(2L+1)(2L-1)}} \sqrt{\frac{(L-|j|)!(L-1-|j|)!}{(L+|j|)!(L-1+|j|)!}} \\
&\quad \times \frac{\sqrt{(2L+1)(2L-1)}}{4\pi} \left({}_NP_{L,j}(t_1)\ {}_NP_{L-1,j}(t_2) - {}_NP_{L-1,j}(t_1)\ {}_NP_{L,j}(t_2) \right).
\end{aligned}$$

Then, we obtain the following Christoffel-Darboux formula for the associated spin-weighted Legendre functions

$$\begin{aligned}
&(t_1 - t_2) \sum_{n=|N|}^{L-1} (2n+1) \frac{(n-|j|)!}{(n+|j|)!}\ {}_NP_{n,j}(t_1)\ {}_NP_{n,j}(t_2) \\
&= \beta_{L,|j|}^N \frac{(L-|j|)!}{(L+|j|)!} \left({}_NP_{L,j}(t_1)\ {}_NP_{L-1,j}(t_2) - {}_NP_{L-1,j}(t_1)\ {}_NP_{L,j}(t_2) \right).
\end{aligned}$$

□

Lemma 3.7.15. *Then, the Christoffel-Darboux formula for the spin-weighted Legendre polynomials is given for $t_1, t_2 \in [-1,1]$ by*

$$\begin{aligned}
&(t_1 - t_2) \sum_{n=|N|}^{L-1} (2n+1)\ {}_NP_n(t_1)\ {}_NP_n(t_2) \\
&= \sqrt{(L-N)(L+N)} \left({}_NP_L(t_1)\ {}_NP_{L-1}(t_2) - {}_NP_{L-1}(t_1)\ {}_NP_L(t_2) \right)
\end{aligned}$$

for all $N \in \mathbb{Z}$.

Proof. With the previous lemma and with $j = 0$, we get for $t_1, t_2 \in [-1,1]$, $N \in \mathbb{Z}$

$$\begin{aligned}
&(t_1 - t_2) \sum_{n=|N|}^{L-1} (2n+1)\ {}_NP_n(t_1)\ {}_NP_n(t_2) w_N^{\frac{1}{2}}(t_1) w_N^{\frac{1}{2}}(t_2) \\
&= w_N^{\frac{1}{2}}(t_1) w_N^{\frac{1}{2}}(t_2) \beta_{L,0}^N \left({}_NP_L(t_1)\ {}_NP_{L-1}(t_2) - {}_NP_{L-1}(t_1)\ {}_NP_L(t_2) \right).
\end{aligned}$$

Then, we obtain the following Christoffel-Darboux formula for the spin-weighted Legendre polynomials

$$\begin{aligned}
&(t_1 - t_2) \sum_{n=|N|}^{L-1} (2n+1)\ {}_NP_n(t_1)\ {}_NP_n(t_2) \\
&= \sqrt{(L-N)(L+N)} \left({}_NP_L(t_1)\ {}_NP_{L-1}(t_2) - {}_NP_{L-1}(t_1)\ {}_NP_L(t_2) \right).
\end{aligned}$$

□

Lemma 3.7.16. *With t_k a root of ${}_NP_L$, we get for $N \in \mathbb{Z}$*

$$ {}_NP_L'(t_k)\ {}_NP_{L-1}(t_k) = \frac{{}_NA_L\ {}_N\gamma_{L-1}}{{}_NA_{L-1}} \sum_{n=|N|}^{L-1} \frac{{}_NP_n^2(t_k)}{{}_N\gamma_n}. $$

Proof. With the previous lemma, we get for $N \in \mathbb{Z}$ and $t_1, t_2 \in [-1,1]$ with $t_1 \neq t_2$

$$ \sum_{n=|N|}^{L-1} \frac{2n+1}{2} {}_NP_n(t_1)\ {}_NP_n(t_2) = \frac{\sqrt{(L-N)(L+N)}}{2} \frac{{}_NP_L(t_1)\ {}_NP_{L-1}(t_2) - {}_NP_{L-1}(t_1)\ {}_NP_L(t_2)}{t_1 - t_2}. $$

Further, with

$$ \begin{aligned} \frac{{}_NA_L\ {}_N\gamma_{L-1}}{{}_NA_{L-1}} &= \frac{(-1)^N (2L)!}{2^L L! \sqrt{(L+N)!(L-N)!}} \frac{2^{L-1}(L-1)!\sqrt{(L-1+N)!(L-1-N)!}}{(-1)^N (2L-2)!} \frac{2}{2L-1} \\ &= \frac{2L(2L-1)}{2L\sqrt{(L+N)(L-N)}} \frac{2}{2L-1} \\ &= \frac{2}{\sqrt{(L+N)(L-N)}}, \end{aligned} \tag{3.28} $$

we obtain

$$ \sum_{n=|N|}^{L-1} \frac{{}_NP_n(t_1)\ {}_NP_n(t_2)}{{}_N\gamma_n} = \frac{{}_NA_{L-1}}{{}_NA_L\ {}_N\gamma_{L-1}} \frac{{}_NP_L(t_1)\ {}_NP_{L-1}(t_2) - {}_NP_{L-1}(t_1)\ {}_NP_L(t_2)}{t_1 - t_2}. $$

For $t := t_1 = t_2$, we get

$$ \begin{aligned} &\sum_{n=|N|}^{L-1} \frac{{}_NP_n^2(t)}{{}_N\gamma_n} \\ &= \frac{{}_NA_{L-1}}{{}_NA_L\ {}_N\gamma_{L-1}} \lim_{t_2 \to t_1} \frac{{}_NP_L(t_1)\ {}_NP_{L-1}(t_2) - {}_NP_{L-1}(t_1)\ {}_NP_L(t_2)}{t_1 - t_2} \\ &= \frac{{}_NA_{L-1}}{{}_NA_L\ {}_N\gamma_{L-1}} \lim_{t_2 \to t_1} \frac{({}_NP_L(t_1) - {}_NP_L(t_2))\ {}_NP_{L-1}(t_2) - ({}_NP_{L-1}(t_1) - {}_NP_{L-1}(t_2))\ {}_NP_L(t_2)}{t_1 - t_2} \\ &= \frac{{}_NA_{L-1}}{{}_NA_L\ {}_N\gamma_{L-1}} \left({}_NP_L'(t)\ {}_NP_{L-1}(t) - {}_NP_{L-1}'(t)\ {}_NP_L(t)\right). \end{aligned} $$

For $t = t_k$ with t_k a root of ${}_NP_L$, we receive

$$ {}_NP_L'(t_k)\ {}_NP_{L-1}(t_k) = \frac{{}_NA_L\ {}_N\gamma_{L-1}}{{}_NA_{L-1}} \sum_{n=|N|}^{L-1} \frac{{}_NP_n^2(t_k)}{{}_N\gamma_n}. $$

□

Now, we calculate the weighted integral of a spin-weighted Legendre polynomial.

Lemma 3.7.17. *The weighted integral over the spin-weighted Legendre polynomials is given by*

$$ \int_{-1}^{1} {}_NP_n(t) w_N(t)\ \mathrm{d}t = \sum_{k=0}^{n-N} \sum_{l=0}^{n-k} \sum_{m=0}^{k} {}_Nb_n^{k,l,m} \cdot \begin{cases} 2, & l = m = 0 \\ \frac{2}{l+m+1}, & l+m \text{ even} \\ 0, & l+m \text{ odd} \end{cases}, $$

where

$$ {}_Nb_n^{k,l,m} = \frac{(-1)^{n+l+k}\sqrt{(n+N)!(n-N)!}}{2^n(n-N-k)!(N+k)!}\binom{n}{k}\binom{n-k}{l}\binom{k}{m} $$

for $n \in \mathbb{N}_0$ *and* $N \in \mathbb{Z}$ *with* $n \geq |N|$.

Proof. Let $t \in [-1,1]$, $n \in \mathbb{N}_0$, and $N \in \mathbb{Z}$ with $n \geq |N|$. We can calculate the spin-weighted Legendre polynomials from Definition 3.7.1 with the Leibniz rule by

$$ \begin{aligned} {}_NP_n(t) &= \underbrace{\frac{(-1)^n}{2^n\sqrt{(n+N)!(n-N)!}}}_{={}_Nc_n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\left[(1-t)^{n-N}(1+t)^{n+N}\right] \\ &= {}_Nc_n\sum_{k=0}^{n}\binom{n}{k}\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k(1-t)^{n-N}\right)\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-k}(1+t)^{n+N}\right) \\ &= \sum_{k=0}^{n-N}\underbrace{{}_Nc_n\binom{n}{k}\frac{(n-N)!}{(n-N-k)!}\frac{(n+N)!}{(N+k)!}(-1)^k}_{=:{}_Na_n^k}(1-t)^{n-N-k}(1+t)^{N+k} \\ &= \left(\frac{1+t}{1-t}\right)^N\sum_{k=0}^{n-N}{}_Na_n^k(1-t)^{n-k}(1+t)^k. \end{aligned} $$

Then, we get with the binomial formula

$$ \begin{aligned} {}_NP_n(t)w_N(t) &= \sum_{k=0}^{n-N}{}_Na_n^k(1-t)^{n-k}(1+t)^k \\ &= \sum_{k=0}^{n-N}{}_Na_n^k\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^lt^l\sum_{m=0}^{k}\binom{k}{m}t^m \\ &= \sum_{k=0}^{n-N}\sum_{l=0}^{n-k}\sum_{m=0}^{k}\underbrace{{}_Na_n^k\binom{n-k}{l}\binom{k}{m}(-1)^l}_{=:{}_Nb_n^{k,l,m}}t^{l+m}. \end{aligned} $$

So, we can calculate the integral by

$$ \int_{-1}^{1}{}_NP_n(t)w_N(t)\,\mathrm{d}t = \sum_{k=0}^{n-N}\sum_{l=0}^{n-k}\sum_{m=0}^{k}{}_Nb_n^{k,l,m}\underbrace{\int_{-1}^{1}t^{l+m}\,\mathrm{d}t,}_{=\begin{cases}2, & l=m=0\\ \frac{2}{l+m+1}, & l+m \text{ even}\\ 0, & l+m \text{ odd}\end{cases}} $$

where

$$ {}_Nb_n^{k,l,m} = \frac{(-1)^{n+l+k}\sqrt{(n+N)!(n-N)!}}{2^n(n-N-k)!(N+k)!}\binom{n}{k}\binom{n-k}{l}\binom{k}{m}. $$

□

3.8 Additional Properties of the Spin-Weighted Spherical Harmonics

In this chapter, we summarize additional properties of the spin-weighted spherical harmonics.

Remark 3.8.1. *From [93], we get the confirmation of Corollary 3.3.7 with Theorem 3.4.9, because*

$$J^2 D^n_{j,N}(\alpha,\beta,\gamma) = n(n+1) D^n_{j,N}(\alpha,\beta,\gamma),$$

where $\alpha,\gamma \in [0,2\pi]$, $\beta \in [0,\pi]$ *and*

$$J^2 = -\left[\partial_\beta^2 + \cot\beta\partial_\beta + \frac{1}{\sin^2\beta}\left(\partial_\alpha^2 - 2\cos\beta\partial_\alpha\partial_\gamma + \partial_\gamma^2\right)\right].$$

With $t = \cos\beta$, *this leads us with Definition 3.4.1 to*

$$\begin{aligned}
& J^2 D^n_{j,N}(\alpha,\beta,\gamma) = -\left[\left(1-t^2\right)\partial_t^2 - 2t\partial_t - \frac{j^2 - 2jNt + N^2}{1-t^2}\right] D^n_{j,N}(\alpha,\beta,\gamma) \\
\Leftrightarrow\quad & J^2 \overline{D^n_{j,N}(\alpha,\beta,\gamma)} = -\left[\left(1-t^2\right)\partial_t^2 - 2t\partial_t - \frac{j^2 - 2jNt + N^2}{1-t^2}\right] \overline{D^n_{j,N}(\alpha,\beta,\gamma)} \\
\Leftrightarrow\quad & J^2 \overline{D^n_{j,-N}(\alpha,\beta,\gamma)} = -\left[\left(1-t^2\right)\partial_t^2 - 2t\partial_t - \frac{j^2 + 2jNt + N^2}{1-t^2}\right] \overline{D^n_{j,-N}(\alpha,\beta,\gamma)} \\
\Leftrightarrow\quad & J^2 \overline{D^n_{j,-N}(\alpha,\beta,\gamma)} = -\left[\partial_t\left(\left(1-t^2\right)\partial_t\right) + \frac{\partial_\alpha^2 + 2iNt\partial_\alpha - N^2}{1-t^2}\right] \overline{D^n_{j,-N}(\alpha,\beta,\gamma)}
\end{aligned}$$

for $\xi = \xi(t,\alpha) \in \Omega$. *Then, we obtain with Theorem 3.4.9*

$$_NY_{n,j}(\xi) = (-1)^N \sqrt{\frac{2n+1}{4\pi}}\, \overline{D^n_{j,-N}(\alpha,\beta,0)}$$

that

$$J^2 \ _NY_{n,j}(\xi) = -\Delta_\xi^{*,N} \ _NY_{n,j}(\xi).$$

Now, we collect additional recursion relations for the spin-weighted spherical harmonics. We borrow them from [93], where these recursion relations are listed for the Wigner D-function. Now, we recapitulate them directly for the spin-weighted spherical harmonics with help of Theorem 3.4.9. For further details, see [93].

Lemma 3.8.2. *For the spin-weighted spherical harmonics, the following five differential relations are given [93] for* $\xi = \xi(t,\varphi) \in \Omega$ *and for* $n \in \mathbb{N}_0$, $N, j \in \mathbb{Z}$ *with* $n \geq |N|$ *and* $n \geq |j|$

$$\left(t^2-1\right)\partial_t \ _NY_{n,j}(\xi) = -(n+1)\alpha^N_{n,j} \ _NY_{n-1,j}(\xi) + \frac{jN}{n(n+1)} \ _NY_{n,j}(\xi) + n\alpha^N_{n+1,j} \ _NY_{n+1,j}(\xi),$$

$$\begin{aligned}
\sqrt{1-t^2}\,\partial_t \ _NY_{n,j}(\xi) &= \frac{1}{2}\sqrt{(n+j)(n-j+1)}\, e^{-i\varphi} \ _NY_{n,j-1}(\xi) \\
&\quad - \frac{1}{2}\sqrt{(n-j)(n+j+1)}\, e^{i\varphi} \ _NY_{n,j+1}(\xi), \\
&= \frac{1}{2}\sqrt{(n-N)(n+N+1)} \ _{N+1}Y_{n,j}(\xi) \\
&\quad - \frac{1}{2}\sqrt{(n+N)(n-N+1)} \ _{N-1}Y_{n,j}(\xi),
\end{aligned}$$

$$= \pm \frac{N + jt}{\sqrt{1-t^2}} \; {}_NY_{n,j}(\xi) \pm \sqrt{(n \pm j)(n \mp j + 1)} \; e^{\mp i\varphi} \; {}_NY_{n,j\mp 1}(\xi),$$
$$= \mp \frac{j + Nt}{\sqrt{1-t^2}} \; {}_NY_{n,j}(\xi) \pm \sqrt{(n \mp N)(n \pm N + 1)} \; {}_{N\pm 1}Y_{n,j}(\xi),$$

where ${}_NY_{-1,j}(\xi) := 0$ *and* ${}_NY_{n,j}(\xi) := 0$ *for* $n < |N|$ *or* $n < |j|$.

The second relation is also mentioned in [12] and the third one in [12, 66]. Furthermore, in the fifth differential relation, we see the alternative definition of the spin-weighted spherical harmonics from Lemma 3.6.2.

Lemma 3.8.3. *For the spin-weighted spherical harmonics, we get the recursion relations [93] for* $\xi = \xi(t, \varphi) \in \Omega$ *and for* $n \in \mathbb{N}_0$, $N, j \in \mathbb{Z}$ *with* $n \geq |N|$ *and* $n \geq |j|$

$$\frac{j + Nt}{\sqrt{1-t^2}} \; {}_NY_{n,j}(\xi) = \frac{1}{2}\sqrt{(n-N)(n+N+1)} \; {}_{N+1}Y_{n,j}(\xi) + \frac{1}{2}\sqrt{(n+N)(n-N+1)} \; {}_{N-1}Y_{n,j}(\xi),$$

$$-\frac{N + jt}{\sqrt{1-t^2}} \; {}_NY_{n,j}(\xi) = \frac{1}{2}\sqrt{(n+j)(n-j+1)} \; e^{-i\varphi} \; {}_NY_{n,j-1}(\xi) + \frac{1}{2}\sqrt{(n-j)(n+j+1)} \; e^{i\varphi} \; {}_NY_{n,j+1}(\xi),$$

$$e^{\pm i\varphi} \; {}_NY_{n,j\pm 1}(\xi) = \mp \sqrt{\frac{(n-N)(n+N+1)}{(n\mp j)(n\pm j+1)}} \; \frac{1 \pm t}{2} \; {}_{N+1}Y_{n,j}(\xi) - \frac{N\sqrt{1-t^2}}{\sqrt{(n\mp j)(n\pm j+1)}} \; {}_NY_{n,j}(\xi) \pm \sqrt{\frac{(n+N)(n-N+1)}{(n\mp j)(n\pm j+1)}} \; \frac{1 \mp t}{2} \; {}_{N-1}Y_{n,j}(\xi),$$

$$- \; {}_{N\mp 1}Y_{n,j}(\xi) = \pm \sqrt{\frac{(n+j)(n-j+1)}{(n\pm N)(n\mp N+1)}} \; \frac{1 \pm t}{2} \; e^{-i\varphi} \; {}_NY_{n,j-1}(\xi) - \frac{j\sqrt{1-t^2}}{\sqrt{(n\pm N)(n\mp N+1)}} \; {}_NY_{n,j}(\xi) \mp \sqrt{\frac{(n-j)(n+j+1)}{(n\pm N)(n\mp N+1)}} \; \frac{1 \mp t}{2} \; e^{i\varphi} \; {}_NY_{n,j+1}(\xi),$$

$$\sqrt{1-t^2} \; e^{\pm i\varphi} \; {}_NY_{n,j\pm 1}(\xi) = \mp \sqrt{\frac{n \pm j + 1}{n \mp j}} \; \alpha_{n,j}^N \; {}_NY_{n-1,j}(\xi) - \frac{N\sqrt{(n\mp j)(n\pm j+1)}}{n(n+1)} \; {}_NY_{n,j}(\xi) \pm \sqrt{\frac{n \mp j}{n \pm j + 1}} \; \alpha_{n+1,j}^N \; {}_NY_{n+1,j}(\xi),$$

$$-\sqrt{1-t^2} \; {}_{N\mp 1}Y_{n,j}(\xi) = \pm \sqrt{\frac{n \mp N + 1}{n \pm N}} \; \alpha_{n,j}^N \; {}_NY_{n-1,j}(\xi) - \frac{j\sqrt{(n\pm N)(n\mp N+1)}}{n(n+1)} \; {}_NY_{n,j}(\xi)$$

$$\mp \sqrt{\frac{n \pm N}{n \mp N + 1}}\ \alpha^N_{n+1,j}\ {}_N Y_{n+1,j}(\xi),$$

$$-(1 \pm t)\ e^{i\varphi}\ {}_{N\mp 1}Y_{n,j+1}(\xi) = \sqrt{\frac{(n+j+1)(n \mp N + 1)}{(n-j)(n \pm N)}}\ \alpha^N_{n,j}\ {}_N Y_{n-1,j}(\xi)$$
$$\pm \frac{\sqrt{(n-j)(n+j+1)(n \pm N)(n \mp N + 1)}}{n(n+1)}\ {}_N Y_{n,j}(\xi)$$
$$+ \sqrt{\frac{(n-j)(n \pm N)}{(n+j+1)(n \mp N + 1)}}\ \alpha^N_{n+1,j}\ {}_N Y_{n+1,j}(\xi),$$

$$-(1 \pm t)\ e^{-i\varphi}\ {}_{N\pm 1}Y_{n,j-1}(\xi) = \sqrt{\frac{(n-j+1)(n \pm N + 1)}{(n+j)(n \mp N)}}\ \alpha^N_{n,j}\ {}_N Y_{n-1,j}(\xi)$$
$$\pm \frac{\sqrt{(n+j)(n-j+1)(n \mp N)(n \pm N + 1)}}{n(n+1)}\ {}_N Y_{n,j}(\xi)$$
$$+ \sqrt{\frac{(n+j)(n \mp N)}{(n-j+1)(n \pm N + 1)}}\ \alpha^N_{n+1,j}\ {}_N Y_{n+1,j}(\xi),$$

where ${}_N Y_{-1,j}(\xi) := 0$ *and* ${}_N Y_{n,j}(\xi) := 0$ *for* $n < |N|$ *or* $n < |j|$.

The first relation is also mentioned in [12, 66] and the second in [12].

Lemma 3.8.4. *Another recursion relation is given for* $\xi = \xi(t, \varphi) \in \Omega$, $n \in \mathbb{N}_0$, *and* $N, j \in \mathbb{Z}$ *with* $n \geq |N|$ *and* $n \geq |j|$ *by [97]*

$$ {}_N Y_{n,j}(\xi) = \sqrt{\frac{n \mp N + 1}{n \pm N}}\ \frac{j \mp nt}{n\sqrt{1-t^2}}\ {}_{N\mp 1}Y_{n,j}(\xi)$$
$$\pm \sqrt{\frac{2n+1}{2n-1} \frac{(n \pm N - 1)\,(n^2 - j^2)}{n \pm N}}\ \frac{1}{n\sqrt{1-t^2}}\ {}_{N\mp 1}Y_{n-1,j}(\xi),$$

where ${}_N Y_{-1,j}(\xi) := 0$ *and* ${}_N Y_{n,j}(\xi) := 0$ *for* $n < |N|$ *or* $n < |j|$.

3.9 Relation to the Scalar, Vector, and Tensor Spherical Harmonics

The scalar, vector, and tensor spherical harmonics and the spin-weighted spherical harmonics are related to each other [90]. In this chapter, these relations are to be shown.

From the definition of the spin-weighted spherical harmonics, we know that

$${}_0 Y_{n,j} = Y_{n,j}.$$

Next, we look at the vector spherical harmonics.

Theorem 3.9.1. *The tangential vector spherical harmonics can be combined such that*

$$\pm \frac{1}{\sqrt{2}}\ \left(-y^{(2)}_{n,j}(\xi) \pm i y^{(3)}_{n,j}(\xi)\right) = {}_{\pm 1}Y_{n,j}(\xi)\tau_{\pm},$$

where

$$\tau_{\pm} := -\frac{1}{\sqrt{2}}\ \left(\varepsilon^t \pm i\varepsilon^{\varphi}\right)$$

with $\tau_\pm \cdot \overline{\tau_\pm} = 1$ *and* $\tau_\pm \cdot \overline{\tau_\mp} = 0$ *for* $\xi \in \Omega$.

Proof. The proof is straight forward. For $\xi = \xi(t,\varphi) \in \Omega$, we get

$$\begin{aligned}
&\pm \frac{1}{\sqrt{2}} \left(-y_{n,j}^{(2)}(\xi) \pm i y_{n,j}^{(3)}(\xi)\right) \\
&= \pm \frac{1}{\sqrt{2n(n+1)}} \left(-\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}} \partial_\varphi - \varepsilon^t \sqrt{1-t^2}\partial_t \mp i\varepsilon^\varphi \sqrt{1-t^2}\partial_t \pm i\varepsilon^t \frac{1}{\sqrt{1-t^2}} \partial_\varphi\right) Y_{n,j}(\xi) \\
&= \pm \frac{1}{\sqrt{n(n+1)}} \left[\left(\sqrt{1-t^2}\partial_t - \frac{i}{\sqrt{1-t^2}} (\pm\partial_\varphi)\right) Y_{n,j}(\xi)\right] \left[\frac{-1}{\sqrt{2}} \left(\varepsilon^t \pm i\varepsilon^\varphi\right)\right].
\end{aligned}$$

The proposition follows from (3.2) such that

$$\pm \frac{1}{\sqrt{2}} \left(-y_{n,j}^{(2)}(\xi) \pm i y_{n,j}^{(3)}(\xi)\right) = {}_{\pm1}Y_{n,j}(\xi)\tau_\pm.$$

The orthonormality of the vectors $\tau_\pm$ is obviously given by the orthonormality of the unit sphere vectors ε^t and ε^φ. □

Now, we represent the tensor spherical harmonics by the spin-weighted spherical harmonics.

Corollary 3.9.2. *In the same way, we get for* $\xi \in \Omega$ *for the left normal/right tangential tensor spherical harmonics*

$$\pm \frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(1,2)}(\xi) \pm i \boldsymbol{y}_{n,j}^{(1,3)}(\xi)\right) = {}_{\pm1}Y_{n,j}(\xi) \left(\xi \otimes \tau_\pm\right)$$

and the left tangential/right normal tensor spherical harmonics

$$\pm \frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(2,1)}(\xi) \pm i \boldsymbol{y}_{n,j}^{(3,1)}(\xi)\right) = {}_{\pm1}Y_{n,j}(\xi) \left(\tau_\pm \otimes \xi\right).$$

Theorem 3.9.3. *In analogy, we can also combine for* $\xi \in \Omega$

$$-\frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(2,3)}(\xi) \pm i \boldsymbol{y}_{n,j}^{(3,2)}(\xi)\right) = {}_{\pm2}Y_{n,j}(\xi) \left(\tau_\pm \otimes \tau_\pm\right).$$

Proof. The proof is also straight forward. For $\xi = \xi(t,\varphi) \in \Omega$, we obtain

$$\begin{aligned}
&-\frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(2,3)}(\xi) \pm i \boldsymbol{y}_{n,j}^{(3,2)}(\xi)\right) \\
&= \frac{1}{\sqrt{2}} \left(\boldsymbol{y}_{n,j}^{(2,3)}(\xi) \mp i \boldsymbol{y}_{n,j}^{(3,2)}(\xi)\right) \\
&= \frac{1}{2} \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}} \left(\nabla_\xi^* \otimes \nabla_\xi^* Y_{n,j}(\xi) - L_\xi^* \otimes L_\xi^* Y_{n,j}(\xi) + 2\nabla_\xi^* Y_{n,j}(\xi) \otimes \xi\right. \\
&\quad \left.\mp i \nabla_\xi^* \otimes L_\xi^* Y_{n,j}(\xi) \mp i L_\xi^* \otimes \nabla_\xi^* Y_{n,j}(\xi) \mp 2i L_\xi^* Y_{n,j}(\xi) \otimes \xi\right) \\
&= \frac{1}{2} \frac{1}{\sqrt{n(n+1)(n(n+1)-2)}} \left[(\varepsilon^\varphi \otimes \partial_\varphi \varepsilon^\varphi) \left(\frac{1}{1-t^2} \partial_\varphi \pm i\partial_t\right) Y_{n,j}(\xi)\right. \\
&\quad + (\varepsilon^\varphi \otimes \varepsilon^\varphi) \left(\frac{1}{1-t^2} \partial_\varphi^2 - t\partial_t - \left(1-t^2\right)\partial_t^2 + t\partial_t \pm i\partial_\varphi\partial_t \pm i\frac{t}{1-t^2} \partial_\varphi \pm i\partial_t\partial_\varphi\right. \\
&\qquad \left.\pm i\frac{t}{1-t^2} \partial_\varphi\right) Y_{n,j}(\xi)
\end{aligned}$$

$$+\left(\varepsilon^\varphi \otimes \varepsilon^t\right)\Bigg(\partial_\varphi\partial_t + \partial_t\partial_\varphi + \underbrace{\frac{t}{1-t^2}\ \partial_\varphi \mp i\frac{1}{1-t^2}}_{=\frac{2t}{1-t^2}\ \partial_\varphi - \frac{t}{1-t^2}\ \partial_\varphi}\ \partial_\varphi^2 \pm i\left(1-t^2\right)\partial_t^2 \mp it\partial_t\Bigg)Y_{n,j}(\xi)$$

$$+\left(\varepsilon^t \otimes \varepsilon^\varphi\right)\Bigg(\partial_t\partial_\varphi + \frac{t}{1-t^2}\ \partial_\varphi + \partial_\varphi\partial_t + \frac{t}{1-t^2}\ \partial_\varphi \pm i\left(1-t^2\right)\partial_t^2 \mp it\partial_t \mp i\frac{1}{1-t^2}\ \partial_\varphi^2$$

$$\pm\, it\partial_t\Bigg)Y_{n,j}(\xi)$$

$$+\left(\varepsilon^t \otimes \xi\right)\left(-\sqrt{1-t^2}\partial_t + 2\sqrt{1-t^2}\partial_t \pm i\frac{1}{\sqrt{1-t^2}}\ \partial_\varphi \mp 2i\frac{1}{\sqrt{1-t^2}}\ \partial_\varphi\right)Y_{n,j}(\xi)$$

$$+\left(\varepsilon^t \otimes \varepsilon^t\right)\Bigg(\left(1-t^2\right)\partial_t^2 - t\partial_t - \frac{1}{1-t^2}\ \partial_\varphi^2 \mp i\partial_t\partial_\varphi \quad \underbrace{\mp i\frac{t}{1-t^2}\ \partial_\varphi}_{\mp i\frac{2t}{1-t^2}\ \partial_\varphi \pm i\frac{t}{1-t^2}\ \partial_\varphi} \quad \mp i\partial_\varphi\partial_t\Bigg)Y_{n,j}(\xi)$$

$$+\left(\varepsilon^\varphi \otimes \xi\right)\left(-\frac{1}{\sqrt{1-t^2}}\ \partial_\varphi + \frac{2}{\sqrt{1-t^2}}\ \partial_\varphi \mp i\sqrt{1-t^2}\partial_t \pm 2i\sqrt{1-t^2}\partial_t\right)Y_{n,j}(\xi)$$

$$+\left(\varepsilon^t \otimes \partial_\varphi\varepsilon^\varphi\right)\left(\partial_t \mp i\frac{1}{1-t^2}\ \partial_\varphi\right)Y_{n,j}(\xi)\Bigg]$$

$$=\frac{1}{2}\frac{1}{\sqrt{n(n+1)(n(n+1)-2)}}\ \Bigg[\left(\left(1-t^2\right)\partial_t^2 - \frac{1}{1-t^2}\ \partial_\varphi^2 \mp i\left(\partial_t\partial_\varphi + \partial_\varphi\partial_t + \frac{2t}{1-t^2}\ \partial_\varphi\right)\right)$$

$$\times Y_{n,j}(\xi)\left(-\varepsilon^\varphi\otimes\varepsilon^\varphi + \varepsilon^t\otimes\varepsilon^t \pm i\left(\varepsilon^t\otimes\varepsilon^\varphi + \varepsilon^\varphi\otimes\varepsilon^t\right)\right) + \left(\frac{1}{1-t^2}\ \partial_\varphi \pm i\partial_t\right)Y_{n,j}(\xi)$$

$$\times\left(\varepsilon^\varphi\otimes\partial_\varphi\varepsilon^\varphi \mp i\varepsilon^t\otimes\partial_\varphi\varepsilon^\varphi\right) - t\left(\frac{1}{1-t^2}\ \partial_\varphi \pm i\partial_t\right)Y_{n,j}(\xi)\left(\varepsilon^\varphi\otimes\varepsilon^t \mp i\varepsilon^t\otimes\varepsilon^t\right)$$

$$+\sqrt{1-t^2}\left(\frac{1}{1-t^2}\ \partial_\varphi \pm i\partial_t\right)Y_{n,j}(\xi)\left(\varepsilon^\varphi\otimes\xi \mp i\varepsilon^t\otimes\xi\right)\Bigg]$$

$$=\frac{1}{\sqrt{n(n+1)(n(n+1)-2)}}\ \Bigg[\left(\left(1-t^2\right)\partial_t^2 - \frac{1}{1-t^2}\ \partial_\varphi^2 \mp i\left(\partial_t\partial_\varphi + \partial_\varphi\partial_t + \frac{2t}{1-t^2}\ \partial_\varphi\right)\right)$$

$$\times Y_{n,j}(\xi)\left(\tau_\pm\otimes\tau_\pm\right) + \frac{1}{2}\ \left(\frac{1}{1-t^2}\ \partial_\varphi \pm i\partial_t\right)Y_{n,j}(\xi)\left(\varepsilon^\varphi \mp i\varepsilon^t\right)\otimes\left(\partial_\varphi\varepsilon^\varphi - t\varepsilon^t + \sqrt{1-t^2}\xi\right)\Bigg].$$

With

$$\partial_\varphi\varepsilon^\varphi - t\varepsilon^t + \sqrt{1-t^2}\xi = \begin{pmatrix}-\cos(\varphi)\\ -\sin(\varphi)\\ 0\end{pmatrix} - t\begin{pmatrix}-t\cos(\varphi)\\ -t\sin(\varphi)\\ \sqrt{1-t^2}\end{pmatrix} + \sqrt{1-t^2}\begin{pmatrix}\sqrt{1-t^2}\cos(\varphi)\\ \sqrt{1-t^2}\sin(\varphi)\\ t\end{pmatrix} = 0,$$

we obtain

$$-\frac{1}{\sqrt{2}}\ \left(-\boldsymbol{y}_{n,j}^{(2,3)}(\xi) \pm i\boldsymbol{y}_{n,j}^{(3,2)}(\xi)\right)$$

$$=\frac{1}{\sqrt{n(n+1)(n(n+1)-2)}}\ \Bigg[\left(\left(1-t^2\right)\partial_t^2 - \frac{1}{1-t^2}\ \partial_\varphi^2 \mp 2i\left(\partial_\varphi\partial_t + \frac{t}{1-t^2}\ \partial_\varphi\right)\right)$$

$$\times\ Y_{n,j}(\xi)\left(\tau_\pm\otimes\tau_\pm\right)\Bigg]$$

and with (3.3)

$$-\frac{1}{\sqrt{2}}\ \left(-\boldsymbol{y}_{n,j}^{(2,3)}(\xi) \pm i\boldsymbol{y}_{n,j}^{(3,2)}(\xi)\right) = {}_{\pm 2}Y_{n,j}(\xi)\left(\tau_\pm\otimes\tau_\pm\right).$$

□

Chapter 4

Scalar Slepian Functions on the Sphere

Before we construct the tensor Slepian functions, where we use the properties of the spin-weighted spherical harmonics, an explaination of the method of spherical Slepian functions is advised. Therefore, we start with the already known Slepian functions for scalar fields on the sphere. Here, we follow mainly [78, 82].

It is well known that functions cannot have a temporal (or spatial) and spectral finite support at the same time. However, often we want to find and represent signals that are not only timelimited but also optimally concentrated in the spectral domain. This concentration problem was solved by Slepian, Landau, and Pollak in the early 1960s for Euclidean domains like the real line [52, 84, 85]. In this thesis, we do not deal with this case and refer instead to [37, 78].
Indeed, in geosciences we get measurement data and models on the planetary surface or on parts of it. On the unit sphere, the concentration problem was solved by Albertella, Sansò, and Sneeuw [2] and by Simons, Dahlen, and Wieczorek [78, 82]. The obtained basis functions are called the (scalar) spherical Slepian functions. The first investigations are done in [37].

As we have stated before twice, the scalar Slepian functions on the sphere have a huge field of applications. Especially in physical, computational, and biomedical fields like in geodesy and geophysics, gravimetry, geodynamics, cosmology, planetary science, biomedical science, and in computer science (see [43, 67] and the references therein).

Slepian functions form an orthogonal family of functions. They are defined on a domain (e.g. a region on the Earth's surface), in which they are either optimally concentrated or within which they are exactly limited and simultaneously exactly limited by a certain bandwidth or maximally concentrated. In this chapter, we want to find the bandlimited scalar functions that are optimally concentrated within a region of interest. We will discuss the case of spacelimited functions, which are spatially concentrated by a bandlimit in Chapter 7.

An important fact of the Slepian functions is that they are orthonormal on the entire sphere and orthogonal on the concentration domain.

We start with a basic notation.

Definition 4.0.1. *With $L \in \mathbb{N}_0$, we denote the bandlimit. This means that L is the maximal degree of a bandlimited function $F \in \mathrm{Harm}_{0\ldots L}(\Omega)$ respectively of a bandlimited vector field*

$f \in \mathrm{harm}_{0\ldots L}(\Omega)$ *or a bandlimited tensor field* $\boldsymbol{f} \in \mathbf{harm}_{0\ldots L}(\Omega)$.

We choose the measurable set $R \subset \Omega$ *as the region of interest.*

4.1 Derivation

In this section, we want to construct the scalar Slepian functions on the sphere. For this purpose, we formulate the problem as concentration problem, eigenvalue problem, and integral equation.

Let $R \subset \Omega$ be the region of interest. Then, every by L bandlimited function $F \in \mathrm{L}^2(\Omega)$, shortly $F \in \mathrm{Harm}_{0\ldots L}(\Omega)$, can be written in the basis of the fully normalized spherical harmonics. This means that

$$F = \sum_{n=0}^{L} \sum_{j=-n}^{n} F_{n,j} Y_{n,j},$$

where $F_{n,j} = \langle F, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)}$ and $F_{n,j} = 0$ for all $L < n \leq \infty$ and all $-n \leq j \leq n$.

So, there are $(L+1)^2$ coefficients $F_{n,j}$ and basis functions $Y_{n,j}$, $n = 0, \ldots, L$, $j = -n, \ldots, n$.

We want to find the by L bandlimited functions F that are optimally concentrated within the region R. The optimally concentrated signal is the one with the least energy outside the region.

Problem 4.1.1. *We maximize the concentration ratio and get the concentration problem*

$$\lambda = \frac{\int_R F(\xi)\overline{F(\xi)}\, \mathrm{d}\omega(\xi)}{\int_\Omega F(\xi)\overline{F(\xi)}\, \mathrm{d}\omega(\xi)} = \max.$$

With

$$\int_R F(\xi)\overline{F(\xi)}\, \mathrm{d}\omega(\xi) = \sum_{n=0}^{L} \sum_{j=-n}^{n} \sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} F_{n,j}\overline{F_{n',j'}} \underbrace{\int_R Y_{n,j}(\xi)\overline{Y_{n',j'}(\xi)}\, \mathrm{d}\omega(\xi)}_{=:\overline{K_{nj,n'j'}}=K_{n'j',nj}}$$

and

$$\int_\Omega F(\xi)\overline{F(\xi)}\, \mathrm{d}\omega(\xi) = \sum_{n=0}^{L} \sum_{j=-n}^{n} |F_{n,j}|^2,$$

we obtain the formulation

$$\lambda = \frac{\bar{G}^{\mathrm{T}} K G}{\bar{G}^{\mathrm{T}} G},$$

where $G := (F_{00}, \ldots, F_{LL})^{\mathrm{T}}$ and $K := \begin{pmatrix} K_{00,00} & \ldots & K_{00,LL} \\ \vdots & \ddots & \vdots \\ K_{LL,00} & \ldots & K_{LL,LL} \end{pmatrix}$.

Then, we can conclude directly from Problem 4.1.1 the following eigenvalue problem.

Problem 4.1.2. *We have to solve the eigenvalue problem*

$$KG = \lambda G.$$

So, we consider

$$\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} K_{nj,n'j'}F_{n',j'} = \lambda F_{n,j} \tag{4.1}$$

for all $n = 0, \ldots, L$ and all $j = -n, \ldots, n$, where the kernel matrix is given by

$$K_{nj,n'j'} := \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi)\ \mathrm{d}\omega(\xi) =: K^0_{nj,n'j'}.$$

We choose the notation such that it is unique for the general case with spin-weighted spherical harmonics (compare Chapter 8).

The kernel matrix is Hermitian and positive definite and the eigenvalues are real (see Lemma 8.1.4). However, the kernel matrix is also ill-conditioned. So, the solution of the eigenvalue problem is numerically unstable. For special regions, we can solve the eigenvalue problem by replacement by a commuting eigenvalue problem. For this purpose, we have to reformulate equation (4.1).
Upon multiplying by $Y_{n,j}(\eta)$, $\eta \in \Omega$, and summing over all $n = 0, \ldots, L$ and $j = -n, \ldots, n$, we get

$$\sum_{n=0}^{L}\sum_{j=-n}^{n} Y_{n,j}(\eta) \sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi)\ \mathrm{d}\omega(\xi)\ F_{n',j'} = \lambda \sum_{n=0}^{L}\sum_{j=-n}^{n} Y_{n,j}(\eta) F_{n,j}.$$

By interchanging summation and integration, this leads to

$$\int_R \sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\eta) \sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} F_{n',j'} Y_{n',j'}(\xi)\ \mathrm{d}\omega(\xi) = \lambda \sum_{n=0}^{L}\sum_{j=-n}^{n} F_{n,j} Y_{n,j}(\eta).$$

With these considerations, we can reformulate the eigenvalue problem of Problem 4.1.2.

Problem 4.1.3. *We obtain a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel, this means that*

$$\int_R \mathscr{K}(\xi,\eta) F(\xi)\ \mathrm{d}\omega(\xi) = \lambda F(\eta),$$

where

$$\begin{aligned}\mathscr{K}(\xi,\eta) &:= \sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\eta)\\ &= \sum_{n=0}^{L} \frac{2n+1}{4\pi}\ P_n(\xi\cdot\eta)\\ &=: \mathscr{K}^0(\xi,\eta)\end{aligned}$$

and $\xi, \eta \in \Omega$.

Definition 4.1.4. *We know from Problem 4.1.2 that G are the eigenvectors of K. Furthermore, we know that the matrix K is Hermitian and positive definite. Therefore, we get an infinite number of solutions of the eigenvalue problem, which can be described by $(L+1)^2$ pairwise linearly independent solutions. These so-called eigenvectors can be chosen such that they form an orthogonal basis and hence, an orthonormal basis of* $\mathrm{Harm}_{0\ldots L}(\Omega)$ *with help of* *the Gram-Schmidt algorithm. Further, we sort them by decreasing eigenvalues. This means that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{(L+1)^2}$. Then, we denote these sorted orthonormal eigenvectors of the eigenvalue problem (4.1) by $G_1^0, \ldots, G_{(L+1)^2}^0$ and the sorted eigenvalues by $\lambda_1^0, \ldots \lambda_{(L+1)^2}^0$.*

With $\mathscr{G}_1^0, \ldots, \mathscr{G}_{(L+1)^2}^0$, we denote the sorted orthonormal eigenfunctions of the integral equation from Problem 4.1.3. The spherical harmonic coefficients of the eigenfunctions are also the components of the sorted orthonormal eigenvectors. We denote them by $(G_{n,j}^0)_1, \ldots,$ $(G_{n,j}^0)_{(L+1)^2}$. The eigenfunctions are the scalar Slepian functions of spin weight zero on the sphere and we get for this Slepian basis the same number of basis functions like for the basis of spherical harmonics.

To determine the eigenfunctions $\mathscr{G}_\alpha^0$ and eigenvalues λ_α^0 of the integral equation with $\alpha = 1, \ldots, (L+1)^2$, we can find a commuting differential operator $\mathscr{I}_\xi^0$ to the kernel function $\mathscr{K}^0$, for example, for the spherical cap (a circularly symmetric cap of colatitudinal radius ϑ) as region of interest. This operator is given by [37, 78]

$$\mathscr{I}_\xi^0 = (b-t)\Delta_\xi^* + \left(t^2-1\right)\partial_t - L(L+2)t,$$

where $\xi = \xi(\varphi, t) \in \Omega$, $b = \cos\vartheta \leq t \leq 1$. Then, the eigenfunctions can be obtained by

$$\mathscr{I}_\xi^0 \mathscr{G}_\alpha^0(\xi) = \chi_\alpha \mathscr{G}_\alpha^0(\xi),$$

where χ_α and λ_α^0 are not necessarily equal. This means that the kernel function and its commuting differential operator have the same eigenfunctions but need not to have the same eigenvalues. In Chapter 8, we go into details. For arbitrary regions, we have to solve the eigenvalue problem with the ill-conditioned kernel matrix from equation (4.1) numerically (compare Chapter 8.1).

Summary 4.1.5. *In summary, we start with the concentration problem*

$$\lambda = \frac{\int_R \mathscr{G}(\xi)\overline{\mathscr{G}(\xi)}\ \mathrm{d}\omega(\xi)}{\int_\Omega \mathscr{G}(\xi)\overline{\mathscr{G}(\xi)}\ \mathrm{d}\omega(\xi)} = \max.$$

From this problem, we can follow that the scalar Slepian functions are the solution of the eigenvalue problem

$$KG = \lambda G$$

and of the integral equation

$$\int_R \mathscr{K}(\xi,\eta)\mathscr{G}(\xi)\ \mathrm{d}\omega(\xi) = \lambda\mathscr{G}(\eta) \tag{4.2}$$

for all $\eta \in \Omega$. The kernel matrix is given by

$$K_{nj,n'j'} := \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi)\ \mathrm{d}\omega(\xi) =: K_{nj,n'j'}^0.$$

and the kernel function by

$$\mathscr{K}(\xi,\eta) = \sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\eta).$$

The scalar Slepian functions can be written in the basis of the spherical harmonics by

$$\mathscr{G}(\xi) = \sum_{n=0}^{L} \sum_{j=-n}^{n} G_{n,j} Y_{n,j}(\xi) \tag{4.3}$$

for all $\xi \in \Omega$ *with*

$$G_{n,j} = \langle \mathscr{G}, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)}.$$

Furthermore, we denote the scalar Slepian functions by $\mathscr{G}_\alpha := \mathscr{G}_\alpha^0$ *for all* $\alpha = 1, \ldots, (L+1)^2$. *The scalar eigenvectors, the solution of the eigenvalue problem*

$$\sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} K_{nj,n'j'} G_{n',j'} = \lambda G_{n,j} \tag{4.4}$$

for all $n = 0, \ldots, L$ *and* $j = -n, \ldots, n$, *we denote by* $G_\alpha := G_\alpha^0$ *and the eigenvalues by* $\lambda_\alpha := \lambda_\alpha^0$.

4.2 Properties

Now that we know how to calculate the eigenvalues, eigenvectors, and eigenfunctions of the concentration problem, thus the scalar Slepian functions on the sphere. We choose them to be sorted by the eigenvalues like $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{(L+1)^2}$.

Here, we want to show the needed properties of the scalar Slepian functions.

Theorem 4.2.1. *An important property of the Slepian functions is that the eigenvectors and eigenfunctions are orthonormal on the unit sphere and orthogonal on the region of interest R [82]. This means that*

$$\sum_{n=0}^{L} \sum_{j=-n}^{n} \left(G_{n,j}\right)_\alpha \overline{\left(G_{n,j}\right)_\beta} = \delta_{\alpha,\beta}, \tag{4.5}$$

$$\sum_{n=0}^{L} \sum_{j=-n}^{n} \sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} \left(G_{n,j}\right)_\alpha \overline{K_{nj,n'j'}} \overline{\left(G_{n',j'}\right)_\beta} = \lambda_\alpha \delta_{\alpha,\beta}, \tag{4.6}$$

$$\langle \mathscr{G}_\alpha, \mathscr{G}_\beta \rangle_{\mathrm{L}^2(\Omega)} = \delta_{\alpha,\beta}, \tag{4.7}$$

$$\langle \mathscr{G}_\alpha, \mathscr{G}_\beta \rangle_{\mathrm{L}^2(R)} = \lambda_\alpha \delta_{\alpha,\beta} \tag{4.8}$$

for all $\alpha, \beta = 1, \ldots, (L+1)^2$.

Proof. This proof is straight forward.

- The left-hand side of the first property is the Euclidean scalar product of two vectors, so we get

$$\sum_{n=0}^{L} \sum_{j=-n}^{n} \left(G_{n,j}\right)_\alpha \overline{\left(G_{n,j}\right)_\beta} = \overline{G_\beta}^{\mathrm{T}} G_\alpha.$$

  Because we choose them to be the sorted orthonormal eigenvectors of the eigenvalue problem, we get (4.5).

- Then, the second property, which is property (4.6), can be proven with the help of (4.5) by

$$\begin{aligned}\sum_{n=0}^{L}\sum_{j=-n}^{n}\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'}(G_{n,j})_\alpha\ \overline{K_{nj,n'j'}}\ \overline{(G_{n',j'})_\beta} &= \sum_{n=0}^{L}\sum_{j=-n}^{n}(G_{n,j})_\alpha\ \overline{\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'}K_{nj,n'j'}\,(G_{n',j'})_\beta}\\ &= \sum_{n=0}^{L}\sum_{j=-n}^{n}(G_{n,j})_\alpha\ \underbrace{\lambda_\beta}_{\in\mathbb{R}}\ \overline{(G_{n,j})_\beta}\\ &= \lambda_\beta\sum_{n=0}^{L}\sum_{j=-n}^{n}(G_{n,j})_\alpha\ \overline{(G_{n,j})_\beta}\\ &= \lambda_\beta\delta_{\alpha,\beta}\\ &= \lambda_\alpha\delta_{\alpha,\beta},\end{aligned}$$

  where we use that the eigenvalues λ_β, $\beta = 1,\ldots,(L+1)^2$ are real (see Lemma 8.1.4).

- The third property, which is property (4.7), can also be proven with the help of Theorem 3.6.14, the Parseval identity for spin-weighted spherical harmonics, and of (4.5) by

$$\begin{aligned}\langle\mathscr{G}_\alpha,\mathscr{G}_\beta\rangle_{\mathrm{L}^2(\Omega)} &= \int_\Omega \mathscr{G}_\alpha(\xi)\overline{\mathscr{G}_\beta(\xi)}\ \mathrm{d}\omega(\xi)\\ &= \sum_{n=0}^{L}\sum_{j=-n}^{n}(G_{n,j})_\alpha\ \overline{(G_{n,j})_\beta}\\ &= \delta_{\alpha,\beta}.\end{aligned}$$

  So, we see that the scalar Slepian functions form a orthonormal system and hence, with the comparison of the dimension they also form an orthonormal basis of $\mathrm{Harm}_{0\ldots L}(\Omega)$.

- Analogously, we obtain for the fourth property, which is property (4.8), with the help of (4.3) and (4.6) that

$$\begin{aligned}\langle\mathscr{G}_\alpha,\mathscr{G}_\beta\rangle_{\mathrm{L}^2(R)} &= \int_R \mathscr{G}_\alpha(\xi)\overline{\mathscr{G}_\beta(\xi)}\ \mathrm{d}\omega(\xi)\\ &= \int_R\left(\sum_{n=0}^{L}\sum_{j=-n}^{n}(G_{n,j})_\alpha\ Y_{n,j}(\xi)\right)\left(\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'}\overline{(G_{n',j'})_\beta}\ \overline{Y_{n',j'}(\xi)}\right)\ \mathrm{d}\omega(\xi)\\ &= \sum_{n=0}^{L}\sum_{j=-n}^{n}\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'}(G_{n,j})_\alpha\ \overline{(G_{n',j'})_\beta}\underbrace{\int_R Y_{n,j}(\xi)\overline{Y_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi)}_{=\overline{K_{nj,n'j'}}}\\ &= \lambda_\alpha\delta_{\alpha,\beta}.\end{aligned}$$

□

Theorem 4.2.2. *The scalar Slepian functions* $\{\mathscr{G}_\alpha\}_{\alpha=1,\ldots,(L+1)^2}$ *form a complete orthonormal basis system on* $\mathrm{Harm}_{0\ldots L}(\Omega)$ *[2] and therefore, we can write every* $F\in\mathrm{Harm}_{0\ldots L}(\Omega)$ *in the basis of the spherical harmonics and in the basis of the scalar Slepian functions. This means* *that for* $\xi \in \Omega$

$$F(\xi) = \sum_{n=0}^{L} \sum_{j=-n}^{n} \underbrace{\langle F, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)}}_{=:F_{n,j}} Y_{n,j}(\xi)$$
$$= \sum_{\alpha=1}^{(L+1)^2} \underbrace{\langle F, \mathscr{G}_\alpha \rangle_{\mathrm{L}^2(\Omega)}}_{=:F_\alpha} \mathscr{G}_\alpha(\xi).$$

Proof. We know the orthonormality of the scalar Slepian functions from the previous theorem. Furthermore, we know from Theorem 2.4.35 that the spherical harmonics up to degree L are a complete orthonormal basis of $\left(\mathrm{Harm}_{0\ldots L}(\Omega), \langle \cdot, \cdot \rangle_{\mathrm{L}^2(\Omega)}\right)$, which consists out of $(L+1)^2$ basis functions. Then, the completeness of the scalar Slepian functions follows directly, because of the same number of basis functions. □

Theorem 4.2.3. *We can also write the spherical harmonics up to degree L in the basis of the scalar Slepian functions*

$$Y_{n,j} = \sum_{\alpha=1}^{(L+1)^2} \overline{(G_{n,j})_\alpha} \mathscr{G}_\alpha \tag{4.9}$$

and

$$\sum_{\alpha=1}^{(L+1)^2} (G_{n,j})_\alpha \overline{(G_{n',j'})_\alpha} = \delta_{n,n'} \delta_{j,j'} \tag{4.10}$$

[80].

Proof. We can prove the two properties (4.9) and (4.10) together. Therefore, we look at a function $F \in \mathrm{Harm}_{0\ldots L}(\Omega)$ written in the basis of the scalar Slepian functions by

$$F = \sum_{\alpha=1}^{(L+1)^2} \langle F, \mathscr{G}_\alpha \rangle_{\mathrm{L}^2(\Omega)} \mathscr{G}_\alpha.$$

We use (4.3) and get then F written in the basis of the spherical harmonics by a basis transformation by

$$F = \sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} F_{n',j'} Y_{n',j'},$$

where

$$F_{n',j'} = \sum_{\alpha=1}^{(L+1)^2} \langle F, \mathscr{G}_\alpha \rangle_{\mathrm{L}^2(\Omega)} \langle \mathscr{G}_\alpha, Y_{n',j'} \rangle_{\mathrm{L}^2(\Omega)}.$$

Furthermore, we know already that $F_{n',j'} = \langle F, Y_{n',j'} \rangle_{\mathrm{L}^2(\Omega)}$. With $F = Y_{n,j}$ for all $n = 0, \ldots, L$ and all $j = -n, \ldots, n$, we obtain (4.9) with the help of (4.3) by

$$Y_{n,j} = \sum_{\alpha=1}^{(L+1)^2} \langle Y_{n,j}, \mathscr{G}_\alpha \rangle_{\mathrm{L}^2(\Omega)} \mathscr{G}_\alpha$$
$$= \sum_{\alpha=1}^{(L+1)^2} \overline{\langle \mathscr{G}_\alpha, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)}} \mathscr{G}_\alpha$$

$$= \sum_{\alpha=1}^{(L+1)^2} \overline{(G_{n,j})_\alpha} \mathscr{G}_\alpha.$$

Moreover, we get on the one hand that

$$F_{n',j'} = \langle Y_{n',j'}, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)} = \delta_{n,n'}\delta_{j,j'}$$

and on the other hand that

$$\begin{aligned} F_{n',j'} &= \sum_{\alpha=1}^{(L+1)^2} \langle Y_{n,j}, \mathscr{G}_\alpha \rangle_{\mathrm{L}^2(\Omega)} \langle \mathscr{G}_\alpha, Y_{n',j'} \rangle_{\mathrm{L}^2(\Omega)} \\ &= \sum_{\alpha=1}^{(L+1)^2} \overline{\langle \mathscr{G}_\alpha, Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)}} \langle \mathscr{G}_\alpha, Y_{n',j'} \rangle_{\mathrm{L}^2(\Omega)} \\ &= \sum_{\alpha=1}^{(L+1)^2} \overline{(G_{n,j})_\alpha} \left(G_{n',j'}\right)_\alpha . \end{aligned}$$

Together, this yields (4.10).

Note that the proposition of this theorem is equivalent to the principal axes transformation. □

Theorem 4.2.4. *The scalar Slepian functions also fulfill the following properties [11]*

$$\sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha (G_{n,j})_\alpha \overline{(G_{n',j'})_\alpha} = K_{nj,n'j'}, \tag{4.11}$$

$$\sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha \mathscr{G}_\alpha(\xi) \overline{\mathscr{G}_\alpha(\eta)} = \sum_{n=0}^{L} \sum_{j=-n}^{n} \sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} Y_{n,j}(\xi) K_{nj,n'j'} \overline{Y_{n',j'}(\eta)} \tag{4.12}$$

for all $n, n' = 0, \ldots, L$, *all* $j = -n, \ldots, n$ *,all* $j' = -n', \ldots, n'$, *and all* $\xi, \eta \in \Omega$ *and [31]*

$$\mathscr{K}(\xi, \eta) = \sum_{\alpha=1}^{(L+1)^2} \overline{\mathscr{G}_\alpha(\xi)} \mathscr{G}_\alpha(\eta).$$

Proof. This proof is straight forward.

- We obtain the first property with (4.1) and with (4.10) by

$$\begin{aligned} \sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha (G_{n,j})_\alpha \overline{(G_{n',j'})_\alpha} &= \sum_{\alpha=1}^{(L+1)^2} \sum_{l=0}^{L} \sum_{m=-l}^{l} K_{nj,lm} (G_{l,m})_\alpha \overline{(G_{n',j'})_\alpha} \\ &= \sum_{l=0}^{L} \sum_{m=-l}^{l} K_{nj,lm} \underbrace{\sum_{\alpha=1}^{(L+1)^2} (G_{l,m})_\alpha \overline{(G_{n',j'})_\alpha}}_{=\delta_{l,n'}\delta_{m,j'}} \\ &= K_{nj,n'j'}. \end{aligned}$$

- For the second property, we need (4.3) and (4.11). Then, we get

$$\begin{aligned}\sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha \mathscr{G}_\alpha(\xi)\overline{\mathscr{G}_\alpha(\eta)} &= \sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha \sum_{n=0}^{L}\sum_{j=-n}^{n} (G_{n,j})_\alpha Y_{n,j}(\xi) \sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} \overline{(G_{n',j'})_\alpha Y_{n',j'}(\eta)} \\ &= \sum_{n=0}^{L}\sum_{j=-n}^{n}\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} \underbrace{\left(\sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha (G_{n,j})_\alpha \overline{(G_{n',j'})_\alpha}\right)}_{=K_{nj,n'j'}} Y_{n,j}(\xi)\overline{Y_{n',j'}(\eta)} \\ &= \sum_{n=0}^{L}\sum_{j=-n}^{n}\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} Y_{n,j}(\xi) K_{nj,n'j'} \overline{Y_{n',j'}(\eta)}.\end{aligned}$$

- We obtain the third property with (4.3) and with (4.10) by

$$\begin{aligned}\sum_{\alpha=1}^{(L+1)^2} \overline{\mathscr{G}_\alpha(\xi)}\mathscr{G}_\alpha(\eta) &= \sum_{\alpha=1}^{(L+1)^2} \sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{(G_{n,j})_\alpha Y_{n,j}(\xi)} \sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} (G_{n',j'})_\alpha Y_{n',j'}(\eta) \\ &= \sum_{n=0}^{L}\sum_{j=-n}^{n}\sum_{n'=0}^{L}\sum_{j'=-n'}^{n'} \underbrace{\left(\sum_{\alpha=1}^{(L+1)^2} (G_{n',j'})_\alpha \overline{(G_{n,j})_\alpha}\right)}_{=\delta_{n,n'}\delta_{j,j'}} \overline{Y_{n,j}(\xi)} Y_{n',j'}(\eta) \\ &= \sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n',j'}(\eta) \\ &= \mathscr{K}(\xi,\eta).\end{aligned}$$

□

Another important property of the Slepian functions, shown in numerical experiments, is that there are often only eigenvalues $\lambda \approx 1$ and $\lambda \approx 0$. So, we can conclude on the one hand that the matrix K is ill-conditioned. On the other hand, we can differentiate between functions that are well-concentrated within the region of interest and functions of lower concentration. For this purpose, we explain the definition of the Shannon number in the next section.

4.3 Shannon Number

The Shannon number, given by

$$S = \sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha = \mathrm{tr}(K),$$

is usually a good estimate for the number of significant eigenvalues, the eigenvalues $\lambda \approx 1$. Therefore, S gives (approximately) the dimension of the space of signals that are bandlimited by L and optimally concentrated in R at the same time. This space has as basis the eigenfunctions $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_S$.

Lemma 4.3.1. *The Shannon number of the scalar Slepian functions on the sphere is given* *by*

$$S = (L+1)^2 \, \frac{A}{4\pi},$$

where A denotes the area of the region R on the unit sphere Ω.

Proof. With Corollary 3.4.25 from the addition theorem and with Remark 3.2.2, we get that

$$\begin{aligned} S &= \sum_{\alpha=1}^{(L+1)^2} \lambda_\alpha = \operatorname{tr}(K) \\ &= \sum_{n=0}^{L} \sum_{j=-n}^{n} K_{nj,nj} \\ &= \int_R \sum_{n=0}^{L} \sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\xi) \mathrm{d}\omega(\xi) \\ &= \frac{1}{4\pi} \sum_{n=0}^{L} (2n+1) \int_R \mathrm{d}\omega(\xi) \\ &= (L+1)^2 \, \frac{A}{4\pi}. \end{aligned}$$

□

It is obvious that for a small area of the region R with respect to the area of the sphere, $A \ll 4\pi$, the number of Slepian functions with insignificant eigenvalues is considerably larger than the number of those with significant eigenvalues ($S \ll (L+1)^2$). The other way around, for an area of the region R nearly covering the whole sphere, $A \approx 4\pi$, there are many more Slepian functions with significant eigenvalues than those with insignificant eigenvalues ($S \approx (L+1)^2$) [11].

Chapter 5

Vector Slepian Functions on the Sphere

After explaining the scalar Slepian functions in the previous chapter, we can now take a look at the vector Slepian functions. They have previously been investigated by [43, 67, 83]. Bandlimited vector fields localized to a spherical cap were first used in biomedical science [43]. In this chapter, the derivation of the vector Slepian functions follows mainly [43], whereas the properties and the Shannon number follow mainly [67].

5.1 Derivation

For the construction of the vector Slepian functions on the sphere, we need the spin-weighted spherical harmonics and their properties from Chapter 3 to solve the concentration problem. Then, we can also formulate the problem not only as concentration problem, but also as eigenvalue problem and as integral equation. We solve this problem separately for the type $i = 1, 2, 3$ of the vector spherical harmonics in parts.

Let $R \subset \Omega$ be the region of interest. Then, every by L bandlimited vector field $f \in \mathrm{l}^2(\Omega)$, shortly $f \in \mathrm{harm}_{0\ldots L}(\Omega)$, can be written in the basis of the vector spherical harmonics of Hill from Definition 2.5.1, this means that

$$f = f_{\mathrm{nor}} + f_{\mathrm{tan}} = \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} f_{n,j}^{(i)} y_{n,j}^{(i)},$$

where

$$\begin{aligned} f_{\mathrm{nor}} &= \sum_{n=0}^{L} \sum_{j=-n}^{n} f_{n,j}^{(1)} y_{n,j}^{(1)}, \\ f_{\mathrm{tan}} &= \sum_{n=1}^{L} \sum_{j=-n}^{n} \left(f_{n,j}^{(2)} y_{n,j}^{(2)} + f_{n,j}^{(3)} y_{n,j}^{(3)} \right), \\ f_{n,j}^{(i)} &= \int_{\Omega} f(\xi) \cdot \overline{y_{n,j}^{(i)}(\xi)} \, \mathrm{d}\omega(\xi), \\ 0_i &= \begin{cases} 0, & i = 1 \\ 1, & i = 2, 3 \end{cases}. \end{aligned}$$

Furthermore,

$$f_{n,j}^{(i)} = 0$$

for all $L < n \leq \infty$, all $-n \leq j \leq n$, and all $i = 1, 2, 3$.

So, there are $(L+1)^2$ coefficients $f_{n,j}^{(1)}$, $n = 0, \ldots, L$, $j = -n, \ldots, n$, $(L+1)^2 - 1$ coefficients $f_{n,j}^{(2)}$, $n = 1, \ldots, L$, $j = -n, \ldots, n$, and $(L+1)^2 - 1$ coefficients $f_{n,j}^{(3)}$, $n = 1, \ldots, L$, $j = -n, \ldots, n$. All in all, these are $3(L+1)^2 - 2$ coefficients and basis functions.

Problem 5.1.1. *The concentration problem is given by*

$$\lambda = \frac{\int_R f(\xi) \cdot \overline{f(\xi)} \, \mathrm{d}\omega(\xi)}{\int_\Omega f(\xi) \cdot \overline{f(\xi)} \, \mathrm{d}\omega(\xi)} = \max .$$

With

$$\int_R f(\xi) \cdot \overline{f(\xi)} \, \mathrm{d}\omega(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \sum_{i'=1}^{3} \sum_{n'=0_{i'}}^{L} \sum_{j'=-n'}^{n'} f_{n,j}^{(i)} \overline{f_{n',j'}^{(i')}} \underbrace{\int_R y_{n,j}^{(i)}(\xi) \cdot \overline{y_{n',j'}^{(i')}(\xi)} \, \mathrm{d}\omega(\xi)}_{=:\overline{k_{nj,n'j'}^{ii'}} = k_{n'j',nj}^{i'i}}$$

and

$$\int_\Omega f(\xi) \cdot \overline{f(\xi)} \, \mathrm{d}\omega(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \left| f_{n,j}^{(i)} \right|^2 ,$$

we obtain the formulation

$$\lambda = \frac{\bar{g}^{\mathrm{T}} k g}{\bar{g}^{\mathrm{T}} g},$$

where $g := (f_{00}^{(1)}, \ldots, f_{LL}^{(1)}, f_{1,-1}^{(2)}, \ldots, f_{LL}^{(2)}, f_{1,-1}^{(3)}, \ldots, f_{LL}^{(3)})^{\mathrm{T}}$,

$$k := \begin{pmatrix} k^{(11)} & 0 & 0 \\ 0 & k^{(22)} & k^{(23)} \\ 0 & k^{(32)} & k^{(33)} \end{pmatrix} = \begin{pmatrix} k^{\mathrm{nor}} & 0 \\ 0 & k^{\mathrm{tan}} \end{pmatrix}, \qquad k^{(32)} = \overline{\left(k^{(23)}\right)}^{\mathrm{T}},$$

$$k^{(ii')} := \begin{pmatrix} k_{0_i,-0_i,0_{i'},-0_{i'}}^{(ii')} & \cdots & k_{0_i,-0_i,LL}^{(ii')} \\ \vdots & \ddots & \vdots \\ k_{LL,0_{i'},-0_{i'}}^{(ii')} & \cdots & k_{LL,LL}^{(ii')} \end{pmatrix},$$

and

$$k_{nj,n'j'}^{(ii')} := \int_R \overline{y_{n,j}^{(i)}(\xi)} \cdot y_{n',j'}^{(i')}(\xi) \, \mathrm{d}\omega(\xi)$$

for all $i, i' = 1, 2, 3$, all $n = 0_i, \ldots, L$, all $n' = 0_{i'}, \ldots, L$, all $j = -n, \ldots, n$ and all $j' = -n', \ldots, n'$.

We gain this structure of k, because f_{nor} and f_{tan} are pointwise orthogonal in the Euclidean sense. So, the normal and the tangential part are independent from each other. Therefore, we consider them separately.

5.1.1 The Normal Part

First, we look at the normal part of the vector field f_{nor}. Then, we get

$$\begin{aligned} k^{\mathrm{nor}}_{nj,n'j'} &= \int_R \overline{y^{(1)}_{n,j}(\xi)} \cdot y^{(1)}_{n',j'}(\xi) \; \mathrm{d}\omega(\xi) \\ &= \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi) \, (\xi \cdot \xi) \; \mathrm{d}\omega(\xi) \\ &= \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi) \; \mathrm{d}\omega(\xi) \\ &=: K^0_{nj,n'j'} \end{aligned}$$

for all $n, n' = 0, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$.

Problem 5.1.2. *This is equal to the scalar case. So, we get the eigenvalue problem*

$$k^{\mathrm{nor}} g_{\mathrm{nor}} = \lambda g_{\mathrm{nor}}$$

and particularly

$$\sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} k^{\mathrm{nor}}_{nj,n'j'} f^{(1)}_{n',j'} = \lambda f^{(1)}_{n,j}, \tag{5.1}$$

where $g_{\mathrm{nor}} = \left(f^{(1)}_{00}, \ldots, f^{(1)}_{LL} \right)^{\mathrm{T}}$ *and for all* $n = 0, \ldots, L$ *and all* $j = -n, \ldots, n$.

We know already that $k^{\mathrm{nor}} = K^0$ is Hermitian, positive definite, supposed to be ill-conditioned, and its eigenvalues are real. Therefore, we continue, like in the previous chapter, for special regions. So, upon multiplying by $Y_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $n = 0, \ldots, L$ and $j = -n, \ldots, n$, and interchanging summation and integration, we obtain again a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel.

Problem 5.1.3. *The integral equation is given by*

$$\int_R \mathscr{k}^{\mathrm{nor}}(\xi, \eta) F(\xi) \; \mathrm{d}\omega(\xi) = \lambda F(\eta),$$

where

$$\begin{aligned} \mathscr{k}^{\mathrm{nor}}(\xi, \eta) &:= \sum_{n=0}^{L} \sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\eta) \\ &= \sum_{n=0}^{L} \frac{2n+1}{4\pi} \; P_n(\xi \cdot \eta) \\ &=: \mathscr{K}^0(\xi, \eta) \end{aligned}$$

and

$$F(\xi) := \sum_{n=0}^{L} \sum_{j=-n}^{n} f^{(1)}_{n,j} Y_{n,j}(\xi)$$

with $\xi, \eta \in \Omega$.

Note that the vector problem is now reduced to a scalar eigenvalue problem (5.1). Here, we see directly that

$$f_{\mathrm{nor}}(\xi) = \xi F(\xi).$$

This scalar problem is equal to the problem for the scalar Slepian function from Chapter 4. So, we know already an orthonormal basis of solutions $\mathscr{G}_1^0, \ldots, \mathscr{G}_{(L+1)^2}^0$ of the integral equation sorted with respect to decreasing eigenvalues. So, we obtain the corresponding orthonormal eigenfunctions $(g_{\mathrm{nor}})_\alpha$ and eigenvalues $(\lambda_{\mathrm{nor}})_\alpha$ for $\alpha = 1, \ldots, (L+1)^2$ of the normal part of the vector field by

$$(g_{\mathrm{nor}})_\alpha(\xi) = \xi \mathscr{G}_\alpha^0(\xi)$$

and

$$(\lambda_{\mathrm{nor}})_\alpha = \lambda_\alpha^0$$

for $\alpha = 1, \ldots, (L+1)^2$ and all $\xi \in \Omega$.

5.1.2 The Tangential Part

The tangential part of the vector field f_{tan} is more difficult. In general, we construct a new basis for the tangential vector space, because

$$y_{n,j}^{(2)}(\xi) \cdot \overline{y_{n',j'}^{(3)}(\xi)} \neq 0 \qquad \text{and} \qquad y_{n,j}^{(3)}(\xi) \cdot \overline{y_{n',j'}^{(2)}(\xi)} \neq 0$$

for $\xi \in \Omega$.

Definition 5.1.4. *We use the results of Chapter 3.9 and define*

$$y_{n,j}^{\pm}(\xi) := \pm\frac{1}{\sqrt{2}} \left(-y_{n,j}^{(2)}(\xi) \pm i y_{n,j}^{(3)}(\xi) \right) = {}_{\pm 1}Y_{n,j}(\xi) \tau_{\pm}$$

and write

$$f_{\mathrm{tan}}(\xi) = \sum_{n=1}^{L} \sum_{j=-n}^{n} \left(f_{n,j}^{+} y_{n,j}^{+}(\xi) + f_{n,j}^{-} y_{n,j}^{-}(\xi) \right),$$

where $\tau_\pm \cdot \overline{\tau_\pm} = 1$ and $\tau_\pm \cdot \overline{\tau_\mp} = 0$.

Then, $y_{n,j}^{\pm}(\xi) \cdot \overline{y_{n',j'}^{\mp}(\xi)} = 0$ and

$$k^{\mathrm{tan}} = \begin{pmatrix} k^+ & 0 \\ 0 & k^- \end{pmatrix},$$

where

$$k^{\pm} := \begin{pmatrix} k_{1,-1,1,-1}^{\pm} & \ldots & k_{1,-1,LL}^{\pm} \\ \vdots & \ddots & \vdots \\ k_{LL,1,-1}^{\pm} & \ldots & k_{LL,LL}^{\pm} \end{pmatrix}$$

and

$$\begin{aligned} k_{nj,n'j'}^{\pm} &:= \int_R \overline{y_{n,j}^{\pm}(\xi)} \cdot y_{n',j'}^{\pm}(\xi) \, \mathrm{d}\omega(\xi) \\ &= \int_R \overline{{}_{\pm 1}Y_{n,j}(\xi)} \, {}_{\pm 1}Y_{n',j'}(\xi) \, \mathrm{d}\omega(\xi) \\ &=: K_{nj,n'j'}^{\pm 1}. \end{aligned}$$

We choose the notation such that it is unique for the general case with spin-weighted spherical harmonics (compare Chapter 8).

Problem 5.1.5. *So, we get the eigenvalue problems*

$$k^{\pm} g^{\pm}_{\tan} = \lambda g^{\pm}_{\tan}$$

and particularly

$$\sum_{n'=1}^{L} \sum_{j'=-n'}^{n'} k^{\pm}_{nj,n'j'} f^{\pm}_{n',j'} = \lambda f^{\pm}_{n,j}, \tag{5.2}$$

where $g^{\pm}_{\tan} = \left(f^{\pm}_{00}, \ldots, f^{\pm}_{LL}\right)^{\mathrm{T}}$.

This kernel matrix $k^{\pm} = K^{\pm 1}$ is also Hermitian, positive definite, and the eigenvalues are real (see Lemma 8.1.4). However, the kernel matrix is also ill-conditioned. As a consequence, the solution of the eigenvalue problem is numerically unstable. For special regions, we can solve the eigenvalue problem by replacing it with a commuting eigenvalue problem. Therefore, we have to reformulate equation (5.2).

Upon multiplying by ${}_{\pm 1}Y_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $n = 1, \ldots, L$ and $j = -n, \ldots, n$, and interchanging summation and integration, we obtain a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel.

Problem 5.1.6. *This integral equation is given by*

$$\int_R \mathscr{k}^{\pm}(\xi, \eta) F^{\pm}(\xi) \, \mathrm{d}\omega(\xi) = \lambda F^{\pm}(\eta),$$

where

$$\mathscr{k}^{\pm}(\xi, \eta) := \sum_{n=1}^{L} \sum_{j=-n}^{n} \overline{{}_{\pm 1}Y_{n,j}(\xi)} \; {}_{\pm 1}Y_{n,j}(\eta) =: \mathscr{K}^{\pm 1}(\xi, \eta)$$

and

$$F^{\pm}(\xi) := \sum_{n=1}^{L} \sum_{j=-n}^{n} f^{\pm}_{n,j} \; {}_{\pm 1}Y_{n,j}(\xi)$$

with $\xi, \eta \in \Omega$.

The vector problem is reduced to a scalar eigenvalue problem (5.2). Here, we see directly that

$$f_{\tan}(\xi) = \tau_{+} F^{+}(\xi) + \tau_{-} F^{-}(\xi).$$

Definition 5.1.7. *We know from Problem 5.1.5 that* $g^{\pm}_{\tan}$ *are the eigenvectors of* $k^{\pm}$. *Furthermore, we know that the matrix* $k^{\pm} = K^{\pm 1}$ *is Hermitian and positive definite. Therefore, we get an infinite number of solutions of the eigenvalue problem, which can be described by* $(L+1)^2 - 1$ *pairwise linearly independent solutions. These eigenvectors can be chosen such that they form an orthogonal basis and hence, an orthonormal basis of* $\mathrm{harm}^{\pm}_{1\ldots L}(\Omega)$ *with help of the Gram-Schmidt algorithm, where*

$$\mathrm{harm}^{\pm}_{1\ldots L}(\Omega) := \mathrm{span}\left\{ y^{\pm}_{n,j} \right\}_{n=1,\ldots,L, j=-n,\ldots,n}.$$

Further, we sort them by decreasing eigenvalues. Then, we denote these sorted orthonormal eigenvectors of the eigenvalue problem (5.2) by $G^{\pm 1}_{1}, \ldots, G^{\pm 1}_{(L+1)^2-1}$ *and the sorted eigenvalues by* $\lambda^{\pm 1}_{1}, \ldots, \lambda^{\pm 1}_{(L+1)^2-1}$.

With $\mathscr{G}^{\pm 1}_{1}, \ldots, \mathscr{G}^{\pm 1}_{(L+1)^2-1}$, *we denote the sorted orthonormal eigenfunctions of the integral equation from Problem 5.1.6. The spin-weighted spherical harmonic coefficients of the eigenfunctions are also the components of the sorted orthonormal eigenvectors. We denote them* *by* $(G^{\pm 1}_{n,j})_1, \ldots, (G^{\pm 1}_{n,j})_{(L+1)^2-1}$. *The eigenfunctions are the Slepian functions of spin weight* ± 1 *on the sphere. We get for this spin-weighted Slepian basis the same number of basis functions like for the basis of the type* 2 *respectively* 3 *vector spherical harmonics of Hill.*

To determine the eigenfunctions $\mathscr{G}^{\pm 1}_\alpha$ and eigenvalues $\lambda^{\pm 1}_\alpha$ of the integral equation with $\alpha = 1, \ldots, (L+1)^2 - 1$, we can also find a commuting differential operator $\mathscr{I}^{\pm 1}_\xi$ to the kernel function $\mathscr{K}^{\pm 1}$ for the case of the sphere. For example, for the spherical cap (a circularly symmetric cap of colatitudinal radius ϑ), this operator is given by [43]. As we see here, this can be put into the more general context of the spin-weighted spherical harmonics by

$$\mathscr{I}^{\pm 1}_\xi = (b-t)\Delta^{\pm 1}_\xi + \left(t^2-1\right)\partial_t - L(L+2)t,$$

where $\xi = \xi(\varphi, t) \in \Omega$, $b = \cos\vartheta \le t \le 1$,

$$\Delta^{\pm 1}_\xi = \Delta^*_\xi - \frac{1 - 2it\,(\pm\partial_\varphi)}{1-t^2}$$

and we know from Corollary 3.3.7 that

$$\Delta^{\pm 1}_\xi \, {}_{\pm 1}Y_{n,j}(\xi) = -n(n+1)\, {}_{\pm 1}Y_{n,j}(\xi).$$

Then, the eigenfunctions of spin weight ± 1 can be obtained by

$$\mathscr{I}^{\pm 1}_\xi \mathscr{G}^{\pm 1}_\alpha(\xi) = \chi_\alpha \mathscr{G}^{\pm 1}_\alpha(\xi),$$

where χ_α and $\lambda^{\pm 1}_\alpha$ are not necessarily equal. This means that the kernel function and its commuting differential operator have the same eigenfunctions but need not to have the same eigenvalues. So, we obtain the eigenfunctions $\left(\mathscr{g}^{\pm}_{\tan}\right)_\alpha$ and eigenvalues $(\lambda_{\tan^\pm})_\alpha$ for $\alpha = 1, \ldots, (L+1)^2 - 1$ of the tangential part of the vector field by

$$\left(\mathscr{g}^{\pm}_{\tan}\right)_\alpha(\xi) = \tau_\pm \mathscr{G}^{\pm 1}_\alpha(\xi)$$

and

$$(\lambda_{\tan^\pm})_\alpha = \lambda^{\pm 1}_\alpha$$

for $\alpha = 1, \ldots, (L+1)^2 - 1$. In Chapter 8, we go into details. For arbitrary regions, we have to solve the eigenvalue problem with the ill-conditioned kernel matrix from equation (5.2) numerically (compare Chapter 8.1).

5.1.3 Complete Vector Solution

By combining and summarizing the results from the previous sections, we obtain the complete vector solution as follows.

Altogether, we have to solve the concentration problem, Problem 5.1.1,

$$\lambda = \frac{\int_R \mathscr{g}(\xi) \cdot \overline{\mathscr{g}(\xi)}\, d\omega(\xi)}{\int_\Omega \mathscr{g}(\xi) \cdot \overline{\mathscr{g}(\xi)}\, d\omega(\xi)} = \max,$$

with

$$\mathscr{g}(\xi) = \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} g^i_{n,j} y^i_{n,j,}(\xi) \tag{5.3}$$

for all $\xi \in \Omega$ and

$$g^i_{n,j} = \int_\Omega g\!\!\!/(\xi) \cdot \overline{y^i_{n,j}(\xi)} \, \mathrm{d}\omega(\xi)$$

for all $(i, n, j) \in J$, where

$$0_i = \begin{cases} 0 & , i = 1 \\ 1 & , i = 2, 3 \end{cases} .$$

Here, we now use the following notations.

Definition 5.1.8. *We define the set of indices*

$$J := \{(i, n, j) \mid i = 1, 2, 3; n = 0_i, \ldots, L; j = -n, \ldots, n\} .$$

Definition 5.1.9. *Now, we use as basis the transformed vector spherical harmonics given by*

$$\begin{aligned} y^1_{n,j}(\xi) &:= y^{(1)}_{n,j}(\xi) = \xi Y_{n,j}(\xi), \\ y^2_{n,j}(\xi) &:= y^{+}_{n,j}(\xi) = \tau_{+} \ {}_{+1}Y_{n,j}(\xi), \\ y^3_{n,j}(\xi) &:= y^{-}_{n,j}(\xi) = \tau_{-} \ {}_{-1}Y_{n,j}(\xi) \end{aligned}$$

for $\xi \in \Omega$.

Note that this denotes a different basis than the vector spherical harmonics of Hill $y^{(i)}_{n,j}$, $i = 1, 2, 3$. These new basis functions are also orthonormal on the sphere with respect to $\mathrm{L}^2(\Omega)$.

Lemma 5.1.10. *The transformed vector spherical harmonics are orthonormal on the sphere. This means that*

$$\int_\Omega y^i_{n,j}(\xi) \cdot \overline{y^{i'}_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi) = \delta_{i,i'} \delta_{n,n'} \delta_{j,j'} \tag{5.4}$$

for all (i, n, j) *and* $(i', n', j') \in J$.

Proof. With $\xi \cdot \xi = 1 = \tau_\pm \cdot \overline{\tau_\pm}$ and $\xi \cdot \overline{\tau_\pm} = 0 = \tau_\pm \cdot \overline{\tau_\mp}$, we obtain that

$$\int_\Omega y^i_{n,j}(\xi) \cdot \overline{y^{i'}_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi) = \delta_{i,i'} \int_\Omega {}_{N_i}Y_{n,j}(\xi) \ \overline{{}_{N_i}Y_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi),$$

where

$$N_i := \begin{cases} 0 & , i = 1 \\ +1 & , i = 2 \\ -1 & , i = 3 \end{cases} .$$

With Theorem 3.4.21, we get directly that

$$\int_\Omega y^i_{n,j}(\xi) \cdot \overline{y^{i'}_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi) = \delta_{i,i'} \delta_{n,n'} \delta_{j,j'}$$

for all (i, n, j) and $(i', n', j') \in J$. □

Theorem 5.1.11 (Completeness and Parseval Identity for the Transformed Vector Spherical Harmonics)**.** *The transformed vector spherical harmonics form a complete orthonormal basis of* $\mathrm{l}^2(\Omega)$*. Consequently, for* $f \in \mathrm{l}^2(\Omega)$*, we obtain*

$$\lim_{L \to \infty} \left\| f - \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \langle f, y^i_{n,j} \rangle_{\mathrm{l}^2(\Omega)} \, y^i_{n,j} \right\|_{\mathrm{l}^2(\Omega)} = 0.$$

This means that every function $f \in \mathrm{l}^2(\Omega)$ can be written uniquely in the $\mathrm{l}^2(\Omega)$-sense in terms of a Fourier series by

$$f = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=-n}^{n} \left\langle f, y_{n,j}^i \right\rangle_{\mathrm{l}^2(\Omega)} y_{n,j}^i.$$

Furthermore, for every function $f, g \in \mathrm{l}^2(\Omega)$, the Parseval identity is fulfilled such that

$$\langle f, g\rangle_{\mathrm{l}^2(\Omega)} = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=-n}^{n} \left\langle f, y_{n,j}^i \right\rangle_{\mathrm{l}^2(\Omega)} \overline{\left\langle g, y_{n,j}^i \right\rangle_{\mathrm{l}^2(\Omega)}}.$$

Proof. We know from Theorem 2.5.10 that the vector spherical harmonics of Hill [40] form a complete orthonormal basis. Furthermore, we know already from (5.4) that the transformed vector spherical harmonics of Section 5.1.3 are orthonormal. Now, only the proof for completeness remains. Therefore, we have to show that for every function $f \in \mathrm{l}^2(\Omega)$ with $\left\langle f, y_{n,j}^l \right\rangle_{\mathrm{l}^2(\Omega)} = 0$ for all $l = 1, 2, 3$, all $n \geq 0_l$, and all $j = -n, \ldots, n$, it follows that $f = 0$.

For $l = 1$, we know that $y_{n,j}^1 = y_{n,j}^{(1)}$ for all $n = 0, \ldots, L$ and all $j = -n, \ldots, n$. With the completeness of the vector spherical harmonics of Hill, the proposition is clear.
For $l = 2, 3$, we know that $y_{n,j}^l = \pm\frac{1}{\sqrt{2}}\left(-y_{n,j}^{(2)} \pm i y_{n,j}^{(3)}\right)$, where we use "+" if $l = 2$ and "−" if $l = 3$ for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$. Then, we get for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$

$$0 = \left\langle f, y_{n,j}^l \right\rangle_{\mathrm{l}^2(\Omega)} = \pm\frac{1}{\sqrt{2}}\left(-\left\langle f, y_{n,j}^{(2)} \right\rangle_{\mathrm{l}^2(\Omega)} \pm i \left\langle f, y_{n,j}^{(3)} \right\rangle_{\mathrm{l}^2(\Omega)}\right).$$

This leads to the system of linear equations

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \left\langle f, y_{n,j}^{(2)} \right\rangle_{\mathrm{l}^2(\Omega)} \\ \left\langle f, y_{n,j}^{(3)} \right\rangle_{\mathrm{l}^2(\Omega)} \end{pmatrix} = \begin{pmatrix} \left\langle f, y_{n,j}^2 \right\rangle_{\mathrm{l}^2(\Omega)} \\ \left\langle f, y_{n,j}^3 \right\rangle_{\mathrm{l}^2(\Omega)} \end{pmatrix}$$

for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$, where

$$\det \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = -\frac{i}{2} - \frac{i}{2} = -i \neq 0.$$

Therefore, we obtain

$$\left\langle f, y_{n,j}^{(2)} \right\rangle_{\mathrm{l}^2(\Omega)} = 0$$

and

$$\left\langle f, y_{n,j}^{(3)} \right\rangle_{\mathrm{l}^2(\Omega)} = 0.$$

With the completeness of the vector spherical harmonics of Hill, we know that $f = 0$. So, the transformed vector spherical harmonics are complete and form an orthonormal basis of $\mathrm{l}^2(\Omega)$. With the completeness, the Parseval identity follows directly. □

Problem 5.1.12. *Then, the concentration problem leads to the eigenvalue problem*

$$kg = \lambda g,$$

where

$$k := \begin{pmatrix} k^1 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^3 \end{pmatrix} := \begin{pmatrix} K^0 & 0 & 0 \\ 0 & K^{+1} & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \in \mathbb{R}^{\left[3(L+1)^2-2\right]\times\left[3(L+1)^2-2\right]}$$

and

$$k^{ii}_{nj,n'j'} := k^i_{nj,n'j'} := \int_R \overline{y^i_{n,j}(\xi)} \cdot y^i_{n',j'}(\xi)\ \mathrm{d}\omega(\xi)$$

with $k^{ii'}_{nj,n'j'} = \delta_{i,i'} k^{ii}_{nj,n'j'}$ *for all* (i,n,j) *and* $(i',n',j') \in J$.

Then, we see that

$$\sum_{i'=1}^{3} \sum_{n'=0_i}^{L} \sum_{j'=-n'}^{n'} k^{ii'}_{nj,n'j'} g^{i'}_{n',j'} = \lambda g^i_{n,j} \tag{5.5}$$

for all $(i,n,j) \in J$.

Upon multiplying by $y^i_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $(i,n,j) \in J$, and interchanging summation and integration, we obtain

$$\begin{aligned} &\int_R \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \sum_{i'=1}^{3} \sum_{n'=0_{i'}}^{L} \sum_{j'=-n'}^{n'} y^i_{n,j}(\eta) \left(\overline{y^i_{n,j}(\xi)} \cdot y^{i'}_{n',j'}(\xi) \right) g^{i'}_{n',j'}\ \mathrm{d}\omega(\xi) \\ &= \lambda \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} g^i_{n,j} y^i_{n,j}(\eta). \end{aligned}$$

With Lemma 2.2.8, this leads to

$$\begin{aligned} &\int_R \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \left(y^i_{n,j}(\eta) \otimes \overline{y^i_{n,j}(\xi)} \right) \sum_{i'=1}^{3} \sum_{n'=0_{i'}}^{L} \sum_{j'=-n'}^{n'} g^{i'}_{n',j'} y^{i'}_{n',j'}(\xi)\ \mathrm{d}\omega(\xi) \\ &= \lambda \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} g^i_{n,j} y^i_{n,j}(\eta). \end{aligned}$$

Then, we can reformulate Problem 5.1.12 to the following integral equation.

Problem 5.1.13. *We get the integral equation*

$$\int_R \mathscr{k}(\xi,\eta) \mathscr{g}(\xi)\ \mathrm{d}\omega(\xi) = \lambda \mathscr{g}(\eta) \tag{5.6}$$

for all $\eta \in \Omega$ *with the kernel function*

$$\mathscr{k}(\xi,\eta) = \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} y^i_{n,j}(\eta) \otimes \overline{y^i_{n,j}(\xi)}.$$

Altogether, with our previous results and with $d := (L+1)^2$, the eigenvectors of Problem

5.1.12 are given by

$$g_\alpha = \begin{cases} g_\alpha^1 := \begin{pmatrix} G_\alpha^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & , \alpha = 1, \ldots, d \\ g_\alpha^2 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G_{\alpha-d}^{+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} & , \alpha = d+1, \ldots, 2d-1 \; . \\ g_\alpha^3 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G_{\alpha-(2d-1)}^{-1} \end{pmatrix} & , \alpha = 2d, \ldots, 3d-2 \end{cases}$$

Note that G_α^N are also vectors, where $G_\alpha^0 \in \mathbb{R}^d$ and $G_\alpha^{\pm 1} \in \mathbb{R}^{d-1}$.

With

$$\begin{aligned} \left(g_{\mathrm{nor}}\right)_\alpha(\xi) &= \xi \mathscr{G}_\alpha^0(\xi), \\ \left(g_{\tan}^{\pm}\right)_\alpha(\xi) &= \tau_\pm \mathscr{G}_\alpha^{\pm 1}(\xi), \\ \tau_\pm &= -\frac{1}{\sqrt{2}}\left(\varepsilon^t \pm i\varepsilon^\varphi\right), \end{aligned}$$

the vector Slepian functions are given by

$$g_\alpha(\xi) = \begin{cases} g_\alpha^1(\xi) := \xi \mathscr{G}_\alpha^0(\xi) & , \alpha = 1, \ldots, d \\ g_\alpha^2(\xi) := \tau_+ \mathscr{G}_{\alpha-d}^{+1}(\xi) & , \alpha = d+1, \ldots, 2d-1 \\ g_\alpha^3(\xi) := \tau_- \mathscr{G}_{\alpha-(2d-1)}^{-1}(\xi) & , \alpha = 2d, \ldots, 3d-2 \end{cases}$$

and the associated eigenvalues by

$$\lambda_\alpha = \begin{cases} \lambda_\alpha^0 & , \alpha = 1, \ldots, d \\ \lambda_{\alpha-d}^{+1} & , \alpha = d+1, \ldots, 2d-1 \\ \lambda_{\alpha-(2d-1)}^{-1} & , \alpha = 2d, \ldots, 3d-2 \end{cases}$$

for $\alpha = 1, \ldots, 3(L+1)^2 - 2$.

5.2 Properties

Now, we know how to calculate the eigenvalues, eigenvectors and eigenfunctions of the concentration problem, thus the vector Slepian functions on the sphere. Now, we look at the properties of the vector Slepian functions. We can show the same theorems as for the scalar

Slepian functions.

Theorem 5.2.1. *The vector Slepian functions and their eigenvectors are orthonormal on the unit sphere and orthogonal on the region of interest R. This means that*

$$\sum_{i=1}^{3}\sum_{n=0_i}^{L}\sum_{j=-n}^{n}\left(g_{n,j}^{i}\right)_{\alpha}\overline{\left(g_{n,j}^{i}\right)_{\beta}}=\delta_{\alpha,\beta}, \tag{5.7}$$

$$\sum_{i=1}^{3}\sum_{n=0_i}^{L}\sum_{j=-n}^{n}\sum_{i'=1}^{3}\sum_{n'=0_{i'}}^{L}\sum_{j'=-n'}^{n'}\left(g_{n,j}^{i}\right)_{\alpha}\overline{k_{nj,n'j'}^{ii'}}\,\overline{\left(g_{n',j'}^{i'}\right)_{\beta}}=\lambda_\alpha\delta_{\alpha,\beta}, \tag{5.8}$$

$$\langle g_\alpha, g_\beta\rangle_{\mathrm{l}^2(\Omega)}=\delta_{\alpha,\beta}, \tag{5.9}$$

$$\langle g_\alpha, g_\beta\rangle_{\mathrm{l}^2(R)}=\lambda_\alpha\delta_{\alpha,\beta} \tag{5.10}$$

for all $\alpha,\beta=1,\ldots,3(L+1)^2-2$ *and all* $(i,n,j),(i',n',j')\in J$.

Theorem 5.2.2. *The vector Slepian functions* $\{g_\alpha\}_{\alpha=1,\ldots,3(L+1)^2-2}$ *form a complete orthonormal basis system of* $\left(\mathrm{harm}_{0\ldots L}(\Omega),\langle\cdot,\cdot\rangle_{\mathrm{l}^2(\Omega)}\right)$ *and therefore, we can write every by L bandlimited vector field* $f\in\mathrm{l}^2(\Omega)$ *in the basis of the transformed vector spherical harmonics and in the basis of the vector Slepian functions. This means that for* $\xi\in\Omega$

$$\begin{aligned} f(\xi) &= \sum_{i=1}^{3}\sum_{n=0_i}^{L}\sum_{j=-n}^{n}\underbrace{\langle f, y_{n,j}^{i}\rangle_{\mathrm{l}^2(\Omega)}}_{=:f_{n,j}^{i}}\, y_{n,j}^{i}(\xi)\\ &= \sum_{\alpha=1}^{3(L+1)^2-2}\underbrace{\langle f, g_\alpha\rangle_{\mathrm{l}^2(\Omega)}}_{=:f_\alpha}\, g_\alpha(\xi). \end{aligned}$$

Theorem 5.2.3. *We can also write the transformed vector spherical harmonics in the basis of the vector Slepian functions*

$$y_{n,j}^{i}=\sum_{\alpha=1}^{3(L+1)^2-2}\overline{\left(g_{n,j}^{i}\right)_{\alpha}}\,g_\alpha, \tag{5.11}$$

$$\sum_{\alpha=1}^{3(L+1)^2-2}\left(g_{n,j}^{i}\right)_{\alpha}\overline{\left(g_{n',j'}^{i'}\right)_{\alpha}}=\delta_{i,i'}\delta_{n,n'}\delta_{j,j'}. \tag{5.12}$$

Theorem 5.2.4. *The vector Slepian functions also fulfill the following properties*

$$\sum_{\alpha=1}^{3(L+1)^2-2}\lambda_\alpha\left(g_{n,j}^{i}\right)_{\alpha}\overline{\left(g_{n',j'}^{i'}\right)_{\alpha}}=k_{nj,n'j'}^{ii'},$$

$$\sum_{\alpha=1}^{3(L+1)^2-2}\lambda_\alpha g_\alpha(\xi)\overline{g_\alpha(\eta)}=\sum_{i=1}^{3}\sum_{n=0_i}^{L}\sum_{j=-n}^{n}\sum_{i'=1}^{3}\sum_{n'=0_{i'}}^{L}\sum_{j'=-n'}^{n'}y_{n,j}^{i}(\xi)k_{nj,n'j'}^{ii'}\overline{y_{n',j'}^{i'}(\eta)}$$

for all $(i,n,j),(i',n',j')\in J$, *all* $\xi,\eta\in\Omega$, *and*

$$k(\xi,\eta)=\sum_{\alpha=1}^{3(L+1)^2-2}\overline{g_\alpha(\xi)}\,g_\alpha(\eta).$$

The proofs are analogous to the proofs of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, and Theorem 4.2.4. The only two differences occur firstly in the additional indices $i, i' = 1, 2, 3$, over which we summate and secondly in the starting point of the summation of n and n'. For the completeness of the vector Slepian functions, we need the completeness of the transformed vector spherical harmonics from Theorem 5.1.11. Then, we get the properties simultaneously.

We choose the vector Slepian functions to be sorted by the eigenvalues like $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{3(L+1)^2-2}$ just as before, for the scalar Slepian functions. Here, we also get from numerical experiments that there are often only eigenvalues $\lambda \approx 1$ and $\lambda \approx 0$. Therefore, we calculate the Shannon number for the vector case in the following section.

5.3 Shannon Number

For the vector Slepian functions on the sphere, the Shannon number S is a good estimate for significant eigenvalues, the eigenvalues $\lambda \approx 1$. Therefore, S gives (approximately) the dimension of the space of signals that are bandlimited by L and optimally concentrated in R at the same time. This space has as basis the eigenfunctions $g_1, g_2, \ldots, g_S$.

Lemma 5.3.1. *The Shannon number of the vector Slepian functions on the sphere is given by*

$$S = \left(3(L+1)^2 - 2\right) \frac{A}{4\pi},$$

where A denotes the area of the region R on the unit sphere Ω.

Proof. With Corollary 3.4.25 and with Remark 3.2.2, we get that

$$\begin{aligned}
S &= \sum_{\alpha=1}^{3(L+1)^2-2} \lambda_\alpha = \operatorname{tr}(k) \\
&= \sum_{n=0}^{L} \sum_{j=-n}^{n} K^{0}_{nj,nj} + \sum_{n=1}^{L} \sum_{j=-n}^{n} \left(K^{+1}_{nj,nj} + K^{-1}_{nj,nj}\right) \\
&= \int_R \left(\sum_{n=0}^{L} \sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\xi) + \sum_{n=1}^{L} \sum_{j=-n}^{n} \left(\overline{{}_{+1}Y_{n,j}(\xi)}\ {}_{+1}Y_{n,j}(\xi) + \overline{{}_{-1}Y_{n,j}(\xi)}\ {}_{-1}Y_{n,j}(\xi) \right) \right) \mathrm{d}\omega(\xi) \\
&= \frac{1}{4\pi} \left(\sum_{n=0}^{L} (2n+1) + 2 \sum_{n=1}^{L} (2n+1) \right) \int_R \mathrm{d}\omega(\xi) \\
&= \left(3(L+1)^2 - 2\right) \frac{A}{4\pi}.
\end{aligned}$$

□

Like for the scalar case, it is obvious that, for $A \ll 4\pi$, we get $S \ll 3(L+1)^2 - 2$ and, for $A \approx 4\pi$, we get $S \approx 3(L+1)^2 - 2$.

Chapter 6

Tensor Slepian Functions on the Sphere

In this chapter, we develop the new tensor Slepian functions on the sphere. A different ansatz has already been proposed by [22], but for the basis of the tensor spherical harmonics of Martinec [56]. As an alternative, we construct them for the basis of the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27], which we can transform to a spin-weighted basis system. This enables us to use the properties of spin-weighted spherical harmonics and solve the concentration problem separately for the different types of the tensor spherical harmonics in parts. Furthermore, this facilitates us to construct a commuting operator for special regions. This is useful for the application of the tensor Slepian functions. For example, the satellite mission GOCE delivers tensor data from satellite gravity gradiometry. Here, the components of the Hessian tensor of the gravitational potential are measured [21, 30, 32]. Furthermore, the cosmic microwave background (CMB) polarization is given as tensor data on the spherical cap. Therefore, the extension of the method of Slepian functions for tensor fields is useful for the local analysis of GOCE data and for the analysis of the CMB polarization (see Chapter 9).

6.1 Derivation

For the construction of the tensor Slepian functions on the sphere, we need the spin-weighted spherical harmonics and their properties from Chapter 3 to solve the concentration problem. In that case, we can also formulate the problem not only as concentration problem, but also as eigenvalue problem and integral equation. Then, we can solve the problem separately from the type of the tensor spherical harmonics. As we will see that for a bandlimit L, this offers us the advantage to solve five eigenvalue problems for smaller matrices (the matrix sizes are $[(L+1)^2 - N^2] \times [(L+1)^2 - N^2]$) instead of one eigenvalue problem for a matrix of size $[9(L+1)^2 - 12] \times [9(L+1)^2 - 12]$. The five problems in our calculation are given by $N = 0, \pm 1, \pm 2$. In addition, it has the advantage that we obtain the Slepian functions separated by type. This enables us to look at problems like for the CMB polarization (see Chapter 9).

Let $R \subset \Omega$ be the region of interest. Then, every by L bandlimited tensor field $\boldsymbol{f} \in \mathbf{l}^2(\Omega)$, shortly $\boldsymbol{f} \in \mathbf{harm}_{0\ldots L}(\Omega)$, can be written in the basis of the tensor spherical harmonics of

Freeden, Gervens, and Schreiner from Definition 2.6.2. This means that

$$\boldsymbol{f} = \boldsymbol{f}_{\text{nor,nor}} + \boldsymbol{f}_{\text{nor,tan}} + \boldsymbol{f}_{\text{tan,nor}} + \boldsymbol{f}_{\text{tan,tan}} = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{L} \sum_{j=-n}^{n} \boldsymbol{f}_{n,j}^{(i,k)} \boldsymbol{y}_{n,j}^{(i,k)},$$

where

$$\begin{aligned}
\boldsymbol{f}_{\text{nor,nor}} &= \sum_{n=0}^{L} \sum_{j=-n}^{n} \boldsymbol{f}_{n,j}^{(1,1)} \boldsymbol{y}_{n,j}^{(1,1)}, \\
\boldsymbol{f}_{\text{nor,tan}} &= \sum_{n=1}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{f}_{n,j}^{(1,2)} \boldsymbol{y}_{n,j}^{(1,2)} + \boldsymbol{f}_{n,j}^{(1,3)} \boldsymbol{y}_{n,j}^{(1,3)} \right), \\
\boldsymbol{f}_{\text{tan,nor}} &= \sum_{n=1}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{f}_{n,j}^{(2,1)} \boldsymbol{y}_{n,j}^{(2,1)} + \boldsymbol{f}_{n,j}^{(3,1)} \boldsymbol{y}_{n,j}^{(3,1)} \right), \\
\boldsymbol{f}_{\text{tan,tan}} &= \sum_{n=0_{ik}}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{f}_{n,j}^{(2,2)} \boldsymbol{y}_{n,j}^{(2,2)} + \boldsymbol{f}_{n,j}^{(2,3)} \boldsymbol{y}_{n,j}^{(2,3)} + \boldsymbol{f}_{n,j}^{(3,2)} \boldsymbol{y}_{n,j}^{(3,2)} + \boldsymbol{f}_{n,j}^{(3,3)} \boldsymbol{y}_{n,j}^{(3,3)} \right), \\
\boldsymbol{f}_{n,j}^{(i,k)} &= \int_{\Omega} \boldsymbol{f}(\xi) : \overline{\boldsymbol{y}_{n,j}^{(i,k)}(\xi)} \, \mathrm{d}\omega(\xi), \\
0_{ik} &= \begin{cases} 0, & (i,k) = (1,1),(2,2),(3,3) \\ 1, & (i,k) = (1,2),(1,3),(2,1),(3,1) \\ 2, & (i,k) = (2,3),(3,2) \end{cases}.
\end{aligned}$$

Furthermore,

$$\boldsymbol{f}_{n,j}^{(i,k)} = 0$$

for all $L < n \leq \infty$, all $-n \leq j \leq n$, and all $i,k = 1,2,3$.

So, there are $(L+1)^2$ coefficients $\boldsymbol{f}_{n,j}^{(i,k)}$, $n = 0,\ldots,L$, $j = -n,\ldots,n$ for $(i,k) = (1,1),(2,2),(3,3)$, $(L+1)^2-1$ coefficients $\boldsymbol{f}_{n,j}^{(i,k)}$, $n = 1,\ldots,L$, $j = -n,\ldots,n$ for $(i,k) = (1,2),(1,3),(2,1),(3,1)$, and $(L+1)^2-4$ coefficients $\boldsymbol{f}_{n,j}^{(i,k)}$, $n = 2,\ldots,L$, $j = -n,\ldots,n$ for $(i,k) = (2,3),(3,2)$. All in all, these are $9(L+1)^2 - 4\cdot 1 - 2\cdot 4 = 9(L+1)^2 - 12$ coefficients and basis functions.

Problem 6.1.1. *The concentration problem is given by*

$$\lambda = \frac{\int_R \boldsymbol{f}(\xi) : \overline{\boldsymbol{f}(\xi)} \, \mathrm{d}\omega(\xi)}{\int_\Omega \boldsymbol{f}(\xi) : \overline{\boldsymbol{f}(\xi)} \, \mathrm{d}\omega(\xi)} = \max.$$

With

$$\begin{aligned}
\int_R \boldsymbol{f}(\xi) : \overline{\boldsymbol{f}(\xi)} \, \mathrm{d}\omega(\xi) &= \sum_{i,k,l,m=1}^{3} \sum_{n=0_{ik}}^{L} \sum_{j=-n}^{n} \sum_{n'=0_{i'k'}}^{L} \sum_{j'=-n'}^{n'} \boldsymbol{f}_{n,j}^{(i,k)} \overline{\boldsymbol{f}_{n',j'}^{(i',k')}} \\
&\quad \times \underbrace{\int_R \boldsymbol{y}_{n,j}^{(i,k)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(i',k')}(\xi)} \, \mathrm{d}\omega(\xi)}_{=:\overline{\boldsymbol{k}_{nj,n'j'}^{ik,i'k'}} = \boldsymbol{k}_{n'j',nj}^{i'k',ik}}, \\
\int_\Omega \boldsymbol{f}(\xi) : \overline{\boldsymbol{f}(\xi)} \, \mathrm{d}\omega(\xi) &= \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{L} \sum_{j=-n}^{n} \left| \boldsymbol{f}_{n,j}^{(i,k)} \right|^2,
\end{aligned}$$

we obtain the formulation

$$\lambda = \frac{\overline{\boldsymbol{g}}^{\mathrm{T}} \boldsymbol{k} \boldsymbol{g}}{\overline{\boldsymbol{g}}^{\mathrm{T}} \boldsymbol{g}},$$

where $\boldsymbol{g} := (\boldsymbol{f}_{00}^{(1,1)}, \dots, \boldsymbol{f}_{LL}^{(1,1)}, \boldsymbol{f}_{1,-1}^{(1,2)}, \dots, \boldsymbol{f}_{LL}^{(1,2)}, \dots, \boldsymbol{f}_{00}^{(3,3)}, \dots, \boldsymbol{f}_{LL}^{(3,3)})^{\mathrm{T}}$,

$$\boldsymbol{k} := \begin{pmatrix} \boldsymbol{k}^{(11,11)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{k}^{(12,12)} & \boldsymbol{k}^{(12,13)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{k}^{(13,12)} & \boldsymbol{k}^{(13,13)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{k}^{(21,21)} & \boldsymbol{k}^{(21,31)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{k}^{(31,21)} & \boldsymbol{k}^{(31,31)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^{(22,22)} & \boldsymbol{k}^{(22,23)} & \boldsymbol{k}^{(22,32)} & \boldsymbol{k}^{(22,33)} \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^{(23,22)} & \boldsymbol{k}^{(23,23)} & \boldsymbol{k}^{(23,32)} & \boldsymbol{k}^{(23,33)} \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^{(32,22)} & \boldsymbol{k}^{(32,23)} & \boldsymbol{k}^{(32,32)} & \boldsymbol{k}^{(32,33)} \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^{(33,22)} & \boldsymbol{k}^{(33,23)} & \boldsymbol{k}^{(33,32)} & \boldsymbol{k}^{(33,33)} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{k}^{\mathrm{nor,nor}} & 0 & 0 & 0 \\ 0 & \boldsymbol{k}^{\mathrm{nor,tan}} & 0 & 0 \\ 0 & 0 & \boldsymbol{k}^{\mathrm{tan,nor}} & 0 \\ 0 & 0 & 0 & \boldsymbol{k}^{\mathrm{tan,tan}} \end{pmatrix},$$

$$\boldsymbol{k}^{(ik,i'k')} = \overline{\left(\boldsymbol{k}^{(i'k',ik)}\right)}^{\mathrm{T}}, \qquad \boldsymbol{k}^{(ik,i'k')} := \begin{pmatrix} \boldsymbol{k}^{(ik,i'k')}_{\mathbf{0}_{ik},-\mathbf{0}_{ik},\mathbf{0}_{i'k'},-\mathbf{0}_{i'k'}} & \dots & \boldsymbol{k}^{(ik,i'k')}_{\mathbf{0}_{ik},-\mathbf{0}_{ik},LL} \\ \vdots & \ddots & \vdots \\ \boldsymbol{k}^{(ik,i'k')}_{LL,\mathbf{0}_{i'k'},-\mathbf{0}_{i'k'}} & \dots & \boldsymbol{k}^{(ik,i'k')}_{LL,LL} \end{pmatrix},$$

and

$$\boldsymbol{k}^{(ik,i'k')}_{nj,n'j'} := \int_R \overline{\boldsymbol{y}^{(i,k)}_{n,j}(\xi)} : \boldsymbol{y}^{(i',k')}_{n',j'}(\xi) \; \mathrm{d}\omega(\xi)$$

for all $i, k, i', k' = 1, 2, 3$, all $n = 0_{ik}, \dots, L$, all $n' = 0_{i'k'}, \dots, L$, all $j = -n, \dots, n$, and all $j' = -n', \dots, n'$.

We gain the structure of the matrix $\boldsymbol{k}$, because $\boldsymbol{f}_{\mathrm{nor,nor}}$, $\boldsymbol{f}_{\mathrm{nor,tan}}$, $\boldsymbol{f}_{\mathrm{tan,nor}}$, and $\boldsymbol{f}_{\mathrm{tan,tan}}$ are pointwise orthogonal [33] in the sense that

$$\begin{aligned} &\boldsymbol{f}_{\mathrm{nor,nor}} \perp \boldsymbol{f}_{\mathrm{nor,tan}}, \quad \boldsymbol{f}_{\mathrm{nor,nor}} \perp \boldsymbol{f}_{\mathrm{tan,nor}}, \quad \boldsymbol{f}_{\mathrm{nor,nor}} \perp \boldsymbol{f}_{\mathrm{tan,tan}}, \\ &\boldsymbol{f}_{\mathrm{nor,tan}} \perp \boldsymbol{f}_{\mathrm{tan,nor}}, \quad \boldsymbol{f}_{\mathrm{nor,tan}} \perp \boldsymbol{f}_{\mathrm{tan,tan}}, \quad \boldsymbol{f}_{\mathrm{tan,nor}} \perp \boldsymbol{f}_{\mathrm{tan,tan}}. \end{aligned}$$

6.1.1 The Normal Part

First, we look at the normal part of the tensor field $\boldsymbol{f}_{\mathrm{nor,nor}}$. This is analogous to the normal part of the vector Slepian functions. Therefore, it is analogous to the scalar Slepian functions. Then, we get

$$\begin{aligned} \boldsymbol{k}^{\mathrm{nor,nor}}_{nj,n'j'} &= \int_R \overline{\boldsymbol{y}^{(1,1)}_{n,j}(\xi)} : \boldsymbol{y}^{(1,1)}_{n',j'}(\xi) \; \mathrm{d}\omega(\xi) \\ &= \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi) \, (\xi \otimes \xi) : (\xi \otimes \xi) \; \mathrm{d}\omega(\xi) \\ &= \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi) \; \mathrm{d}\omega(\xi) \\ &=: K^0_{nj,n'j'} \end{aligned}$$

for all $n, n' = 0, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$. This is equal to the scalar case.

Then, we get from Problem 6.1.1 the following eigenvalue problem.

Problem 6.1.2. *So, we get the eigenvalue problem*

$$\boldsymbol{k}^{\mathrm{nor,nor}} \boldsymbol{g}_{\mathrm{nor,nor}} = \lambda \boldsymbol{g}_{\mathrm{nor,nor}}$$

and particularly

$$\sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} \boldsymbol{k}_{nj,n'j'}^{\mathrm{nor,nor}} \boldsymbol{f}_{n',j'}^{(1,1)} = \lambda \boldsymbol{f}_{n,j}^{(1,1)}, \tag{6.1}$$

where $\boldsymbol{g}_{\mathrm{nor,nor}} = \left(\boldsymbol{f}_{00}^{(1,1)}, \ldots, \boldsymbol{f}_{LL}^{(1,1)} \right)^{\mathrm{T}}$ *and for all* $n = 0, \ldots, L$ *and all* $j = -n, \ldots, n$.

We know already that $\boldsymbol{k}^{\mathrm{nor,nor}} = K^0$ is Hermitian, positive definite, supposed to be ill-conditioned, and its eigenvalues are real. Therefore, we continue, like in the previous chapter, for special regions. Upon multiplying by $Y_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $n = 0, \ldots, L$ and over $j = -n, \ldots, n$, and interchanging summation and integration, we obtain a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel.

Problem 6.1.3. *This integral equation is given by*

$$\int_R \boldsymbol{\mathscr{k}}^{\mathrm{nor,nor}}(\xi, \eta) F(\xi) \, \mathrm{d}\omega(\xi) = \lambda F(\eta),$$

where

$$\begin{aligned} \boldsymbol{\mathscr{k}}^{\mathrm{nor,nor}}(\xi, \eta) &:= \sum_{n=0}^{L} \sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\eta) \\ &= \sum_{n=0}^{L} \frac{2n+1}{4\pi} \, P_n(\xi \cdot \eta) \\ &=: \mathscr{K}^0(\xi, \eta) \end{aligned}$$

and

$$F(\xi) := \sum_{n=0}^{L} \sum_{j=-n}^{n} \boldsymbol{f}_{n,j}^{(1,1)} Y_{n,j}(\xi)$$

for $\xi, \eta \in \Omega$.

Note that the tensor problem is now reduced to a scalar eigenvalue problem (6.1). Here, we see clearly that

$$\boldsymbol{f}_{\mathrm{nor,nor}}(\xi) = (\xi \otimes \xi) \, F(\xi).$$

This scalar problem is equal to the problem for the scalar Slepian function form Chapter 4. So, we know already an orthonormal basis of solutions $\mathscr{G}_1^0, \ldots, \mathscr{G}_{(L+1)^2}^0$ of the integral equation sorted with respect to decreasing eigenvalues. So, we obtain the corresponding orthonormal eigenfunctions $(\boldsymbol{\mathscr{g}}_{\mathrm{nor,nor}})_\alpha$ and eigenvalues $(\lambda_{\mathrm{nor,nor}})_\alpha$ for $\alpha = 1, \ldots, (L+1)^2$ of the normal part of the tensor field by

$$(\boldsymbol{\mathscr{g}}_{\mathrm{nor,nor}})_\alpha (\xi) = (\xi \otimes \xi) \, \mathscr{G}_\alpha^0(\xi)$$

and

$$\left(\lambda_{\mathrm{nor,nor}}\right)_\alpha = \lambda^0_\alpha$$

for $\alpha = 1, \ldots, (L+1)^2$.

6.1.2 The Left Normal/Right Tangential Part

The left normal/right tangential part of the tensor field $\boldsymbol{f}_{\mathrm{nor,tan}}$ is analogous to the tangential part of vector Slepian functions. Because in general,

$$\boldsymbol{y}^{(1,2)}_{n,j}(\xi) : \overline{\boldsymbol{y}^{(1,3)}_{n',j'}(\xi)} \neq 0, \qquad \boldsymbol{y}^{(1,3)}_{n,j}(\xi) : \overline{\boldsymbol{y}^{(1,2)}_{n',j'}(\xi)} \neq 0,$$

and always

$$\boldsymbol{y}^{(1,2)}_{n,j}(\xi) : \overline{\boldsymbol{y}^{(1,2)}_{n',j'}(\xi)} = y^{(2)}_{n,j}(\xi) \cdot \overline{y^{(2)}_{n',j'}(\xi)}$$

and

$$\boldsymbol{y}^{(1,3)}_{n,j}(\xi) : \overline{\boldsymbol{y}^{(1,3)}_{n',j'}(\xi)} = y^{(3)}_{n,j}(\xi) \cdot \overline{y^{(3)}_{n',j'}(\xi)}$$

for all $\xi \in \Omega$, all $n, n' = 1, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$, we also construct a new basis for the left normal/right tangential tensor space.

Definition 6.1.4. *We use the results of Chapter 3.9 and define*

$$\boldsymbol{y}^{(1,\pm)}_{n,j}(\xi) := \pm\frac{1}{\sqrt{2}}\left(-\boldsymbol{y}^{(1,2)}_{n,j}(\xi) \pm i\boldsymbol{y}^{(1,3)}_{n,j}(\xi)\right) = {}_{\pm 1}Y_{n,j}(\xi)\,(\xi \otimes \tau_\pm)$$

for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$ and write

$$\boldsymbol{f}_{\mathrm{nor,tan}}(\xi) := \sum_{n=1}^{L}\sum_{j=-n}^{n}\left(\boldsymbol{f}^{(1,+)}_{n,j}\boldsymbol{y}^{(1,+)}_{n,j}(\xi) + \boldsymbol{f}^{(1,-)}_{n,j}\boldsymbol{y}^{(1,-)}_{n,j}(\xi)\right),$$

where $\tau_\pm \cdot \overline{\tau_\pm} = 1$ and $\tau_\pm \cdot \overline{\tau_\mp} = 0$.

Then,

$$\boldsymbol{y}^{(1,\pm)}_{n,j}(\xi) : \overline{\boldsymbol{y}^{(1,\mp)}_{n',j'}(\xi)} = 0$$

for all $\xi \in \Omega$ and

$$\boldsymbol{k}^{\mathrm{nor,tan}} = \begin{pmatrix} \boldsymbol{k}^{(1,+)} & 0 \\ 0 & \boldsymbol{k}^{(1,-)} \end{pmatrix},$$

where

$$\boldsymbol{k}^{(1,\pm)} := \begin{pmatrix} \boldsymbol{k}^{(1,\pm)}_{1,-1,1,-1} & \cdots & \boldsymbol{k}^{(1,\pm)}_{1,-1,LL} \\ \vdots & \ddots & \vdots \\ \boldsymbol{k}^{(1,\pm)}_{LL,1,-1} & \cdots & \boldsymbol{k}^{(1,\pm)}_{LL,LL} \end{pmatrix}$$

and

$$\begin{aligned} \boldsymbol{k}^{(1,\pm)}_{nj,n'j'} &:= \int_R \overline{\boldsymbol{y}^{(1,\pm)}_{n,j}(\xi)} : \boldsymbol{y}^{(1,\pm)}_{n',j'}(\xi)\; \mathrm{d}\omega(\xi) \\ &= \int_R \overline{{}_{\pm 1}Y_{n,j}(\xi)}\; {}_{\pm 1}Y_{n',j'}(\xi)\; \mathrm{d}\omega(\xi) \\ &=: K^{\pm 1}_{nj,n'j'} \end{aligned}$$

for all $n, n' = 1, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$.

Then, we can conclude from Problem 6.1.1 the following eigenvalue problem.

Problem 6.1.5. *So, we get the eigenvalue problems*

$$\boldsymbol{k}^{(1,\pm)}\boldsymbol{g}^{(1,\pm)}_{\mathrm{nor,tan}} = \lambda \boldsymbol{g}^{(1,\pm)}_{\mathrm{nor,tan}}$$

and particularly

$$\sum_{n'=1}^{L}\sum_{j'=-n'}^{n'} \boldsymbol{k}^{(1,\pm)}_{nj,n'j'}\boldsymbol{f}^{(1,\pm)}_{n',j'} = \lambda \boldsymbol{f}^{(1,\pm)}_{n,j}, \tag{6.2}$$

where $\boldsymbol{g}^{(1,\pm)}_{\mathrm{nor,tan}} = \left(\boldsymbol{f}^{(1,\pm)}_{1,-1}\dots\boldsymbol{f}^{(1,\pm)}_{LL}\right)^{\mathrm{T}}$ *and for all* $n = 1,\dots,L$ *and all* $j = -n,\dots,n$.

We know already that this kernel matrix $k^{(1,\pm)} = K^{\pm 1}$ is Hermitian, positive definite, supposed to be ill-conditioned, and its eigenvalues are real. Therefore, we continue, like in the previous chapter, for special regions and reformulate equation (6.2).
Upon multiplying by ${}_{\pm 1}Y_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $n = 1,\dots,L$ and $j = -n,\dots,n$, and interchanging summation and integration, we obtain a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel.

Problem 6.1.6. *This integral equation is given by*

$$\int_R \boldsymbol{\mathscr{k}}^{(1,\pm)}(\xi,\eta)F^{(1,\pm)}(\xi)\ \mathrm{d}\omega(\xi) = \lambda F^{(1,\pm)}(\eta),$$

where

$$\boldsymbol{\mathscr{k}}^{(1,\pm)}(\xi,\eta) := \sum_{n=1}^{L}\sum_{j=-n}^{n} \overline{{}_{\pm 1}Y_{n,j}(\xi)}\ {}_{\pm 1}Y_{n,j}(\eta) =: \mathscr{K}^{\pm 1}(\xi,\eta)$$

and

$$F^{(1,\pm)}(\xi) := \sum_{n=1}^{L}\sum_{j=-n}^{n} \boldsymbol{f}^{(1,\pm)}_{n,j}\ {}_{\pm 1}Y_{n,j}(\xi)$$

for $\xi,\eta \in \Omega$.

Note that the tensor problem is now reduced to a scalar eigenvalue problem (6.2). Here, we see clearly that

$$\boldsymbol{f}_{\mathrm{nor,tan}}(\xi) = (\xi\otimes\tau_+)\,F^{(1,+)}(\xi) + (\xi\otimes\tau_-)\,F^{(1,-)}(\xi).$$

This scalar problem is equal to the tangential part of the problem for the vector Slepian function from Chapter 5.1.2. So, we know already an orthonormal basis of solutions $\mathscr{G}^{\pm 1}_1,\dots,\mathscr{G}^{\pm 1}_{(L+1)^2-1}$ of the integral equation sorted with respect to decreasing eigenvalues. So, we obtain the corresponding orthonormal eigenfunctions $\left(\boldsymbol{\mathscr{g}}^{(1,\pm)}_{\mathrm{nor,tan}}\right)_\alpha$ and eigenvalues $\left(\lambda^{(1,\pm)}_{\mathrm{nor,tan}}\right)_\alpha$ for $\alpha = 1,\dots,(L+1)^2-1$ of the left normal/right tangential part of the tensor field by

$$\left(\boldsymbol{\mathscr{g}}^{(1,\pm)}_{\mathrm{nor,tan}}\right)_\alpha(\xi) = (\xi\otimes\tau_\pm)\,\mathscr{G}^{\pm 1}_\alpha(\xi)$$

and

$$\left(\lambda^{(1,\pm)}_{\mathrm{nor,tan}}\right)_\alpha = \lambda^{\pm 1}_\alpha$$

for $\alpha = 1,\dots,(L+1)^2-1$.

6.1.3 The Left Tangential/Right Normal Part

The left tangential/right normal part of the tensor field $\boldsymbol{f}_{\text{tan,nor}}$ is also analogous to the tangential part of vector Slepian functions. Because in general,

$$\boldsymbol{y}_{n,j}^{(2,1)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(3,1)}(\xi)} \neq 0, \qquad \boldsymbol{y}_{n,j}^{(3,1)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(2,1)}(\xi)} \neq 0,$$

and always

$$\boldsymbol{y}_{n,j}^{(2,1)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(2,1)}(\xi)} = y_{n,j}^{(2)}(\xi) \cdot \overline{y_{n',j'}^{(2)}(\xi)}$$

and

$$\boldsymbol{y}_{n,j}^{(3,1)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(3,1)}(\xi)} = y_{n,j}^{(3)}(\xi) \cdot \overline{y_{n',j'}^{(3)}(\xi)}$$

for all $\xi \in \Omega$, all $n, n' = 1, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$, we construct again a new basis for the left normal/right tangential tensor space.

Definition 6.1.7. *We use the results of Chapter 3.9 and define*

$$\boldsymbol{y}_{n,j}^{(\pm,1)}(\xi) := \pm\frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(2,1)}(\xi) \pm i\boldsymbol{y}_{n,j}^{(3,1)}(\xi)\right) = {}_{\pm 1}Y_{n,j}(\xi)\, (\tau_\pm \otimes \xi)$$

for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$ and write

$$\boldsymbol{f}_{\text{tan,nor}}(\xi) := \sum_{n=1}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{f}_{n,j}^{(+,1)} \boldsymbol{y}_{n,j}^{(+,1)}(\xi) + \boldsymbol{f}_{n,j}^{(-,1)} \boldsymbol{y}_{n,j}^{(-,1)}(\xi)\right),$$

where $\tau_\pm \cdot \overline{\tau_\pm} = 1$ and $\tau_\pm \cdot \overline{\tau_\mp} = 0$.

Then,

$$\boldsymbol{y}_{n,j}^{(\pm,1)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(\mp,1)}(\xi)} = 0$$

for all $\xi \in \Omega$ and

$$\boldsymbol{k}^{\text{tan,nor}} = \begin{pmatrix} \boldsymbol{k}^{(+,1)} & 0 \\ 0 & \boldsymbol{k}^{(-,1)} \end{pmatrix},$$

where

$$\boldsymbol{k}^{(\pm,1)} := \begin{pmatrix} \boldsymbol{k}_{1,-1,1,-1}^{(\pm,1)} & \ldots & \boldsymbol{k}_{1,-1,LL}^{(\pm,1)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{k}_{LL,1,-1}^{(\pm,1)} & \ldots & \boldsymbol{k}_{LL,LL}^{(\pm,1)} \end{pmatrix}$$

and

$$\begin{aligned} \boldsymbol{k}_{nj,n'j'}^{(\pm,1)} &:= \int_R \overline{\boldsymbol{y}_{n,j}^{(\pm,1)}(\xi)} : \boldsymbol{y}_{n',j'}^{(\pm,1)}(\xi)\, \mathrm{d}\omega(\xi) \\ &= \int_R \overline{{}_{\pm 1}Y_{n,j}(\xi)}\; {}_{\pm 1}Y_{n',j'}(\xi)\, \mathrm{d}\omega(\xi) \\ &=: K_{nj,n'j'}^{\pm 1} \end{aligned}$$

for all $n, n' = 1, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$.

Then, we get the following eigenvalue problem from Problem 6.1.1.

Problem 6.1.8. *So, we get the eigenvalue problems*

$$\boldsymbol{k}^{(\pm,1)} \boldsymbol{g}_{\text{tan,nor}}^{(\pm,1)} = \lambda \boldsymbol{g}_{\text{tan,nor}}^{(\pm,1)}$$

and particularly

$$\sum_{n'=1}^{L}\sum_{j'=-n'}^{n'} \boldsymbol{k}^{(\pm,1)}_{nj,n'j'} \boldsymbol{f}^{(\pm,1)}_{n',j'} = \lambda \boldsymbol{f}^{(\pm,1)}_{n,j}, \tag{6.3}$$

where $\boldsymbol{g}^{(\pm,1)}_{\mathrm{tan,nor}} = \left(\boldsymbol{f}^{(\pm,1)}_{1,-1},\ldots,\boldsymbol{f}^{(\pm,1)}_{LL}\right)^{\mathrm{T}}$ *for all* $n=1,\ldots,L$ *and all* $j=-n,\ldots,n$.

We know already that $\boldsymbol{k}^{(\pm,1)} = K^{\pm 1}$ is Hermitian, positive definite, supposed to be ill-conditioned, and its eigenvalues are real. Therefore, we continue, like in the previous chapter, for special regions. Upon multiplying by ${}_{\pm1}Y_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $n = 1,\ldots,L$ and $j=-n,\ldots,n$, and interchanging summation and integration, we obtain again a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel.

Problem 6.1.9. *This integral equation is given by*

$$\int_R \mathscr{k}^{(\pm,1)}(\xi,\eta) F^{(\pm,1)}(\xi)\ \mathrm{d}\omega(\xi) = \lambda F^{(\pm,1)}(\eta),$$

where

$$\mathscr{k}^{(\pm,1)}(\xi,\eta) := \sum_{n=1}^{L}\sum_{j=-n}^{n} \overline{{}_{\pm1}Y_{n,j}(\xi)}\ {}_{\pm1}Y_{n,j}(\eta) =: \mathscr{K}^{\pm1}(\xi,\eta)$$

and

$$F^{(\pm,1)}(\xi) := \sum_{n=1}^{L}\sum_{j=-n}^{n} \boldsymbol{f}^{(\pm,1)}_{n,j}\ {}_{\pm1}Y_{n,j}(\xi)$$

Note that the tensor problem is now reduced to a scalar eigenvalue problem (6.3). Here, we see clearly that

$$\boldsymbol{f}_{\mathrm{tan,nor}}(\xi) = (\tau_+ \otimes \xi)\, F^{(+,1)}(\xi) + (\tau_- \otimes \xi)\, F^{(-,1)}(\xi).$$

This scalar problem is again equal to the tangential part of the problem for the vector Slepian function from Chapter 5.1.2. So, we know already an orthonormal basis of solutions $\mathscr{G}^{\pm1}_1,\ldots,\mathscr{G}^{\pm1}_{(L+1)^2-1}$ of the integral equation sorted with respect to decreasing eigenvalues. So, we obtain the corresponding orthonormal eigenfunctions $\left(\boldsymbol{\mathscr{g}}^{(\pm,1)}_{\mathrm{tan,nor}}\right)_\alpha$ and eigenvalues $\left(\lambda^{(\pm,1)}_{\mathrm{tan,nor}}\right)_\alpha$ for $\alpha = 1,\ldots,(L+1)^2-1$ of the left normal/right tangential part of the tensor field by

$$\left(\boldsymbol{\mathscr{g}}^{(\pm,1)}_{\mathrm{tan,nor}}\right)_\alpha(\xi) = (\tau_\pm \otimes \xi)\, \mathscr{G}^{\pm1}_\alpha(\xi)$$

and

$$\left(\lambda^{(\pm,1)}_{\mathrm{tan,nor}}\right)_\alpha = \lambda^{\pm1}_\alpha$$

for $\alpha = 1,\ldots,(L+1)^2-1$.

6.1.4 The Tangential Part

The tangential part of the vector field $\boldsymbol{f}_{\mathrm{tan}}$ is more difficult. Overall, it is not reducible to the scalar or the vector cause. Because in general,

$$\boldsymbol{y}^{(2,2)}_{n,j}(\xi) : \overline{\boldsymbol{y}^{(2,3)}_{n',j'}(\xi)} \neq 0,$$
$$\boldsymbol{y}^{(2,2)}_{n,j}(\xi) : \overline{\boldsymbol{y}^{(3,2)}_{n',j'}(\xi)} \neq 0,$$

$$
\begin{aligned}
\boldsymbol{y}_{n,j}^{(3,3)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(2,3)}(\xi)} &\neq 0, \\
\boldsymbol{y}_{n,j}^{(3,3)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(3,2)}(\xi)} &\neq 0, \\
\boldsymbol{y}_{n,j}^{(2,3)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(3,2)}(\xi)} &\neq 0,
\end{aligned}
$$

but

$$
\begin{aligned}
\boldsymbol{y}_{n,j}^{(2,2)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(3,3)}(\xi)} &= 0, \\
\boldsymbol{y}_{n,j}^{(2,2)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(2,2)}(\xi)} &= Y_{n,j}(\xi)\overline{Y_{n',j'}(\xi)}, \\
\boldsymbol{y}_{n,j}^{(3,3)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(3,3)}(\xi)} &= Y_{n,j}(\xi)\overline{Y_{n',j'}(\xi)},
\end{aligned}
$$

for all $\xi \in \Omega$, all $n = 0_{ik}, \ldots, L$, all $n' = 0_{i'k'}, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$, where $i, k, i', k' = 2, 3$, we construct a new basis for the tangential tensor space.

Definition 6.1.10. *We keep $\boldsymbol{y}_{n,j}^{(2,2)}$ and $\boldsymbol{y}_{n,j}^{(3,3)}$, use the results of Chapter 3.9, and define additionally*

$$
\boldsymbol{y}_{n,j}^{(\pm,\pm)}(\xi) := -\frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(2,3)}(\xi) \pm i\boldsymbol{y}_{n,j}^{(3,2)}(\xi) \right) = {}_{\pm 2}Y_{n,j}(\xi)\,(\tau_\pm \otimes \tau_\pm)
$$

for all $n = 2, \ldots, L$ and all $j = -n, \ldots, n$ and write

$$
\begin{aligned}
\boldsymbol{f}_{\tan,\tan}(\xi) := &\sum_{n=0}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{f}_{n,j}^{(2,2)} \boldsymbol{y}_{n,j}^{(2,2)}(\xi) + \boldsymbol{f}_{n,j}^{(3,3)} \boldsymbol{y}_{n,j}^{(3,3)}(\xi) \right) \\
&+ \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{f}_{n,j}^{(+,+)} \boldsymbol{y}_{n,j}^{(+,+)}(\xi) + \boldsymbol{f}_{n,j}^{(-,-)} \boldsymbol{y}_{n,j}^{(-,-)}(\xi) \right),
\end{aligned}
$$

where $\tau_\pm \cdot \overline{\tau_\pm} = 1$, $\tau_\pm \cdot \overline{\tau_\mp} = 0$.

Lemma 6.1.11. *The tensors $\tau_\pm \otimes \tau_\pm$, $\mathbf{i}_{\tan}$ and $\mathbf{j}_{\tan}$ are pairwise orthonormal. This means that*

$$
(\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\mp \otimes \tau_\mp)} = 0,
$$

and

$$
\begin{aligned}
\mathbf{i}_{\tan} : \overline{(\tau_\pm \otimes \tau_\pm)} &= 0, \\
\mathbf{j}_{\tan} : \overline{(\tau_\pm \otimes \tau_\pm)} &= 0, \\
\mathbf{i}_{\tan} : \mathbf{j}_{\tan} &= 0.
\end{aligned}
$$

Furthermore, we see that

$$
(\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\pm \otimes \tau_\pm)} = 1.
$$

Proof. The proof is straight forward. Let $\xi = \xi(t, \varphi) \in \Omega$. With Lemma 2.2.7 and with Definition 2.3.2, we get

$$
\begin{aligned}
(\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\mp \otimes \tau_\mp)} &= (\tau_\pm \cdot \overline{\tau_\mp})^2 \\
&= \frac{1}{4} \left(\left(\varepsilon^t \pm i\varepsilon^\varphi \right) \cdot \left(\varepsilon^t \pm i\varepsilon^\varphi \right) \right)^2 \\
&= \frac{1}{4} \big(\underbrace{\varepsilon^t \cdot \varepsilon^t}_{=1} \pm i \underbrace{\varepsilon^t \cdot \varepsilon^\varphi}_{=0} \pm i \underbrace{\varepsilon^\varphi \cdot \varepsilon^t}_{=0} - \underbrace{\varepsilon^\varphi \cdot \varepsilon^\varphi}_{=1} \big)^2
\end{aligned}
$$

$$
\begin{aligned}
&= 0,\\
\mathbf{i}_{\tan} : \overline{(\tau_\pm \otimes \tau_\pm)} &= \left(\varepsilon^\varphi \otimes \varepsilon^\varphi + \varepsilon^t \otimes \varepsilon^t\right) : \overline{(\tau_\pm \otimes \tau_\pm)}\\
&= \left(\varepsilon^\varphi \cdot \overline{\tau_\pm}\right)^2 + \left(\varepsilon^t \cdot \overline{\tau_\pm}\right)^2\\
&= -\frac{1}{\sqrt{2}}\left(\left(\varepsilon^\varphi \cdot \left(\varepsilon^t \mp i\varepsilon^\varphi\right)\right)^2 + \left(\varepsilon^t \cdot \left(\varepsilon^t \mp t\varepsilon^\varphi\right)\right)^2\right)\\
&= -\frac{1}{\sqrt{2}}\left((\mp i)^2 + 1^2\right)\\
&= -\frac{1}{\sqrt{2}}(-1+1)\\
&= 0,\\
\mathbf{j}_{\tan} : \overline{(\tau_\pm \otimes \tau_\pm)} &= \left(\varepsilon^t \otimes \varepsilon^\varphi - \varepsilon^\varphi \otimes \varepsilon^t\right) : \overline{(\tau_\pm \otimes \tau_\pm)}\\
&= \left(\varepsilon^t \cdot \overline{\tau_\pm}\right)\left(\varepsilon^\varphi \cdot \overline{\tau_\pm}\right) - \left(\varepsilon^\varphi \cdot \overline{\tau_\pm}\right)\left(\varepsilon^t \cdot \overline{\tau_\pm}\right)\\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{i}_{\tan} : \mathbf{j}_{\tan} &= \left(\varepsilon^\varphi \otimes \varepsilon^\varphi + \varepsilon^t \otimes \varepsilon^t\right) : \left(\varepsilon^t \otimes \varepsilon^\varphi - \varepsilon^\varphi \otimes \varepsilon^t\right)\\
&= \Big(\underbrace{\left(\varepsilon^\varphi \cdot \varepsilon^t\right)}_{=0}\underbrace{\left(\varepsilon^\varphi \cdot \varepsilon^\varphi\right)}_{=1} - \underbrace{\left(\varepsilon^\varphi \cdot \varepsilon^\varphi\right)}_{=1}\underbrace{\left(\varepsilon^\varphi \cdot \varepsilon^t\right)}_{=0} + \underbrace{\left(\varepsilon^t \cdot \varepsilon^t\right)}_{=1}\underbrace{\left(\varepsilon^t \cdot \varepsilon^\varphi\right)}_{=0} - \underbrace{\left(\varepsilon^t \cdot \varepsilon^\varphi\right)}_{=0}\underbrace{\left(\varepsilon^t \cdot \varepsilon^t\right)}_{=1}\Big)\\
&= 0.
\end{aligned}
$$

Furthermore, we obtain again with Lemma 2.2.7 and with Definition 2.3.2

$$
\begin{aligned}
(\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\pm \otimes \tau_\pm)} &= \left(\tau_\pm \cdot \overline{\tau_\pm}\right)^2\\
&= \frac{1}{4}\left(\left(\varepsilon^t \pm i\varepsilon^\varphi\right) \cdot \left(\varepsilon^t \mp i\varepsilon^\varphi\right)\right)^2\\
&= \frac{1}{4}\Big(\underbrace{\varepsilon^t \cdot \varepsilon^t}_{=1} \mp i \underbrace{\varepsilon^t \cdot \varepsilon^\varphi}_{=0} \pm i \underbrace{\varepsilon^\varphi \cdot \varepsilon^t}_{=0} + \underbrace{\varepsilon^\varphi \cdot \varepsilon^\varphi}_{=1}\Big)^2\\
&= 1.
\end{aligned}
$$

□

Derived from the previous lemma, we conclude that

$$
\begin{aligned}
\boldsymbol{y}_{n,j}^{(2,2)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(3,3)}(\xi)} &= 0,\\
\boldsymbol{y}_{n,j}^{(2,2)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(\pm,\pm)}(\xi)} &= 0,\\
\boldsymbol{y}_{n,j}^{(3,3)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(\pm,\pm)}(\xi)} &= 0,\\
\boldsymbol{y}_{n,j}^{(\pm,\pm)}(\xi) : \overline{\boldsymbol{y}_{n',j'}^{(\mp,\mp)}(\xi)} &= 0,
\end{aligned}
$$

for all $\xi \in \Omega$, all $n, n' = 0_{ii}, \ldots, L$, all $j = -n, \ldots, n$, all $j' = -n', \ldots, n'$, where $i = 2, 3, +, -$ with $0_{\pm\pm} = 2$, and

$$
\boldsymbol{k}^{\tan,\tan} = \begin{pmatrix} \boldsymbol{k}^{(2,2)} & 0 & 0 & 0\\ 0 & \boldsymbol{k}^{(3,3)} & 0 & 0\\ 0 & 0 & \boldsymbol{k}^{(+,+)} & 0\\ 0 & 0 & 0 & \boldsymbol{k}^{(-,-)} \end{pmatrix},
$$

where

$$\boldsymbol{k}^{(i,i)} := \begin{pmatrix} \boldsymbol{k}^{(i,i)}_{0_{ii},-0_{ii},0_{ii},-0_{ii}} & \cdots & \boldsymbol{k}^{(i,i)}_{0_{ii},-0_{ii},LL} \\ \vdots & \ddots & \vdots \\ \boldsymbol{k}^{(i,i)}_{LL,0_{ii},-0_{ii}} & \cdots & \boldsymbol{k}^{(i,i)}_{LL,LL} \end{pmatrix}$$

for $i = 2, 3, +, -$. So,

$$\begin{aligned} \boldsymbol{k}^{(i,i)}_{nj,n'j'} &:= \int_R \overline{\boldsymbol{y}^{(i,i)}_{n,j}(\xi)} : \boldsymbol{y}^{(i,i)}_{n',j'}(\xi) \, \mathrm{d}\omega(\xi) \\ &= \int_R \overline{Y_{n,j}(\xi)} Y_{n',j'}(\xi) \, \mathrm{d}\omega(\xi) \\ &=: K^0_{nj,n'j'}, \end{aligned}$$

leads for $i = 2, 3$ and for all $n, n' = 0, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$ to the scalar case and

$$\begin{aligned} \boldsymbol{k}^{(\pm,\pm)}_{nj,n'j'} &:= \int_R \overline{\boldsymbol{y}^{(\pm,\pm)}_{n,j}(\xi)} : \boldsymbol{y}^{(\pm,\pm)}_{n',j'}(\xi) \, \mathrm{d}\omega(\xi) \\ &= \int_R \overline{{}_{\pm 2}Y_{n,j}(\xi)} \; {}_{\pm 2}Y_{n',j'}(\xi) \, \mathrm{d}\omega(\xi) \\ &=: K^{\pm 2}_{nj,n'j'} \end{aligned}$$

leads for all $n, n' = 2, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$ to the case of spin weight ± 2. We choose the notation such that it is unique for the general case with spin-weighted spherical harmonics (compare Chapter 8).

Then, we get from Problem 6.1.1 the following eigenvalue problem.

Problem 6.1.12. *So, we get for $i = 2, 3$ the eigenvalue problems*

$$\boldsymbol{k}^{(i,i)} \boldsymbol{g}^{(i,i)}_{\mathrm{tan,tan}} = \lambda \boldsymbol{g}^{(i,i)}_{\mathrm{tan,tan}}$$

and particularly

$$\sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} \boldsymbol{k}^{(i,i)}_{nj,n'j'} \boldsymbol{f}^{(i,i)}_{n',j'} = \lambda \boldsymbol{f}^{(i,i)}_{n,j}, \tag{6.4}$$

where $\boldsymbol{g}^{(i,i)}_{\mathrm{tan,tan}} = \left(\boldsymbol{f}^{(i,i)}_{00}, \ldots, \boldsymbol{f}^{(i,i)}_{LL} \right)^{\mathrm{T}}$.

We know already that $\boldsymbol{k}^{(i,i)} = K^0$ is Hermitian, positive definite, supposed to be ill-conditioned, and its eigenvalues are real. Therefore, we continue, like in the previous chapter, for special regions. Upon multiplying by $Y_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $n = 0, \ldots, L$ and $j = -n, \ldots, n$, and interchanging summation and integration, we obtain again a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel.

Problem 6.1.13. *This integral equation is given by*

$$\int_R \mathscr{k}^{(i,i)}(\xi, \eta) F^{(i,i)}(\xi) \, \mathrm{d}\omega(\xi) = \lambda F^{(i,i)}(\eta),$$

where

$$\mathscr{k}^{(i,i)}(\xi, \eta) := \sum_{n=0}^{L} \sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\eta) =: \mathscr{K}^0(\xi, \eta)$$

and

$$F^{(i,i)}(\xi) := \sum_{n=0}^{L} \sum_{j=-n}^{n} f_{n,j}^{(i,i)} Y_{n,j}(\xi)$$

for $\xi, \eta \in \Omega$.

Note that the tensor problem is now reduced to a scalar eigenvalue problem (6.4). This scalar problem is equal to the problem for the scalar Slepian function from Chapter 4. So, we know already an orthonormal basis of solutions $\mathscr{G}_1^0, \ldots, \mathscr{G}_{(L+1)^2}^0$ of the integral equation sorted with respect to decreasing eigenvalues. So, we obtain the orthonormal eigenfunctions $\left(\boldsymbol{\mathscr{g}}_{\tan,\tan}^{(i,i)}\right)_\alpha$ and eigenvalues $\left(\lambda_{\tan,\tan}^{(i,i)}\right)_\alpha$ for $i = 2, 3$ and for $\alpha = 1, \ldots, (L+1)^2$ of the tangential part of the tensor field components by

$$\left(\boldsymbol{\mathscr{g}}_{\tan,\tan}^{(2,2)}\right)_\alpha (\xi) = \frac{1}{\sqrt{2}} \, \mathbf{i}_{\tan} \mathscr{G}_\alpha^0(\xi),$$
$$\left(\boldsymbol{\mathscr{g}}_{\tan,\tan}^{(3,3)}\right)_\alpha (\xi) = \frac{1}{\sqrt{2}} \, \mathbf{j}_{\tan} \mathscr{G}_\alpha^0(\xi),$$

and

$$\left(\lambda_{\tan,\tan}^{(i,i)}\right)_\alpha = \lambda_\alpha^0$$

for $\alpha = 1, \ldots, (L+1)^2$ and $i = 2, 3$.

Next, we do the same considerations for $i = \pm$.

Problem 6.1.14. *Additionally, we get the eigenvalue problems*

$$\boldsymbol{k}^{(\pm,\pm)} \boldsymbol{g}_{\tan,\tan}^{(\pm,\pm)} = \lambda \boldsymbol{g}_{\tan,\tan}^{(\pm,\pm)}$$

and particularly

$$\sum_{n'=2}^{L} \sum_{j'=-n'}^{n'} \boldsymbol{k}_{nj,n'j'}^{(\pm,\pm)} \boldsymbol{f}_{n',j'}^{(\pm,\pm)} = \lambda \boldsymbol{f}_{n,j}^{(\pm,\pm)}, \tag{6.5}$$

where $\boldsymbol{g}_{\tan,\tan}^{(\pm,\pm)} = \left(\boldsymbol{f}_{00}^{(\pm,\pm)}, \ldots, \boldsymbol{f}_{LL}^{(\pm,\pm)}\right)^{\mathrm{T}}$.

This kernel matrix $k^{(\pm,\pm)} = K^{\pm 2}$ is also Hermitian and positive definite and the eigenvalues are real (see Lemma 8.1.4). However, the kernel matrix is also ill-conditioned. So, the solution of the eigenvalue problem is numerically unstable. For special regions, we can solve the eigenvalue problem by replacing it with a commuting eigenvalue problem. Therefore, we have to reformulate equation (6.5).
Upon multiplying by ${}_{\pm 2}Y_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $n = 2, \ldots, L$ and $j = -n, \ldots, n$, and interchanging summation and integration, we obtain a homogeneous integral equation of the second kind with a finite-rank, symmetric, and Hermitian kernel.

Problem 6.1.15. *This integral equation is given by*

$$\int_R \boldsymbol{\mathscr{k}}^{(\pm,\pm)}(\xi, \eta) F^{(\pm,\pm)}(\xi) \, \mathrm{d}\omega(\xi) = \lambda F^{(\pm,\pm)}(\eta),$$

where

$$\boldsymbol{\mathscr{k}}^{(\pm,\pm)}(\xi, \eta) := \sum_{n=2}^{L} \sum_{j=-n}^{n} \overline{{}_{\pm 2}Y_{n,j}(\xi)} \; {}_{\pm 2}Y_{n,j}(\eta) =: \mathscr{K}^{\pm 2}(\xi, \eta)$$

(compare Chapter 8) and

$$F^{(\pm,\pm)}(\xi) := \sum_{n=2}^{L} \sum_{j=-n}^{n} \boldsymbol{f}_{n,j}^{(\pm,\pm)} \, {}_{\pm 2}Y_{n,j}(\xi)$$

for $\xi, \eta \in \Omega$.

The tensor problem is reduced to a scalar eigenvalue problem (6.4) and (6.5). Here, we see directly that

$$\boldsymbol{f}_{\tan,\tan}(\xi) = \frac{1}{\sqrt{2}}\, \mathbf{i}_{\tan} F^{(2,2)}(\xi) + \frac{1}{\sqrt{2}}\, \mathbf{j}_{\tan} F^{(3,3)}(\xi) + (\tau_+ \otimes \tau_+)\, F^{(+,+)}(\xi) + (\tau_- \otimes \tau_-)\, F^{(-,-)}(\xi).$$

Definition 6.1.16. *We know from Problem 4.1.2 that* $\boldsymbol{g}_{\tan,\tan}^{(\pm,\pm)}$ *are the eigenvectors of* $\boldsymbol{k}^{(\pm,\pm)}$. *Furthermore, we know that the matrix* $\boldsymbol{k}^{(\pm,\pm)} = K^{\pm 2}$ *is Hermitian and positive definite. Therefore, we get an infinite number of solutions of the eigenvalue problem, which can be described by* $(L+1)^2-4$ *pairwise linearly independent solutions. These eigenvectors can be chosen such that they form an orthogonal basis and hence, an orthonormal basis of* $\mathbf{harm}_{2\ldots L}^{(\pm,\pm)}(\Omega)$ *with help of the Gram-Schmidt algorithm, where*

$$\mathbf{harm}_{2\ldots L}^{(\pm,\pm)}(\Omega) := \operatorname{span}\left\{\boldsymbol{y}_{n,j}^{(\pm,\pm)}\right\}_{n=2,\ldots,L,j=-n,\ldots,n}.$$

Further, we sort them by decreasing eigenvalues. Then, we denote these sorted orthonormal eigenvectors of the eigenvalue problem (6.8) by $G_1^{\pm 2}, \ldots, G_{(L+1)^2-4}^{\pm 2}$ *and the sorted eigenvalues by* $\lambda_1^{\pm 2}, \ldots, \lambda_{(L+1)^2-4}^{\pm 2}$.

With $\mathscr{G}_1^{\pm 2}, \ldots, \mathscr{G}_{(L+1)^2-4}^{\pm 2}$, *we denote the sorted orthonormal eigenfunctions of the integral equation from Problem 6.1.15. The spin-weighted spherical harmonic coefficients of the eigenfunctions are also the components of the sorted orthonormal eigenvectors. We denote them by* $(G_{n,j}^{\pm 2})_1, \ldots, (G_{n,j}^{\pm 2})_{(L+1)^2-4}$. *The eigenfunctions are the Slepian functions of spin weight* ± 2 *on the sphere. We get for this spin-weighted Slepian basis the same number of basis functions like for the basis of the type* $(2,3)$ *respectively* $(3,2)$ *tensor spherical harmonics of Freeden, Gervens, and Schreiner.*

To determine the orthonormal eigenfunctions $\mathscr{G}_\alpha^{\pm 2}$ and eigenvalues $\lambda_\alpha^{\pm 2}$ of the integral equation with $\alpha = 1, \ldots, (L+1)^2 - 4$, we can also find a commuting differential operator $\mathscr{J}_\xi^{\pm 2}$ to the kernel function $\mathscr{K}^{\pm 2}$ for the case on the sphere. For the spherical cap (a circularly symmetric cap of colatitudinal radius ϑ), this operator is given by

$$\mathscr{J}_\xi^{\pm 2} = (b-t)\Delta_\xi^{\pm 2} + \left(t^2-1\right)\partial_t - L(L+2)t,$$

where $\xi = \xi(\varphi, t) \in \Omega$, $b = \cos\vartheta \le t \le 1$,

$$\Delta_\xi^{\pm 2} = \Delta_\xi^* - 4\, \frac{1 - it\,(\pm\partial_\varphi)}{1-t^2}$$

and we know from Corollary 3.3.7 that

$$\Delta_\xi^{\pm 2}\, {}_{\pm 2}Y_{n,j}(\xi) = -n(n+1)\, {}_{\pm 2}Y_{n,j}(\xi).$$

Then, the eigenfunctions of spin weight ± 2 can be obtained by

$$\mathscr{J}_\xi^{\pm 2} \mathscr{G}_\alpha^{\pm 2}(\xi) = \chi_\alpha \mathscr{G}_\alpha^{\pm 2}(\xi),$$

where χ_α and $\lambda_\alpha^{\pm 1}$ are not necessary equal. This means that the kernel function and its commuting differential operator have the same eigenfunctions but need not to have the same eigenvalues. So, we obtain the eigenfunctions $\left(\boldsymbol{g}_{\mathrm{tan,tan}}^{(\pm,\pm)}\right)_\alpha$ and eigenvalues $\left(\lambda_{\mathrm{tan,tan}}^{(\pm,\pm)}\right)_\alpha$ for $\alpha = 1, \ldots, (L+1)^2 - 4$ of the tangential part of the vector field by

$$\left(\boldsymbol{g}_{\mathrm{tan,tan}}^{(\pm,\pm)}\right)_\alpha (\xi) = (\tau_\pm \otimes \tau_\pm)\, \mathscr{G}_\alpha^{\pm 2}(\xi)$$

and

$$\left(\lambda_{\mathrm{tan,tan}}^{(\pm,\pm)}\right)_\alpha = \lambda_\alpha^{\pm 2}$$

for $\alpha = 1, \ldots, (L+1)^2 - 4$. For further details, see Chapter 8. For arbitrary regions, we have to solve the eigenvalue problem with the commonly ill-conditioned kernel matrix from equation (6.5) numerically (compare Chapter 8.1).

6.1.5 Complete Tensor Solution

Now, we combine the results from the previous sections to obtain the complete tensor solution.

Altogether, we have to solve the concentration problem, Problem 6.1.1,

$$\lambda = \frac{\int_R \boldsymbol{g}(\xi) : \overline{\boldsymbol{g}(\xi)}\ \mathrm{d}\omega(\xi)}{\int_\Omega \boldsymbol{g}(\xi) : \overline{\boldsymbol{g}(\xi)}\ \mathrm{d}\omega(\xi)} = \max,$$

with

$$\boldsymbol{g}(\xi) = \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \boldsymbol{g}_{n,j}^{i} \boldsymbol{y}_{n,j}^{i}(\xi) \tag{6.6}$$

for all $\xi \in \Omega$ and

$$\boldsymbol{g}_{n,j}^{i} = \int_\Omega \boldsymbol{g}(\xi) : \overline{\boldsymbol{y}_{n,j}^{i}(\xi)}\ \mathrm{d}\omega(\xi)$$

for all $(i,n,j) \in \boldsymbol{J}$.

Here, we now use the following notations.

Definition 6.1.17. *We define the set of indices*

$$\boldsymbol{J} := \left\{(i,n,j) \mid i = 1, \ldots, 9; n = \mathbf{0}_i, \ldots, L; j = -n, \ldots, n\right\},$$

where

$$\mathbf{0}_i := \begin{cases} 0 & , i = 1,6,7 \\ 1 & , i = 2,3,4,5 \\ 2 & , i = 8,9 \end{cases}.$$

Definition 6.1.18. *Now, we use as basis the transformed tensor spherical harmonics given by*

$$\begin{aligned}
\boldsymbol{y}_{n,j}^{1}(\xi) &:= \boldsymbol{y}_{n,j}^{(1,1)}(\xi) = (\xi \otimes \xi)\, Y_{n,j}(\xi),\\
\boldsymbol{y}_{n,j}^{2}(\xi) &:= \boldsymbol{y}_{n,j}^{(1,+)}(\xi) = (\xi \otimes \tau_+)\ {}_{+1}Y_{n,j}(\xi),\\
\boldsymbol{y}_{n,j}^{3}(\xi) &:= \boldsymbol{y}_{n,j}^{(1,-)}(\xi) = (\xi \otimes \tau_-)\ {}_{-1}Y_{n,j}(\xi),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{y}^4_{n,j}(\xi) &:= \boldsymbol{y}^{(+,1)}_{n,j}(\xi) = (\tau_+ \otimes \xi) \ {}_{+1}Y_{n,j}(\xi), \\
\boldsymbol{y}^5_{n,j}(\xi) &:= \boldsymbol{y}^{(-,1)}_{n,j}(\xi) = (\tau_- \otimes \xi) \ {}_{-1}Y_{n,j}(\xi), \\
\boldsymbol{y}^6_{n,j}(\xi) &:= \boldsymbol{y}^{(2,2)}_{n,j}(\xi) = \frac{1}{\sqrt{2}} \ \mathbf{i}_{\tan} Y_{n,j}(\xi), \\
\boldsymbol{y}^7_{n,j}(\xi) &:= \boldsymbol{y}^{(3,3)}_{n,j}(\xi) = \frac{1}{\sqrt{2}} \ \mathbf{j}_{\tan} Y_{n,j}(\xi), \\
\boldsymbol{y}^8_{n,j}(\xi) &:= \boldsymbol{y}^{(+,+)}_{n,j}(\xi) = (\tau_+ \otimes \tau_+) \ {}_{+2}Y_{n,j}(\xi), \\
\boldsymbol{y}^9_{n,j}(\xi) &:= \boldsymbol{y}^{(-,-)}_{n,j}(\xi) = (\tau_- \otimes \tau_-) \ {}_{-2}Y_{n,j}(\xi)
\end{aligned}$$

for $\xi \in \Omega$.

Note that this denotes a different basis than the tensor spherical harmonics of Freeden, Gervens, and Schreiner $y^{(i,k)}_{n,j}$, $i,k = 1,2,3$.

Lemma 6.1.19. *The transformed tensor spherical harmonics are orthonormal on the sphere. This means that*

$$\int_\Omega \boldsymbol{y}^i_{n,j}(\xi) : \overline{\boldsymbol{y}^{i'}_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi) = \delta_{i,i'}\delta_{n,n'}\delta_{j,j'} \tag{6.7}$$

for all (i,n,j) *and* $(i',n',j') \in \boldsymbol{J}$.

Proof. The spherical unit tensors $\xi \otimes \xi$, $\xi \otimes \tau_\pm$, $\tau_\pm \otimes \xi$, $\frac{1}{\sqrt{2}}\,\mathbf{i}_{\tan}$, $\frac{1}{\sqrt{2}}\,\mathbf{j}_{\tan}$, and $\tau_\pm \otimes \tau_\pm$ are orthonormal to each other with respect to the complex double dot product. This means for example that $(\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\pm \otimes \tau_\pm)} = 1$ and $(\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\mp \otimes \tau_\mp)} = 0$. Therefore, these new basis functions are orthonormal on the sphere such that

$$\int_\Omega \boldsymbol{y}^i_{n,j}(\xi) : \overline{\boldsymbol{y}^{i'}_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi) = \delta_{i,i'} \int_\Omega {}_{\boldsymbol{N}_i}Y_{n,j}(\xi) \ \overline{{}_{\boldsymbol{N}_i}Y_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi),$$

where

$$\boldsymbol{N}_i := \begin{cases} -2 & , i = 9 \\ -1 & , i = 3,5 \\ 0 & , i = 1,6,7 \\ +1 & , i = 2,4 \\ +2 & , i = 8 \end{cases}.$$

With Theorem 3.4.21, we obtain that

$$\int_\Omega \boldsymbol{y}^i_{n,j}(\xi) : \overline{\boldsymbol{y}^{i'}_{n',j'}(\xi)} \, \mathrm{d}\omega(\xi) = \delta_{i,i'}\delta_{n,n'}\delta_{j,j'}$$

for all (i,n,j) and $(i',n',j') \in \boldsymbol{J}$. □

Theorem 6.1.20 (Completeness and Parseval Identity for the Transformed Tensor Spherical Harmonics)**.** *The transformed tensor spherical harmonics form a complete orthonormal basis of* $\mathbf{l}^2(\Omega)$. *Consequently, for* $\boldsymbol{f} \in \mathbf{l}^2(\Omega)$, *we obtain*

$$\lim_{L\to\infty} \left\| \boldsymbol{f} - \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \langle \boldsymbol{f}, \boldsymbol{y}^i_{n,j} \rangle_{\mathbf{l}^2(\Omega)} \boldsymbol{y}^i_{n,j} \right\|_{\mathbf{l}^2(\Omega)} = 0.$$

This means that every function $\boldsymbol{f} \in \mathbf{l}^2(\Omega)$ *can be written uniquely in the* $\mathbf{l}^2(\Omega)$*-sense in terms* *of a Fourier series by*

$$\boldsymbol{f} = \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{\infty} \sum_{j=-n}^{n} \langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^i \rangle_{\mathbf{l}^2(\Omega)} \boldsymbol{y}_{n,j}^i.$$

Furthermore, for every function $\boldsymbol{f}, \boldsymbol{g} \in \mathbf{l}^2(\Omega)$*, the Parseval identity is fulfilled such that*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{\mathbf{l}^2(\Omega)} = \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{\infty} \sum_{j=-n}^{n} \langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^i \rangle_{\mathbf{l}^2(\Omega)} \overline{\langle \boldsymbol{g}, \boldsymbol{y}_{n,j}^i \rangle_{\mathbf{l}^2(\Omega)}}.$$

Proof. We know from Theorem 2.6.12 that the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27] form a complete orthonormal basis. We know already from (6.7) that the transformed tensor spherical harmonics of Section 6.1.5 are orthonormal. Now, we have to prove the completeness. Therefore, we have to show that for every function $\boldsymbol{f} \in \mathbf{l}^2(\Omega)$ with $\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^l \rangle_{\mathbf{l}^2(\Omega)} = 0$ for all $l = 1, \ldots, 9$, all $n \geq \mathbf{0}_l$, and all $j = -n, \ldots, n$ follows that $\boldsymbol{f} = 0$.

For $l = 1, 6, 7$, we know that $\boldsymbol{y}_{n,j}^1 = \boldsymbol{y}_{n,j}^{(1,1)}$, $\boldsymbol{y}_{n,j}^6 = \boldsymbol{y}_{n,j}^{(2,2)}$, and $\boldsymbol{y}_{n,j}^7 = \boldsymbol{y}_{n,j}^{(3,3)}$ for all $n = 0, \ldots, L$ and all $j = -n, \ldots, n$. With the completeness of the tensor spherical harmonics of Freeden, Gervens, and Schreiner, the proposition is clear.
For $l = 2, 3$, we know that $\boldsymbol{y}_{n,j}^l = \pm\frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(1,2)} \pm i\boldsymbol{y}_{n,j}^{(1,3)} \right)$, where we use "+" if $l = 2$ and "−" if $l = 3$ for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$. Then, we get for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$

$$0 = \langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^l \rangle_{\mathbf{l}^2(\Omega)} = \pm\frac{1}{\sqrt{2}} \left(- \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(1,2)} \right\rangle_{\mathbf{l}^2(\Omega)} \pm i \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(1,3)} \right\rangle_{\mathbf{l}^2(\Omega)} \right).$$

This leads to the system of linear equations

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(1,2)} \right\rangle_{\mathbf{l}^2(\Omega)} \\ \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(1,3)} \right\rangle_{\mathbf{l}^2(\Omega)} \end{pmatrix} = \begin{pmatrix} \langle f, \boldsymbol{y}_{n,j}^2 \rangle_{\mathbf{l}^2(\Omega)} \\ \langle f, \boldsymbol{y}_{n,j}^3 \rangle_{\mathbf{l}^2(\Omega)} \end{pmatrix}$$

for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$, where

$$\det \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = -\frac{i}{2} - \frac{i}{2} = -i \neq 0.$$

Therefore, we obtain

$$\left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(1,2)} \right\rangle_{\mathbf{l}^2(\Omega)} = 0$$

and

$$\left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(1,3)} \right\rangle_{\mathbf{l}^2(\Omega)} = 0.$$

Analogous, we get for $l = 4, 5$ with $\boldsymbol{y}_{n,j}^l = \pm\frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(2,1)} \pm i\boldsymbol{y}_{n,j}^{(3,1)} \right)$, where we use "+" if $l = 4$ and "−" if $l = 5$ for all $n = 1, \ldots, L$ and all $j = -n, \ldots, n$ the system of linear equations

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(2,1)} \right\rangle_{\mathbf{l}^2(\Omega)} \\ \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(3,1)} \right\rangle_{\mathbf{l}^2(\Omega)} \end{pmatrix} = \begin{pmatrix} \langle f, \boldsymbol{y}_{n,j}^4 \rangle_{\mathbf{l}^2(\Omega)} \\ \langle f, \boldsymbol{y}_{n,j}^5 \rangle_{\mathbf{l}^2(\Omega)} \end{pmatrix}$$

for all $n = 1, \dots, L$ and all $j = -n, \dots, n$, where

$$\det \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = -\frac{i}{2} - \frac{i}{2} = -i \neq 0.$$

Therefore, we obtain

$$\left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(2,1)} \right\rangle_{\mathbf{l}^2(\Omega)} = 0$$

and

$$\left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(3,1)} \right\rangle_{\mathbf{l}^2(\Omega)} = 0.$$

For $l = 8, 9$, we know that $\boldsymbol{y}_{n,j}^{l} = -\frac{1}{\sqrt{2}} \left(-\boldsymbol{y}_{n,j}^{(2,3)} \pm i \boldsymbol{y}_{n,j}^{(3,2)} \right)$, where we use "+" if $l = 8$ and "−" if $l = 9$ for all $n = 2, \dots, L$ and all $j = -n, \dots, n$. Then, we get for all $n = 2, \dots, L$ and all $j = -n, \dots, n$

$$0 = \langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{l} \rangle_{\mathbf{l}^2(\Omega)} = -\frac{1}{\sqrt{2}} \left(- \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(2,3)} \right\rangle_{\mathbf{l}^2(\Omega)} \pm i \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(3,2)} \right\rangle_{\mathbf{l}^2(\Omega)} \right).$$

This leads to the system of linear equations

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(2,3)} \right\rangle_{\mathbf{l}^2(\Omega)} \\ \left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(3,2)} \right\rangle_{\mathbf{l}^2(\Omega)} \end{pmatrix} = \begin{pmatrix} \langle f, \boldsymbol{y}_{n,j}^{8} \rangle_{\mathbf{l}^2(\Omega)} \\ \langle f, \boldsymbol{y}_{n,j}^{9} \rangle_{\mathbf{l}^2(\Omega)} \end{pmatrix}$$

for all $n = 2, \dots, L$ and all $j = -n, \dots, n$, where

$$\det \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{i}{2} + \frac{i}{2} = i \neq 0.$$

Therefore, we obtain

$$\left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(2,3)} \right\rangle_{\mathbf{l}^2(\Omega)} = 0$$

and

$$\left\langle \boldsymbol{f}, \boldsymbol{y}_{n,j}^{(3,2)} \right\rangle_{\mathbf{l}^2(\Omega)} = 0.$$

So, we receive

$$\left\langle f, \boldsymbol{y}_{n,j}^{(i,k)} \right\rangle_{\mathbf{l}^2(\Omega)} = 0$$

for all $i, k = 1, 2, 3$. We know from the completeness of the tensor spherical harmonics of Freeden, Gervens, and Schreiner that then $\boldsymbol{f} = 0$. So, the transformed tensor spherical harmonics are complete and form a orthonormal basis of $\mathbf{l}^2(\Omega)$. With the completeness, the Parseval identity follows directly. □

Problem 6.1.21. *Then, the concentration problem leads to the eigenvalue problem*

$$\boldsymbol{k}\boldsymbol{g} = \lambda \boldsymbol{g},$$

where

$$\boldsymbol{k} := \begin{pmatrix} \boldsymbol{k}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{k}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{k}^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{k}^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{k}^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{k}^9 \end{pmatrix}$$

$$:= \begin{pmatrix} K^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K^{+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K^{+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K^{+2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K^{-2} \end{pmatrix} \in \mathbb{R}^{\left[9(L+1)^2-12\right]\times\left[9(L+1)^2-12\right]}$$

and

$$\boldsymbol{k}^{ii}_{nj,n'j'} := \boldsymbol{k}^{i}_{nj,n'j'} := \int_R \overline{\boldsymbol{y}^i_{n,j}(\xi)} : \boldsymbol{y}^i_{n',j'}(\xi)\ \mathrm{d}\omega(\xi),$$

because $\boldsymbol{k}^{ii'}_{nj,n'j'} = \delta_{i,i'}\boldsymbol{k}^{ii}_{nj,n'j'}$ *for all* $(i,n,j),(i',n',j') \in \boldsymbol{J}$.

Then, we see that

$$\sum_{i'=1}^{9}\sum_{n'=\mathbf{0}_{i'}}^{L}\sum_{j'=-n'}^{n'} \boldsymbol{k}^{ii'}_{nj,n'j'}\boldsymbol{g}^{i'}_{n',j'} = \lambda \boldsymbol{g}^i_{n,j} \tag{6.8}$$

for all $(i,n,j) \in \boldsymbol{J}$.

Upon multiplying by $\boldsymbol{y}^i_{n,j}(\eta)$, $\eta \in \Omega$, summing over all $(i,n,j) \in \boldsymbol{J}$, and interchanging summation and integration, we obtain

$$\int_R \sum_{i=1}^{9}\sum_{n=\mathbf{0}_i}^{L}\sum_{j=-n}^{n}\sum_{i'=1}^{9}\sum_{n'=\mathbf{0}_{i'}}^{L}\sum_{j'=-n'}^{n'} \boldsymbol{y}^i_{n,j}(\eta)\left(\overline{\boldsymbol{y}^i_{n,j}(\xi)} : \boldsymbol{y}^{i'}_{n',j'}(\xi)\right)\boldsymbol{g}^{i'}_{n',j'}\ \mathrm{d}\omega(\xi)$$
$$= \lambda \sum_{i=1}^{9}\sum_{n=\mathbf{0}_i}^{L}\sum_{j=-n}^{n} \boldsymbol{g}^i_{n,j}\boldsymbol{y}^i_{n,j}(\eta).$$

With Lemma 2.2.9, this leads to

$$\int_R \sum_{i=1}^{9}\sum_{n=\mathbf{0}_i}^{L}\sum_{j=-n}^{n}\left(\boldsymbol{y}^i_{n,j}(\eta)\otimes\overline{\boldsymbol{y}^i_{n,j}(\xi)}\right) : \sum_{i'=1}^{9}\sum_{n'=\mathbf{0}_{i'}}^{L}\sum_{j'=-n'}^{n'} \boldsymbol{g}^{i'}_{n',j'}\boldsymbol{y}^{i'}_{n',j'}(\xi)\ \mathrm{d}\omega(\xi)$$
$$= \lambda \sum_{i=1}^{9}\sum_{n=\mathbf{0}_i}^{L}\sum_{j=-n}^{n} \boldsymbol{g}^i_{n,j}\boldsymbol{y}^i_{n,j}(\eta).$$

Then, we conclude from Problem 6.1.21 the following integral equation.

Problem 6.1.22. *Then, we get the integral equation*

$$\int_R \boldsymbol{k}(\xi,\eta) : \boldsymbol{g}(\xi)\ \mathrm{d}\omega(\xi) = \lambda \boldsymbol{g}(\eta) \tag{6.9}$$

for all $\eta \in \Omega$ *with the kernel function*

$$\boldsymbol{k}(\xi,\eta) = \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \boldsymbol{y}^i_{n,j}(\eta) \otimes \overline{\boldsymbol{y}^i_{n,j}(\xi)}.$$

Altogether, with our previous results and with $d := (L+1)^2$ the orthonormal eigenvectors of Problem 6.1.21 are given by

$$\boldsymbol{g}_\alpha = \begin{cases} \boldsymbol{g}^1_\alpha & ,\alpha = 1,\ldots,d \\ \boldsymbol{g}^2_\alpha & ,\alpha = d+1,\ldots,2d-1 \\ \boldsymbol{g}^3_\alpha & ,\alpha = 2d,\ldots,3d-2 \\ \boldsymbol{g}^4_\alpha & ,\alpha = 3d-1,\ldots,4d-3 \\ \boldsymbol{g}^5_\alpha & ,\alpha = 4d-2,\ldots,5d-4 \\ \boldsymbol{g}^6_\alpha & ,\alpha = 5d-3,\ldots,6d-4 \\ \boldsymbol{g}^7_\alpha & ,\alpha = 6d-3,\ldots,7d-4 \\ \boldsymbol{g}^8_\alpha & ,\alpha = 7d-3,\ldots,8d-8 \\ \boldsymbol{g}^9_\alpha & ,\alpha = 8d-7,\ldots,9d-12 \end{cases},$$

where

$$\boldsymbol{g}^1_\alpha := \begin{pmatrix} G^0_\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad ,\alpha = 1,\ldots,d,$$

$$\boldsymbol{g}^2_\alpha := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{+1}_{\alpha-d} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad ,\alpha = d+1,\ldots,2d-1,$$

$$\boldsymbol{g}^3_\alpha := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{-1}_{\alpha-(2d-1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad ,\alpha = 2d,\ldots,3d-2,$$

$$\boldsymbol{g}_\alpha^4 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{+1}_{\alpha-(3d-2)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad , \alpha = 3d-1, \ldots, 4d-3,$$

$$\boldsymbol{g}_\alpha^5 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{-1}_{\alpha-(4d-3)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad , \alpha = 4d-2, \ldots, 5d-4,$$

$$\boldsymbol{g}_\alpha^6 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{0}_{\alpha-(5d-4)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad , \alpha = 5d-3, \ldots, 6d-4,$$

$$\boldsymbol{g}_\alpha^7 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{0}_{\alpha-(6d-4)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad , \alpha = 6d-3, \ldots, 7d-4,$$

$$\boldsymbol{g}_\alpha^8 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{+2}_{\alpha-(7d-4)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad , \alpha = 7d-3, \ldots, 8d-8,$$

$$\boldsymbol{g}_\alpha^9 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G^{-2}_{\alpha-(8d-8)} \end{pmatrix} \qquad , \alpha = 8d-7, \ldots, 9d-12.$$

Note that G_α^N are also vectors, where $G_\alpha^N \in \mathbb{R}^{d-N^2}$.

With

$$\left(\boldsymbol{\mathscr{g}}_{\mathrm{nor,nor}}\right)_\alpha(\xi) = (\xi \otimes \xi)\, \mathscr{G}_\alpha^0(\xi),$$

$$\left(\boldsymbol{g}^{(1,\pm)}_{\mathrm{nor,tan}}\right)_\alpha(\xi) = (\xi \otimes \tau_\pm)\, \mathscr{G}^{\pm 1}_\alpha(\xi),$$
$$\left(\boldsymbol{g}^{(\pm,1)}_{\mathrm{tan,nor}}\right)_\alpha(\xi) = (\tau_\pm \otimes \xi)\, \mathscr{G}^{\pm 1}_\alpha(\xi),$$
$$\left(\boldsymbol{g}^{(2,2)}_{\mathrm{tan,tan}}\right)_\alpha(\xi) = \frac{1}{\sqrt{2}}\, \mathbf{i}_{\mathrm{tan}} \mathscr{G}^{0}_\alpha(\xi),$$
$$\left(\boldsymbol{g}^{(3,3)}_{\mathrm{tan,tan}}\right)_\alpha(\xi) = \frac{1}{\sqrt{2}}\, \mathbf{j}_{\mathrm{tan}} \mathscr{G}^{0}_\alpha(\xi),$$
$$\left(\boldsymbol{g}^{(\pm,\pm)}_{\mathrm{tan,tan}}\right)_\alpha(\xi) = (\tau_\pm \otimes \tau_\pm)\, \mathscr{G}^{\pm 2}_\alpha(\xi),$$

the tensor Slepian functions are given by

$$\boldsymbol{g}_\alpha(\xi) = \begin{cases} \boldsymbol{g}^1_\alpha(\xi) := (\xi \otimes \xi)\, \mathscr{G}^0_\alpha(\xi) & , \alpha = 1, \dots, d \\ \boldsymbol{g}^2_\alpha(\xi) := (\xi \otimes \tau_+)\, \mathscr{G}^{+1}_{\alpha-d}(\xi) & , \alpha = d+1, \dots, 2d-1 \\ \boldsymbol{g}^3_\alpha(\xi) := (\xi \otimes \tau_-)\, \mathscr{G}^{-1}_{\alpha-(2d-1)}(\xi) & , \alpha = 2d, \dots, 3d-2 \\ \boldsymbol{g}^4_\alpha(\xi) := (\tau_+ \otimes \xi)\, \mathscr{G}^{+1}_{\alpha-(3d-2)}(\xi) & , \alpha = 3d-1, \dots, 4d-3 \\ \boldsymbol{g}^5_\alpha(\xi) := (\tau_- \otimes \xi)\, \mathscr{G}^{-1}_{\alpha-(4d-3)}(\xi) & , \alpha = 4d-2, \dots, 5d-4 \\ \boldsymbol{g}^6_\alpha(\xi) := \frac{1}{\sqrt{2}}\, \mathbf{i}_{\mathrm{tan}} \mathscr{G}^{0}_{\alpha-(5d-4)}(\xi) & , \alpha = 5d-3, \dots, 6d-4 \\ \boldsymbol{g}^7_\alpha(\xi) := \frac{1}{\sqrt{2}}\, \mathbf{j}_{\mathrm{tan}} \mathscr{G}^{0}_{\alpha-(6d-4)}(\xi) & , \alpha = 6d-3, \dots, 7d-4 \\ \boldsymbol{g}^8_\alpha(\xi) := (\tau_+ \otimes \tau_+)\, \mathscr{G}^{+2}_{\alpha-(7d-4)}(\xi) & , \alpha = 7d-3, \dots, 8d-8 \\ \boldsymbol{g}^9_\alpha(\xi) := (\tau_- \otimes \tau_-)\, \mathscr{G}^{-2}_{\alpha-(8d-8)}(\xi) & , \alpha = 8d-7, \dots, 9d-12 \end{cases}$$

and the associated eigenvalues by

$$\lambda_\alpha = \begin{cases} \lambda^0_\alpha & , \alpha = 1, \dots, d \\ \lambda^{+1}_{\alpha-d} & , \alpha = d+1, \dots, 2d-1 \\ \lambda^{-1}_{\alpha-(2d-1)} & , \alpha = 2d, \dots, 3d-2 \\ \lambda^{+1}_{\alpha-(3d-2)} & , \alpha = 3d-1, \dots, 4d-3 \\ \lambda^{-1}_{\alpha-(4d-3)} & , \alpha = 4d-2, \dots, 5d-4 \\ \lambda^{0}_{\alpha-(5d-4)} & , \alpha = 5d-3, \dots, 6d-4 \\ \lambda^{0}_{\alpha-(6d-4)} & , \alpha = 6d-3, \dots, 7d-4 \\ \lambda^{+2}_{\alpha-(7d-4)} & , \alpha = 7d-3, \dots, 8d-8 \\ \lambda^{-2}_{\alpha-(8d-8)} & , \alpha = 8d-7, \dots, 9d-12 \end{cases}$$

for $\alpha = 1, \dots, 9(L+1)^2 - 12$.

6.2 Properties

Now, we know how to calculate the eigenvalues, eigenvectors, and eigenfunctions of the concentration problem, thus the tensor Slepian functions on the sphere. Next, we look at the properties of the tensor Slepian functions.

Theorem 6.2.1. *The tensor Slepian functions and their eigenvectors are orthonormal on the unit sphere and orthogonal on the region of interest R. This means that*

$$\sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{g}^i_{n,j}\right)_\alpha \overline{\left(\boldsymbol{g}^i_{n,j}\right)_\beta} = \delta_{\alpha,\beta}, \tag{6.10}$$

$$\sum_{i=1}^{9}\sum_{n=\mathbf{0}_i}^{L}\sum_{j=-n}^{n}\sum_{i'=1}^{9}\sum_{n'=\mathbf{0}_{i'}}^{L}\sum_{j'=-n'}^{n'}\left(\boldsymbol{g}_{n,j}^{i}\right)_{\alpha}\overline{\boldsymbol{k}_{nj,n'j'}^{ii'}}\ \overline{\left(\boldsymbol{g}_{n',j'}^{i'}\right)_{\beta}}=\lambda_{\alpha}\delta_{\alpha,\beta}, \tag{6.11}$$

$$\langle \boldsymbol{g}_{\alpha},\boldsymbol{g}_{\beta}\rangle_{\mathbf{l}^2(\Omega)}=\delta_{\alpha,\beta}, \tag{6.12}$$

$$\langle \boldsymbol{g}_{\alpha},\boldsymbol{g}_{\beta}\rangle_{\mathbf{l}^2(R)}=\lambda_{\alpha}\delta_{\alpha,\beta} \tag{6.13}$$

for all $\alpha,\beta=1,\ldots,9(L+1)^2-12$.

Theorem 6.2.2. *The tensor Slepian functions* $\{\boldsymbol{g}_{\alpha}\}_{\alpha=1,\ldots,3(L+1)^2-2}$ *form a complete orthonormal basis system of* $\left(\mathbf{harm}_{0\ldots l}(\Omega),\langle\cdot,\cdot\rangle_{\mathbf{l}^2(\Omega)}\right)$ *and therefore, we can write every by L bandlimited tensor field* $\boldsymbol{f}\in\mathbf{l}^2(\Omega)$ *in the basis of the transformed tensor spherical harmonics and in the basis of the tensor Slepian functions. This means that for* $\xi\in\Omega$

$$\begin{aligned}\boldsymbol{f}(\xi)&=\sum_{i=1}^{9}\sum_{n=\mathbf{0}_i}^{L}\sum_{j=-n}^{n}\underbrace{\langle\boldsymbol{f},\boldsymbol{y}_{n,j}^{i}\rangle_{\mathbf{l}^2(\Omega)}}_{=:\boldsymbol{f}_{n,j}^{i}}\boldsymbol{y}_{n,j}^{i}(\xi)\\&=\sum_{\alpha=1}^{9(L+1)^2-12}\underbrace{\langle\boldsymbol{f},\boldsymbol{g}_{\alpha}\rangle_{\mathbf{l}^2(\Omega)}}_{=:\boldsymbol{f}_{\alpha}}\boldsymbol{g}_{\alpha}(\xi).\end{aligned}$$

Theorem 6.2.3. *We can also write the transformed tensor spherical harmonics in the basis of the tensor Slepian functions*

$$\boldsymbol{y}_{n,j}^{i}=\sum_{\alpha=1}^{9(L+1)^2-12}\overline{\left(\boldsymbol{g}_{n,j}^{i}\right)_{\alpha}}\boldsymbol{g}_{\alpha}, \tag{6.14}$$

$$\sum_{\alpha=1}^{9(L+1)^2-12}\left(\boldsymbol{g}_{n,j}^{i}\right)_{\alpha}\overline{\left(\boldsymbol{g}_{n',j'}^{i'}\right)_{\alpha}}=\delta_{i,i'}\delta_{n,n'}\delta_{j,j'}. \tag{6.15}$$

Theorem 6.2.4. *The tensor Slepian functions also fulfill the following properties*

$$\sum_{\alpha=1}^{9(L+1)^2-12}\lambda_{\alpha}\left(\boldsymbol{g}_{n,j}^{i}\right)_{\alpha}\overline{\left(\boldsymbol{g}_{n',j'}^{i'}\right)_{\alpha}}=\boldsymbol{k}_{nj,n'j'}^{ii'},$$

$$\sum_{\alpha=1}^{9(L+1)^2-12}\lambda_{\alpha}\boldsymbol{g}_{\alpha}(\xi)\overline{\boldsymbol{g}_{\alpha}(\eta)}=\sum_{i=1}^{9}\sum_{n=\mathbf{0}_i}^{L}\sum_{j=-n}^{n}\sum_{i'=1}^{9}\sum_{n'=\mathbf{0}_{i'}}^{L}\sum_{j'=-n'}^{n'}\boldsymbol{y}_{n,j}^{i}(\xi)\boldsymbol{k}_{nj,n'j'}^{ii'}\overline{\boldsymbol{y}_{n',j'}^{i'}(\eta)}$$

for all $(i,n,j),(i',n',j')\in\boldsymbol{J}$ *and all* $\xi,\eta\in\Omega$ *and*

$$\boldsymbol{k}(\xi,\eta)=\sum_{\alpha=1}^{9(L+1)^2-12}\overline{\boldsymbol{g}_{\alpha}(\xi)}\boldsymbol{g}_{\alpha}(\eta).$$

The proofs are analogous to the proofs of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, and Theorem 4.2.4. The only difference occurs in the additional indices $i,i'=1,\ldots,9$, over which we summate and the starting point of the summation of n and n'. For the completeness of the tensor Slepian functions, we need the completeness of the transformed tensor spherical harmonics from Theorem 6.1.20. Then, we get the properties simultaneously.

We choose the tensor Slepian functions to be sorted by the eigenvalues like $\lambda_1\geq\lambda_2\geq$

$\cdots \geq \lambda_{9(L+1)^2-12}$ just as before, for the scalar and the vector Slepian functions. Here, we also get from numerical experiments that there are often only eigenvalues $\lambda \approx 1$ and $\lambda \approx 0$. Therefore, we calculate the Shannon number for the tensor case in the following section.

6.3 Shannon Number

For the tensor Slepian functions on the sphere, the Shannon number S, as previously stated in Chapter 4.3, is a good estimate for significant eigenvalues, the eigenvalues $\lambda \approx 1$. Therefore, S gives (approximately) the dimension of the space of signals that are bandlimited by L and optimally concentrated in R at the same time. This space has as basis the eigenfunctions $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_S$.

Lemma 6.3.1. *The Shannon number of the tensor Slepian functions on the sphere is given by*

$$S = \left(9(L+1)^2 - 12\right) \frac{A}{4\pi},$$

where A denotes the area of the region R on the unit sphere Ω.

Proof. With Corollary 3.4.25 and with Remark 3.2.2, we get that

$$\begin{aligned}
S &= \sum_{\alpha=1}^{9(L+1)^2-12} \lambda_\alpha = \operatorname{tr}(\boldsymbol{k}) \\
&= \sum_{n=0}^{L}\sum_{j=-n}^{n} K^{0}_{nj,nj} + \sum_{n=1}^{L}\sum_{j=-n}^{n} \left(K^{+1}_{nj,nj} + K^{-1}_{nj,nj}\right) + \sum_{n=1}^{L}\sum_{j=-n}^{n} \left(K^{+1}_{nj,nj} + K^{-1}_{nj,nj}\right) \\
&\quad + \left[\sum_{n=0}^{L}\sum_{j=-n}^{n} K^{0}_{nj,nj} + \sum_{n=0}^{L}\sum_{j=-n}^{n} K^{0}_{nj,nj} + \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(K^{+2}_{nj,nj} + K^{-2}_{nj,nj}\right)\right] \\
&= \int_R \Bigg(\sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\xi) + \sum_{n=1}^{L}\sum_{j=-n}^{n} \left(\overline{{}_{+1}Y_{n,j}(\xi)}\ {}_{+1}Y_{n,j}(\xi) + \overline{{}_{-1}Y_{n,j}(\xi)}\ {}_{-1}Y_{n,j}(\xi)\right) \\
&\quad + \sum_{n=1}^{L}\sum_{j=-n}^{n} \left(\overline{{}_{+1}Y_{n,j}(\xi)}\ {}_{+1}Y_{n,j}(\xi) + \overline{{}_{-1}Y_{n,j}(\xi)}\ {}_{-1}Y_{n,j}(\xi)\right) + \sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\xi) \\
&\quad + \sum_{n=0}^{L}\sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\xi) + \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\overline{{}_{+2}Y_{n,j}(\xi)}\ {}_{+2}Y_{n,j}(\xi) + \overline{{}_{-2}Y_{n,j}(\xi)}\ {}_{-2}Y_{n,j}(\xi)\right)\Bigg) \,\mathrm{d}\omega(\xi) \\
&= \frac{1}{4\pi}\left(3\sum_{n=0}^{L}(2n+1) + 2\cdot 2\sum_{n=1}^{L}(2n+1) + 2\sum_{n=2}^{L}(2n+1)\right)\int_R \mathrm{d}\omega(\xi) \\
&= \left(9(L+1)^2 - 12\right)\frac{A}{4\pi}.
\end{aligned}$$

□

Like for the scalar and the vector case, it is obvious that, for $A \ll 4\pi$, we get $S \ll 9(L+1)^2 - 12$ and, for $A \approx 4\pi$, we get $S \approx 9(L+1)^2 - 12$.

Chapter 7

Spectral Concentration of Spacelimited Functions

We know already that no function can be strictly bandlimited and strictly spacelimited at the same time. Previously, we looked at the spatial concentration of bandlimited functions. Now, we want to do the same for the opposite problem, the spectral concentration of spacelimited functions. For this purpose, we formulate a new concentration problem. First, we define a spacelimited field.

7.1 Spacelimited Scalar Fields

Again, we start with scalar fields. Here, we follow mainly [11, 82].

Definition 7.1.1. *A on a region $R \subset \Omega$ spacelimited scalar field $\mathscr{H} \in \mathrm{L}^2(\Omega)$ is given by*

$$\mathscr{H} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} H_{n,j} Y_{n,j}$$

with $\mathscr{H} = 0$ on $\Omega \backslash R$ and

$$H_{n,j} = \int_R \mathscr{H}(\xi) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi).$$

for $n \geq 0$ and $j = -n, \ldots, n$.

Problem 7.1.2. *We get the concentration problem for every by R spacelimited scalar field $\mathscr{H} \in \mathrm{L}^2(\Omega)$ spatially optimally concentrated within a spectral interval $0 \leq n \leq L$ by*

$$\lambda = \frac{\sum_{n=0}^{L} \sum_{j=-n}^{n} |H_{n,j}|^2}{\sum_{n=0}^{\infty} \sum_{j=-n}^{n} |H_{n,j}|^2} = \max .$$

Then, we obtain

$$\begin{aligned}\sum_{n=0}^{L} \sum_{j=-n}^{n} |H_{n,j}|^2 &= \sum_{n=0}^{L} \sum_{j=-n}^{n} H_{n,j} \overline{H_{n,j}} \\ &= \sum_{n=0}^{L} \sum_{j=-n}^{n} \int_R \mathscr{H}(\xi) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \int_R \overline{\mathscr{H}(\eta)} Y_{n,j}(\eta) \, \mathrm{d}\omega(\eta)\end{aligned}$$

$$= \int_R \int_R \mathscr{H}(\xi) \underbrace{\left(\sum_{n=0}^{L} \sum_{j=-n}^{n} \overline{Y_{n,j}(\xi)} Y_{n,j}(\eta) \right)}_{=\overline{\mathscr{K}(\eta,\xi)}=\mathscr{K}(\xi,\eta)} \overline{\mathscr{H}(\eta)} \, \mathrm{d}\omega(\xi) \, \mathrm{d}\omega(\eta).$$

With Theorem 3.6.14, the Parseval identity for spin-weighted spherical harmonics, and with $\mathscr{H} = 0$ on $\Omega \backslash R$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} |H_{n,j}|^2 &= \langle \mathscr{H}, \mathscr{H} \rangle_{\mathrm{L}^2(\Omega)} \\ &= \int_\Omega \mathscr{H}(\xi) \overline{\mathscr{H}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= \int_R \mathscr{H}(\xi) \overline{\mathscr{H}(\xi)} \, \mathrm{d}\omega(\xi). \end{aligned}$$

So, the concentration problem can be written by

$$\lambda = \frac{\int_R \int_R \mathscr{H}(\xi) \mathscr{K}(\xi,\eta) \overline{\mathscr{H}(\eta)} \, \mathrm{d}\omega(\xi) \, \mathrm{d}\omega(\eta)}{\int_R \mathscr{H}(\xi) \overline{\mathscr{H}(\xi)} \, \mathrm{d}\omega(\xi)} = \max .$$

Problem 7.1.3. *Therefore, we get the integral equation*

$$\int_R \mathscr{K}(\xi,\eta) \mathscr{H}(\xi) \, \mathrm{d}\omega(\xi) = \lambda \mathscr{H}(\eta), \quad \eta \in R,$$

with the kernel function

$$\mathscr{K}(\xi,\eta) = \sum_{n=0}^{L} \sum_{j=-n}^{n} Y_{n,j}(\eta) \overline{Y_{n,j}(\xi)}.$$

So, we see with (4.2) that the spatially concentrated bandlimited scalar Slepian functions $\mathscr{G}$ and the spectrally concentrated spacelimited scalar Slepian functions $\mathscr{H}$ are related by

$$\mathscr{H}(\xi) = \begin{cases} \mathscr{G}(\xi), & \xi \in R \\ 0, & \text{else} \end{cases} . \tag{7.1}$$

Then, we get for the coefficients with (4.3) for $n \geq 0$ and $j = -n, \ldots, n$

$$\begin{aligned} H_{n,j} &= \int_R \mathscr{H}(\xi) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= \int_R \mathscr{G}(\xi) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi) \\ &= \sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} G_{n',j'} \underbrace{\int_R Y_{n',j'}(\xi) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi)}_{=K_{nj,n'j'}} . \end{aligned}$$

So, we obtain for $n \geq 0$

$$H_{n,j} = \sum_{n'=0}^{L} \sum_{j'=-n'}^{n'} K_{nj,n'j'} G_{n',j'}$$

and in particular with (4.4) for $0 \leq n \leq L$

$$H_{n,j} = \lambda G_{n,j}$$

for all $j = -n, \ldots, n$.

Furthermore, we can formulate the following theorem.

Theorem 7.1.4. *The solutions, which are the spectrally concentrated spacelimited scalar Slepian functions $\mathscr{H}_\alpha$ for $\alpha = 1, \ldots, (L+1)^2$, are orthogonal on the unit sphere Ω and on the region R. This means that*

$$\int_\Omega \mathscr{H}_\alpha(\xi)\overline{\mathscr{H}_\beta(\xi)}\, \mathrm{d}\omega(\xi) = \int_R \mathscr{H}_\alpha(\xi)\overline{\mathscr{H}_\beta(\xi)}\, \mathrm{d}\omega(\xi) = \lambda_\alpha \delta_{\alpha,\beta}.$$

Proof. We know that $\mathscr{H} = 0$ on $\Omega \backslash R$. Then, we get with (7.1) and with (4.8) the proposition

$$\begin{aligned} \int_\Omega \mathscr{H}_\alpha(\xi)\overline{\mathscr{H}_\beta(\xi)}\, \mathrm{d}\omega(\xi) &= \int_R \mathscr{H}_\alpha(\xi)\overline{\mathscr{H}_\beta(\xi)}\, \mathrm{d}\omega(\xi) \\ &= \int_R \mathscr{G}_\alpha(\xi)\overline{\mathscr{G}_\beta(\xi)}\, \mathrm{d}\omega(\xi) \\ &= \lambda_\alpha \delta_{\alpha,\beta}. \end{aligned}$$

□

7.2 Spacelimited Vector Fields

Now, we continue with the spectral concentration of spacelimited vector fields. For the vector spherical harmonics of Hill [40], this has previously been investigated by [67], which we follow here mainly for the transformed vector spherical harmonics. We see that it is analogous to the scalar case.

Definition 7.2.1. *A on a region $R \subset \Omega$ spacelimited vector field $h \in \mathrm{l}^2(\Omega)$ is given by*

$$h = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=-n}^{n} h^i_{n,j} y^i_{n,j}$$

with $h = 0$ on $\Omega \backslash R$ and

$$h^i_{n,j} = \int_R h(\xi) \cdot \overline{y^i_{n,j}(\xi)}\, \mathrm{d}\omega(\xi).$$

for $i = 1, \ldots, 3$, $n \geq 0_i$, and $j = -n, \ldots, n$.

Problem 7.2.2. *We get the concentration problem for every by R spacelimited vector field $h \in \mathrm{l}^2(\Omega)$ spatially optimally concentrated within a spectral interval $0_i \leq n \leq L$ by*

$$\lambda = \frac{\sum_{i=1}^3 \sum_{n=0_i}^L \sum_{j=-n}^n |h^i_{n,j}|^2}{\sum_{i=1}^3 \sum_{n=0_i}^\infty \sum_{j=-n}^n |h^i_{n,j}|^2} = \max.$$

Then, we obtain with Lemma 2.2.8

$$\sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} |h^i_{n,j}|^2 = \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} h^i_{n,j} \overline{h^i_{n,j}}$$

$$= \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \int_R h(\xi) \cdot \overline{y_{n,j}^i(\xi)} \, d\omega(\xi) \int_R \overline{h(\eta)} \cdot y_{n,j}^i(\eta) \, d\omega(\eta)$$

$$= \int_R \int_R h^{\mathrm{T}}(\xi) \underbrace{\left(\sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} \overline{y_{n,j}^i(\xi)} \otimes y_{n,j}^i(\eta) \right)}_{=k^{\mathrm{T}}(\xi,\eta) = \overline{k(\eta,\xi)}} \overline{h(\eta)} \, d\omega(\xi) \, d\omega(\eta).$$

With Theorem 5.1.11, the Parseval identity for the transformed vector spherical harmonics, and with $h = 0$ on $\Omega \backslash R$, we get in analogy to the scalar case

$$\sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=-n}^{n} |h_{n,j}^i|^2 = \int_R h(\xi) \cdot \overline{h(\xi)} \, d\omega(\xi).$$

So, the concentration problem can be written by

$$\lambda = \frac{\int_R \int_R h^{\mathrm{T}}(\xi) k^{\mathrm{T}}(\xi,\eta) \overline{h(\eta)} \, d\omega(\xi) \, d\omega(\eta)}{\int_R h^{\mathrm{T}}(\xi) \overline{h(\xi)} \, d\omega(\xi)} = \max .$$

Problem 7.2.3. *Therefore, we get the integral equation*

$$\int_R k(\xi,\eta) h(\xi) \, d\omega(\xi) = \lambda h(\eta), \quad \eta \in R,$$

with the kernel function

$$k(\xi,\eta) = \sum_{i=1}^{3} \sum_{n=0_i}^{L} \sum_{j=-n}^{n} y_{n,j}^i(\eta) \otimes \overline{y_{n,j}^i(\xi)}.$$

So, we obtain the same results like for the scalar case. With (5.6), the spatially concentrated bandlimited vector Slepian functions g and the spectrally concentrated spacelimited vector Slepian functions h are related by

$$h(\xi) = \begin{cases} g(\xi), & \xi \in R \\ 0, & \text{else} \end{cases} .$$

With (5.3), we get for $n \geq 0_i$

$$h_{n,j}^i = \sum_{i'=1}^{3} \sum_{n'=0_{i'}}^{L} \sum_{j'=-n'}^{n'} k_{nj,n'j'}^{ii'} g_{n',j'}^{i'}$$

and in particalur, with (5.5), the coefficients for $0_i \leq n \leq L$ are related by

$$h_{n,j}^i = \lambda g_{n,j}^i,$$

where $i = 1, 2, 3$, $j = -n, \ldots, n$, and

$$k_{nj,n'j'}^{ii'} = \int_R y_{n',j'}^{i'}(\xi) \cdot \overline{y_{n,j}^i(\xi)} \, d\omega(\xi).$$

Theorem 7.2.4. *The solutions, which are the spectrally concentrated spacelimited vector*

Slepian functions $\mathfrak{h}_\alpha$ for $\alpha = 1, \ldots, 3(L+1)^2 - 2$, are orthogonal on the unit sphere Ω and on the region R. This means that

$$\int_\Omega \mathfrak{h}_\alpha(\xi) \cdot \overline{\mathfrak{h}_\beta(\xi)}\, \mathrm{d}\omega(\xi) = \int_R \mathfrak{h}_\alpha(\xi) \cdot \overline{\mathfrak{h}_\beta(\xi)}\, \mathrm{d}\omega(\xi) = \lambda_\alpha \delta_{\alpha,\beta}.$$

The proof is analogous to the proof of Theorem 7.1.4.

7.3 Spacelimited Tensor Fields

Now, as a new result, we investigate the spectral concentration of spacelimited tensor fields. This is also analogous to the scalar case.

Definition 7.3.1. *A on a region $R \subset \Omega$ spacelimited tensor field $\boldsymbol{\mathfrak{h}} \in \mathbf{l}^2(\Omega)$ is given by*

$$\boldsymbol{\mathfrak{h}} = \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{\infty} \sum_{j=-n}^{n} \boldsymbol{h}_{n,j}^i \boldsymbol{y}_{n,j}^i$$

with $\boldsymbol{\mathfrak{h}} = 0$ on $\Omega \backslash R$ and

$$\boldsymbol{h}_{n,j}^i = \int_R \boldsymbol{\mathfrak{h}}(\xi) : \overline{\boldsymbol{y}_{n,j}^i(\xi)}\, \mathrm{d}\omega(\xi).$$

for $i = 1, \ldots, 9$, $n \geq \mathbf{0}_i$, and $j = -n, \ldots, n$.

Problem 7.3.2. *We get the concentration problem for every by R spacelimited tensor field $\boldsymbol{\mathfrak{h}} \in \mathbf{l}^2(\Omega)$ spatially optimally concentrated within a spectral interval $\mathbf{0}_i \leq n \leq L$ by*

$$\lambda = \frac{\sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} |\boldsymbol{h}_{n,j}^i|^2}{\sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{\infty} \sum_{j=-n}^{n} |\boldsymbol{h}_{n,j}^i|^2} = \max.$$

Then, we obtain with Lemma 2.2.9

$$\begin{aligned}
\sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} |\boldsymbol{h}_{n,j}^i|^2 &= \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \boldsymbol{h}_{n,j}^i \overline{\boldsymbol{h}_{n,j}^i} \\
&= \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \int_R \boldsymbol{\mathfrak{h}}(\xi) : \overline{\boldsymbol{y}_{n,j}^i(\xi)}\, \mathrm{d}\omega(\xi) \int_R \overline{\boldsymbol{\mathfrak{h}}(\eta)} : \boldsymbol{y}_{n,j}^i(\eta)\, \mathrm{d}\omega(\eta) \\
&= \int_R \int_R \boldsymbol{\mathfrak{h}}(\xi) : \Bigg[\underbrace{\left(\sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \overline{\boldsymbol{y}_{n,j}^i(\xi)} \otimes \boldsymbol{y}_{n,j}^i(\eta) \right)}_{=\overline{\boldsymbol{\mathfrak{k}}(\eta,\xi)}} : \overline{\boldsymbol{\mathfrak{h}}(\eta)} \Bigg]\, \mathrm{d}\omega(\xi)\, \mathrm{d}\omega(\eta).
\end{aligned}$$

With Theorem 6.1.20, the Parseval identity for the transformed tensor spherical harmonics, and with $\boldsymbol{\mathfrak{h}} = 0$ on $\Omega \backslash R$, we get in analogy to the scalar and the vector case

$$\sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{\infty} \sum_{j=-n}^{n} |\boldsymbol{h}_{n,j}^i|^2 = \int_R \boldsymbol{\mathfrak{h}}(\xi) : \overline{\boldsymbol{\mathfrak{h}}(\xi)}\, \mathrm{d}\omega(\xi).$$

So, the concentration problem can be written by

$$\lambda = \frac{\int_R \int_R \boldsymbol{h}(\xi) : \left[\overline{\boldsymbol{k}(\eta,\xi)} : \overline{\boldsymbol{h}(\eta)}\right] \, \mathrm{d}\omega(\xi) \, \mathrm{d}\omega(\eta)}{\int_R \boldsymbol{h}(\xi) : \overline{\boldsymbol{h}(\xi)} \, \mathrm{d}\omega(\xi)} = \max .$$

Problem 7.3.3. *Therefore, we get the integral equation*

$$\int_R \boldsymbol{k}(\xi,\eta) : \boldsymbol{h}(\xi) \, \mathrm{d}\omega(\xi) = \lambda \boldsymbol{h}(\eta), \quad \eta \in R,$$

with the kernel function

$$\boldsymbol{k}(\xi,\eta) = \sum_{i=1}^{9} \sum_{n=\mathbf{0}_i}^{L} \sum_{j=-n}^{n} \boldsymbol{y}^i_{n,j}(\eta) \otimes \overline{\boldsymbol{y}^i_{n,j}(\xi)}.$$

So, we obtain the same results like for the scalar and the vector case. With (6.9), the spatially concentrated bandlimited tensor Slepian functions and the spectrally concentrated spacelimited tensor Slepian functions are related by

$$\boldsymbol{h}(\xi) = \begin{cases} \boldsymbol{g}(\xi), & \xi \in R \\ 0, & \text{else} \end{cases}.$$

With (6.6), we get for $n \geq \mathbf{0}_i$

$$\boldsymbol{h}^i_{n,j} = \sum_{i'=1}^{9} \sum_{n'=\mathbf{0}_{i'}}^{L} \sum_{j'=-n'}^{n'} \boldsymbol{k}^{ii'}_{nj,n'j'} \boldsymbol{g}^{i'}_{n',j'}$$

and in particular, with (6.8), the coefficients are related for $\mathbf{0}_i \leq n \leq L$ by

$$\boldsymbol{h}^i_{n,j} = \lambda \boldsymbol{g}^i_{n,j},$$

where $i = 1, \ldots, 9$, $j = -n, \ldots, n$ and

$$\boldsymbol{k}^{ii'}_{nj,n'j'} = \int_R \boldsymbol{y}^{i'}_{n',j'}(\xi) : \overline{\boldsymbol{y}^i_{n,j}(\xi)} \, \mathrm{d}\omega(\xi).$$

Theorem 7.3.4. *The solutions, which are the spectrally concentrated spacelimited tensor Slepian functions $\boldsymbol{h}_\alpha$ for $\alpha = 1, \ldots, 9(L+1)^2 - 12$, are orthogonal on the unit sphere Ω, and on the region R. This means that*

$$\int_\Omega \boldsymbol{h}_\alpha(\xi) : \overline{\boldsymbol{h}_\beta(\xi)} \, \mathrm{d}\omega(\xi) = \int_R \boldsymbol{h}_\alpha(\xi) : \overline{\boldsymbol{h}_\beta(\xi)} \, \mathrm{d}\omega(\xi) = \lambda_\alpha \delta_{\alpha,\beta}.$$

The proof is analogous to the proof of Theorem 7.1.4.

Chapter 8

Slepian Functions on the Spherical Cap

In this chapter, we look at the Slepian functions anew. From the previous chapters, we know that we only have to look at scalar spin-weighted concentration problems. Therefore, we deal here with the spin-weighted Slepian functions of spin weight $N \in \mathbb{Z}$ and go into details for the spherical cap as our region of interest. First, we show how to calculate the elements of the kernel matrix K^N.

8.1 Calculation of the Kernel Matrix

We know that the eigenvalue problem for the spin-weighted problem for spin weight $N \in \mathbb{Z}$ is given by

$$K^N G^N_\alpha = \lambda_\alpha G^N_\alpha,$$

where

$$K^N_{nj,n'j'} = \int_R \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\ \mathrm{d}\omega(\xi).$$

Furthermore, we know that this eigenvalue problem can be transformed into the integral equation

$$\int_R \mathscr{K}^N(\xi,\eta)\mathscr{G}^N(\xi)\ \mathrm{d}\omega(\xi) = \lambda \mathscr{G}^N(\eta),$$

where

$$\mathscr{K}^N(\xi,\eta) := \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta).$$

In this section, we want to calculate the kernel matrix K^N. For the scalar case, the spin weight zero case, this was done by [82]. First, we need further properties.

Lemma 8.1.1. *From [12, 18, 70, 77, 93], we borrow that*

$$\begin{aligned}&D^{j_1}_{m_1,n_1}(\alpha,\beta,\gamma)D^{j_2}_{m_2,n_2}(\alpha,\beta,\gamma)\\ &= \sum_{j,m,n}(-1)^{m+n}(2j+1)\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & -m\end{pmatrix} D^j_{m,n}(\alpha,\beta,\gamma)\begin{pmatrix} j_1 & j_2 & j\\ n_1 & n_2 & -n\end{pmatrix}\end{aligned}$$

for $\alpha,\gamma \in [0,2\pi]$ and $\beta \in [0,\pi]$, where $\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix}$ denotes the Wigner 3j-symbol. Furthermore, j and m or respectively j and n have to be both integers or both half-integers. The *same yields true for j_i and m_i or respectively j_i and n_i with $i = 1, 2$.*

This Wigner 3j-symbol fulfills

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = 0,$$

unless the following conditions yield true.

1. $m_1 \in \{-|j_1|, \ldots, |j_1|\}$, $m_2 \in \{-|j_2|, \ldots, |j_2|\}$, $m \in \{-|j|, \ldots, |j|\}$,
2. $m_1 + m_2 + m = 0$,
3. $|j_1 - j_2| \leq j \leq j_1 + j_2$,
4. $j_1 + j_2 + j$ *is an integer.*

Furthermore, the Wigner 3j-symbol fulfills the relations

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j_1+j_2+j} \begin{pmatrix} j_1 & j & j_2 \\ m_1 & m & m_2 \end{pmatrix}$$

and

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j_1+j_2+j} \begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix}.$$

Now, we take a look at the product of two Wigner D-functions. With 10. from Remark 3.4.4 and with Lemma 8.1.1, we get

$$\begin{aligned} &D^n_{j,-N}(\alpha,\beta,\gamma)\overline{D^{n'}_{j',-N}(\alpha,\beta,\gamma)} \\ &= D^n_{j,-N}(\alpha,\beta,\gamma)(-1)^{N-j'} D^{n'}_{-j',N}(\alpha,\beta,\gamma) \\ &= (-1)^{N-j'} \sum_{k,l,m} (-1)^{l+m}(2k+1) \begin{pmatrix} n & n' & k \\ j & -j' & -l \end{pmatrix} D^k_{l,m}(\alpha,\beta,\gamma) \begin{pmatrix} n & n' & k \\ -N & N & -m \end{pmatrix}. \end{aligned}$$

Because of 2. from Lemma 8.1.1, we see that $j - j' - l = 0$ and $-N + N - m = 0$. So, $l = j - j'$ and $m = 0$, else $\begin{pmatrix} n & n' & k \\ j & -j' & -l \end{pmatrix} = 0$ or $\begin{pmatrix} n & n' & k \\ -N & N & -m \end{pmatrix} = 0$. Then, we obtain with Lemma 8.1.1

$$\begin{aligned} &D^n_{j,-N}(\alpha,\beta,\gamma)\overline{D^{n'}_{j',-N}(\alpha,\beta,\gamma)} \\ &= (-1)^{N-j'} \sum_{k=|n-n'|}^{n+n'} \sum_{l,m=-k}^{k} (-1)^{l+m}(2k+1) \begin{pmatrix} n & n' & k \\ j & -j' & -l \end{pmatrix} \delta_{l,j-j'} D^k_{l,m}(\alpha,\beta,\gamma) \\ &\quad \times \begin{pmatrix} n & n' & k \\ -N & N & -m \end{pmatrix} \delta_{m,0} \\ &= (-1)^{N-j'} \sum_{k=|n-n'|}^{n+n'} (-1)^{j-j'}(2k+1) \begin{pmatrix} n & n' & k \\ j & -j' & j'-j \end{pmatrix} D^k_{j-j',0}(\alpha,\beta,\gamma) \begin{pmatrix} n & n' & k \\ -N & N & 0 \end{pmatrix} \\ &= (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} (2k+1) \begin{pmatrix} n & k & n' \\ j & j'-j & -j' \end{pmatrix} (-1)^{n+k+n'} D^k_{j-j',0}(\alpha,\beta,\gamma) \\ &\quad \times \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} (-1)^{n+k+n'} \end{aligned}$$

$$= (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} (2k+1) \begin{pmatrix} n & k & n' \\ j & j'-j & -j' \end{pmatrix} D^k_{j-j',0}(\alpha,\beta,\gamma) \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix}. \tag{8.1}$$

Theorem 8.1.2. *We can calculate the matrix elements of the kernel matrix K^N by*

$$K^N_{nj,n'j'} = (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} \sqrt{\frac{(2n+1)(2n'+1)(2k+1)}{4\pi}} \begin{pmatrix} n & k & n' \\ j & j'-j & -j' \end{pmatrix}$$
$$\times \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \int_R X_{k,j-j'}(t) e^{-i(j-j')\varphi} \, \mathrm{d}\omega(\xi(t,\varphi))$$

for all $N \in \mathbb{Z}$, all $n, n' \in \mathbb{N}_0$, $n, n' \geq |N|$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$.

Proof. Let $N \in \mathbb{Z}$, $n, n' \in \mathbb{N}_0$, $n, n' \geq |N|$, $j = -n, \ldots, n$, and $j' = -n', \ldots, n'$. With relation (8.1), we can formulate the product of two spin-weighted spherical harmonics, because we know from Theorem 3.4.9 that ${}_NY_{n,j}(\xi) = (-1)^N \sqrt{\frac{2n+1}{4\pi}} \, \overline{D^n_{j,-N}(\varphi,\vartheta,0)}$, where $\xi = \xi(t,\varphi) \in \Omega$ and $t = \cos\vartheta$. So, we have

$$\overline{{}_NY_{n,j}(\xi)} \; {}_NY_{n',j'}(\xi)$$
$$= (-1)^{2N} \frac{\sqrt{(2n+1)(2n'+1)}}{4\pi} \; D^n_{j,-N}(\varphi,\vartheta,0) \overline{D^{n'}_{j',-N}(\varphi,\vartheta,0)}$$
$$= (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} \sqrt{\frac{(2n+1)(2k+1)(2n'+1)}{4\pi}} \sqrt{\frac{2k+1}{4\pi}} D^k_{j-j',0}(\varphi,\vartheta,0)$$
$$\times \begin{pmatrix} n & k & n' \\ j & j'-j & -j' \end{pmatrix} \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix}$$
$$= (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} \sqrt{\frac{(2n+1)(2k+1)(2n'+1)}{4\pi}} \begin{pmatrix} n & k & n' \\ j & j'-j & -j' \end{pmatrix} \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \overline{{}_0Y_{k,j-j'}(\xi)},$$

where we know from Definition 2.4.37 that

$$\overline{{}_0Y_{k,j-j'}(\xi)} = \overline{Y_{k,j-j'}(\xi)} = X_{k,j-j'}(t) e^{-i(j-j')\varphi}.$$

Then,

$$\overline{{}_NY_{n,j}(\xi)} \; {}_NY_{n',j'}(\xi) = (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} \sqrt{\frac{(2n+1)(2n'+1)(2k+1)}{4\pi}}$$
$$\times \begin{pmatrix} n & k & n' \\ j & j'-j & -j' \end{pmatrix} \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} X_{k,j-j'}(t) e^{-i(j-j')\varphi}.$$

With

$$K^N_{nj,n'j'} = \int_R \overline{{}_NY_{n,j}(\xi)} \; {}_NY_{n',j'}(\xi) \, \mathrm{d}\omega(\xi),$$

we get the proposition by interchanging sum and integral. □

So, we have to calculate the following integrals depending on the region R

$$\iint_R X_{k,j-j'}(t) e^{-i(j-j')\varphi} \, \mathrm{d}\varphi \, \mathrm{d}t.$$

For arbitrary regions, this has to be done numerically. This is previously investigated for the scalar Slepian functions by [81]. Furthermore, for special regions, the Slepian functions can be calculated more efficiently by using a commuting operator. For the scalar Slepian function, this method is shown for the spherical cap [37, 78, 82] and for both the spherical double cap and the belt [80]. This method also exists for the vector Slepian functions on the spherical cap [43]. In the following section, we show the same for the more general case of the spin-weighted Slepian functions on the spherical cap. This enables us to construct not only the scalar and the vector Slepian functions for that special region by a commuting operator, but also the tensor Slepian functions.

First, we show how to calculate the Wigner 3j-symbol and different properties of the spin-weighted kernel matrix K^N.

Remark 8.1.3. *To calculate the Wigner 3j-symbol, we use the formula from [12, 18, 70]*

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \Delta(j_1, j_2, j_3)\delta_{m_1+m_2+m_3,0}(-1)^{j_1-j_2-m_3} \\ &\times \sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!} \\ &\times \sum_{k=k_{\min}}^{k_{\max}} \frac{(-1)^k}{k!(j_1+j_2-j_3-k)!(j_1-m_1-k)!(j_2+m_2-k)!} \\ &\times \frac{1}{(j_3-j_2+m_1+k)!(j_3-j_1-m_2+k)!},\end{aligned}$$

where

$$\begin{aligned}\Delta(j_1, j_2, j_3) &= \sqrt{\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}}, \\ k_{\min} &= \max\{-j_3+j_2-m_1; -j_3+j_1+m_2; 0\}, \\ k_{\max} &= \min\{j_1+j_2-j_3; j_1-m_1; j_2+m_2\}.\end{aligned}$$

Now, we look at further properties of the matrix K^N.

Lemma 8.1.4. *The kernel matrix K^N for $N \in \mathbb{Z}$ is Hermitian and positive definite [82]. Then, the eigenvalues are real. Furthermore, for the eigenvalues λ^N, we get that $0 < \lambda^N \leq 1$.*

Proof. Let $N \in \mathbb{Z}$.

- It is obvious that $K^N_{nj,n'j'} = \overline{K^N_{n'j',nj}}$ for $n, n' = |N|, \dots, L$, $j = -n, \dots, n$, and $j' = -n', \dots, n'$. Then, it is clear that K^N is Hermitian.
- Let G^N be an eigenvector and $\mathscr{G}^N$ an eigenfunction of K^N. Then, we obtain from the norm properties and by interchanging summation and integration

$$\overline{G^N}^{\mathrm{T}} K^N G^N = \int_R \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi) = \left\langle \mathscr{G}^N, \mathscr{G}^N \right\rangle_{\mathrm{L}^2(R)} = \left\| \mathscr{G}^N \right\|^2_{\mathrm{L}^2(R)} > 0$$

  for all $\mathscr{G}^N = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} G^N_{n,j}\ {}_N Y_{n,j} \neq 0$ and therefore, for all $G^N \neq 0$. So, K^N is positive definite.
- It is well known from linear algebra that the eigenvalues λ^N of a Hermitian matrix K^N are real [54, page 131 ff.].

- Here, we know that

$$\lambda^N = \frac{\overline{G^N}^{\mathrm{T}} K^N G^N}{\overline{G^N}^{\mathrm{T}} G^N} = \frac{\int_R \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)}{\int_\Omega \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)}.$$

With the positive definiteness of K^N, we get directly that $\lambda^N > 0$. Furthermore, with

$$\mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)} = \left|\mathscr{G}^N(\xi)\right|^2 \geq 0,$$

we get that

$$\int_M \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi) \geq 0$$

for any measurable subset $M \subset \Omega$. Then, we obtain with $R \subset \Omega$ that

$$\begin{aligned}
\lambda^N &= \frac{\overline{G^N}^{\mathrm{T}} K^N G^N}{\overline{G^N}^{\mathrm{T}} G^N} \\
&= \frac{\int_R \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)}{\int_\Omega \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)} \\
&= \frac{\int_\Omega \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi) - \int_{\Omega\setminus R} \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)}{\int_\Omega \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)} \\
&= 1 - \underbrace{\frac{\int_{\Omega\setminus R} \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)}{\int_\Omega \mathscr{G}^N(\xi)\overline{\mathscr{G}^N(\xi)}\, \mathrm{d}\omega(\xi)}}_{\geq 0} \\
&\leq 1.
\end{aligned}$$

Altogether, we get $0 < \lambda^N \leq 1$.

□

Corollary 8.1.5. *The last property of the previous lemma can be enhanced to*

$$0 < \lambda^N < 1,$$

because no bandlimited function can be completely supported within the region R.

Corollary 8.1.6. *The matrix $\overline{K^N}$, the complex conjugation of the kernel matrix K^N, is a Gramian matrix because*

$$\overline{K^N_{nj,n'j'}} = K^N_{n'j',nj} = \langle {}_N Y_{n',j'}, {}_N Y_{n,j} \rangle_{\mathrm{L}^2(\Omega)}$$

for all $N \in \mathbb{Z}$, all $n, n' \in \mathbb{N}_0$, $n, n' \geq |N|$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$. Therefore, we can find a system of eigenvectors of the kernel matrix that forms an orthonormal basis spanning the unit sphere Ω [60].

8.2 Commuting Operator on the Spherical Cap

We know how to calculate the Slepian functions from the spin-weighted eigenvalue problem of spin weight $N \in \mathbb{Z}$

$$K^N G^N = \lambda G^N$$

and from the integral equation

$$\int_R \mathscr{K}^N(\xi,\eta)\mathscr{G}^N(\xi)\,\mathrm{d}\omega(\xi) = \lambda\mathscr{G}^N(\eta)$$

in general. In this chapter, we consider the example of the spherical cap as region of interest. For that case, we can formulate a commuting operator and explain the computation of the Slepian functions. For spin weight zero, this was previously done by [37, 78, 82] and for spin weight ± 1 by [43].

For the spherical cap, the integral over the region of interest R reduces with $b = \cos\theta \le t \le 1$ to

$$\begin{aligned} K^N_{nj,n'j'} &= \int_R \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\,\mathrm{d}\omega(\xi) \\ &= \int_0^{2\pi}\int_0^{\theta} \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \\ &= \int_0^{2\pi}\int_b^{1} \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\,\mathrm{d}t\,\mathrm{d}\varphi \end{aligned}$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|,\dots,L$, all $j = -n,\dots,n$, and all $j' = -n',\dots,n'$. So, the region of interest for the spherical cap is given by $R = [0,2\pi]\times[b,1]$. In Figure 8.1, we see the spherical cap.

Next, we formulate Green's second surface identity for the spin-weighted Beltrami operator on the spherical cap.

Theorem 8.2.1. *Green's second surface identity on the spherical cap for the operator $\Delta^{*,N}$ is given by*

$$\begin{aligned} &\int_R \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi)\right)\,\mathrm{d}\omega(\xi) \\ &= \int_0^{2\pi} \left[\left(1-t^2\right)\left(\overline{G(\xi)}\partial_t F(\xi) - F(\xi)\partial_t\overline{G(\xi)}\right)\right]_{t=b}\,\mathrm{d}\varphi \end{aligned}$$

for $F, G \in \mathrm{X}^2(\Omega_0)$ (see Definition 3.4.17), where $\xi = \xi(t,\varphi) \in \Omega$.

Proof. Let $\Gamma := R/\mathrm{B}_\delta(\varepsilon^3)$, $\delta > 0$. For $F, G \in \mathrm{X}^2(\Gamma)$, we get with Theorem 3.4.28

$$\begin{aligned} &\int_\Gamma \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi)\right)\,\mathrm{d}\omega(\xi) \\ &= \int_{\partial\Gamma}\left(F(\xi)\frac{\partial}{\partial\nu(\xi)}\overline{G(\xi)} - \overline{G(\xi)}\frac{\partial}{\partial\nu(\xi)}F(\xi)\right)\,\mathrm{d}\sigma(\xi) - \underbrace{\int_\Gamma \frac{2iNt}{1-t^2}\,\partial_\varphi\left(F(\xi)\overline{G(\xi)}\right)\,\mathrm{d}\omega(\xi)}_{=0,\text{ because } F,G \text{ are } 2\pi\text{-periodic in } \varphi} \\ &= \int_{\partial R}\left(F(\xi)\frac{\partial}{\partial\nu(\xi)}\overline{G(\xi)} - \overline{G(\xi)}\frac{\partial}{\partial\nu(\xi)}F(\xi)\right)\,\mathrm{d}\sigma(\xi) \\ &\quad + \int_{\partial\mathrm{B}_\delta(\varepsilon^3)\cap\Omega}\left(F(\xi)\frac{\partial}{\partial\nu(\xi)}\overline{G(\xi)} - \overline{G(\xi)}\frac{\partial}{\partial\nu(\xi)}F(\xi)\right)\,\mathrm{d}\sigma(\xi). \end{aligned} \tag{8.2}$$

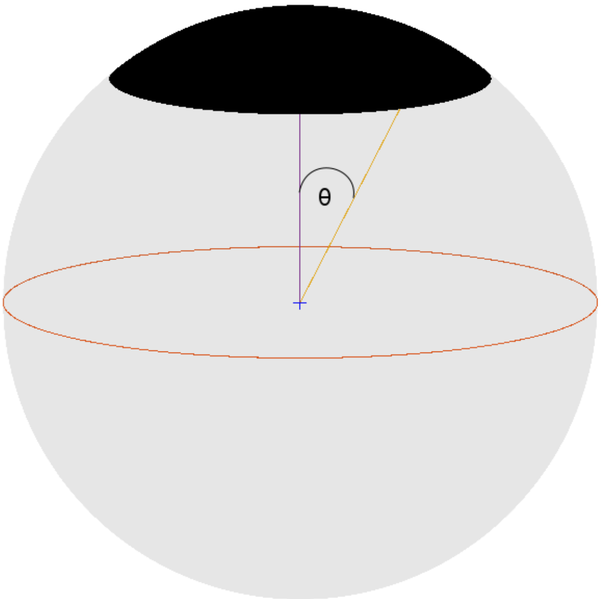


Figure 8.1: The spherical cap of the angle θ.

On ∂R, we obtain that for $\xi = \xi(t,\varphi) \in \Omega$

$$\int_{\partial R} \left(F(\xi) \frac{\partial}{\partial\nu(\xi)} \overline{G(\xi)} - \overline{G(\xi)} \frac{\partial}{\partial\nu(\xi)} F(\xi) \right) \, \mathrm{d}\sigma(\xi)$$
$$= \int_0^{2\pi} \left(F(\xi)\nu(\xi)\cdot\nabla^*_\xi \overline{G(\xi)} - \overline{G(\xi)}\nu(\xi)\cdot\nabla^*_\xi F(\xi) \right) r \, \mathrm{d}\varphi,$$

where $r = \sqrt{1-t^2}$, $\nu(\xi) = -\varepsilon^t$ and consequently $\nu(\xi)\cdot\nabla^*_\xi = -\sqrt{1-t^2}\partial_t$. Then, we get

$$\int_{\partial R} \left(F(\xi) \frac{\partial}{\partial\nu(\xi)} \overline{G(\xi)} - \overline{G(\xi)} \frac{\partial}{\partial\nu(\xi)} F(\xi) \right) \, \mathrm{d}\sigma(\xi)$$
$$= \int_0^{2\pi} \left[(1-t^2) \left(\overline{G(\xi)}\partial_t F(\xi) - F(\xi)\partial_t \overline{G(\xi)} \right) \right]_{t=b} \mathrm{d}\varphi.$$

On $\partial \mathrm{B}_\delta(\varepsilon^3) \cap \Omega$, we obtain that

$$\frac{\partial}{\partial\nu(\xi)} F(\xi) = \nu(\xi)\cdot\nabla^*_\xi F(\xi) = -\varepsilon^t \cdot \nabla^*_\xi F(\xi) = -\sqrt{1-t^2}\partial_t F(\xi)$$

is bounded, because $\partial_t F = \mathcal{O}(\frac{1}{\sqrt{1-t^2}})$. For G, we get the same result analogously. Then, we get for the limit $\delta \to 0_+$ that

$$\int_{\partial \mathrm{B}_\delta(\varepsilon^3)\cap\Omega} \left(F(\xi) \frac{\partial}{\partial\nu(\xi)} \overline{G(\xi)} - \overline{G(\xi)} \frac{\partial}{\partial\nu(\xi)} F(\xi) \right) \, \mathrm{d}\sigma(\xi) \to 0.$$

Furthermore, the left-hand side of (8.2) tends for $\delta \to 0_+$ to the integral over R. This means that

$$\int_\Gamma \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi) \right) \mathrm{d}\omega(\xi) \to \int_R \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi) \right) \mathrm{d}\omega(\xi).$$

Altogether, we obtain

$$\int_R \left(F(\xi)\overline{\Delta^{*,N}_\xi G(\xi)} - \overline{G(\xi)}\Delta^{*,N}_\xi F(\xi) \right) \mathrm{d}\omega(\xi)$$
$$= \int_0^{2\pi} \left[(1-t^2) \left(\overline{G(\xi)}\partial_t F(\xi) - F(\xi)\partial_t \overline{G(\xi)} \right) \right]_{t=b} \mathrm{d}\varphi.$$

□

Now, we define the commuting operator for the spherical cap and prove that this operator commutes with the kernel function.

Theorem 8.2.2. *For the spherical cap with $b = \cos\theta \le t \le 1$, the kernel function, defined by*

$$\mathscr{K}^N(\xi,\eta) := \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_N Y_{n,j}(\xi)} \; {}_N Y_{n,j}(\eta),$$

commutes with the differential operator

$$\mathscr{I}^N_\xi := (b - t_1)\,\Delta^{*,N}_\xi + \left(t_1^2 - 1\right)\partial_{t_1} - L(L+2)t_1$$

for all $N \in \mathbb{Z}$, where $\Delta^{*,N}_\xi$ is defined in Corollary 3.3.7, L is the bandlimit, and $\xi =$

$\xi(t_1, \varphi_1), \eta = \eta(t_2, \varphi_2) \in \Omega$. *This means that for any function* $u \in \mathrm{X}^2(\Omega_0)$*, we obtain*

$$\begin{aligned}\int_R \overline{\mathscr{K}^N(\xi,\eta)} \left[\mathscr{I}_\eta^N u(\eta)\right] \, \mathrm{d}\omega(\eta) &= \int_R \left[\mathscr{I}_\xi^N \overline{\mathscr{K}^N(\xi,\eta)}\right] u(\eta) \, \mathrm{d}\omega(\eta) \\ &= \mathscr{I}_\xi^N \int_R \overline{\mathscr{K}^N(\xi,\eta)} u(\eta) \, \mathrm{d}\omega(\eta).\end{aligned}$$

Before we prove this theorem we have to make additional remarks.

Remark 8.2.3. *Previously, we had for* $N \in \mathbb{Z}$

$$\int_R \mathscr{K}^N(\xi,\eta)\mathscr{G}^N(\xi) \, \mathrm{d}\omega(\xi) = \lambda \mathscr{G}^N(\eta).$$

With $\mathscr{K}^N(\xi,\eta) = \overline{\mathscr{K}^N(\eta,\xi)}$*, this is equal to*

$$\int_R \mathscr{K}^N(\eta,\xi)\mathscr{G}^N(\eta) \, \mathrm{d}\omega(\eta) = \int_R \overline{\mathscr{K}^N(\xi,\eta)}\mathscr{G}^N(\eta) \, \mathrm{d}\omega(\eta) = \lambda \mathscr{G}^N(\xi)$$

for $\xi, \eta \in \Omega$.

Remark 8.2.4. *Let* $N \in \mathbb{Z}$ *and* $\xi, \eta \in \Omega$*. For the proof of Theorem 8.2.2, we have to show the following equations.*

1. *For two functions* u_1 *and* $u_2 \in \mathrm{X}^2(\Omega_0)$*, the differential operator is self-adjoint, this means that it satisfies*

$$\int_R \overline{u_1(\xi)} \left[\mathscr{I}_\xi^N u_2(\xi)\right] \, \mathrm{d}\omega(\xi) = \int_R \overline{\left[\mathscr{I}_\xi^N u_1(\xi)\right]} u_2(\xi) \, \mathrm{d}\omega(\xi)$$

   and particularly, for the spherical cap

$$\int_0^{2\pi} \int_b^1 \overline{u_1(\xi)} \left[\mathscr{I}_\xi^N u_2(\xi)\right] \, \mathrm{d}t \, \mathrm{d}\varphi = \int_0^{2\pi} \int_b^1 \overline{\left[\mathscr{I}_\xi^N u_1(\xi)\right]} u_2(\xi) \, \mathrm{d}t \, \mathrm{d}\varphi,$$

   where $\xi = \xi(t, \varphi) \in \Omega$.

2. $$\overline{\mathscr{I}_\xi^N} \mathscr{K}^N(\xi,\eta) = \mathscr{I}_\eta^N \mathscr{K}^N(\xi,\eta).$$

3. $$\overline{\overline{\mathscr{I}_\xi^N} \mathscr{K}^N(\xi,\eta)} = \mathscr{I}_\xi^N \; \overline{\mathscr{K}^N(\xi,\eta)}.$$

4. *We can interchange the integration and the differential operater, this means that for any* $u \in \mathrm{X}^2(\Omega_0)$

$$\mathscr{I}_\xi^N \int_R \mathscr{K}^N(\xi,\eta) u(\eta) \, \mathrm{d}\omega(\eta) = \int_R \mathscr{I}_\xi^N \mathscr{K}^N(\xi,\eta) u(\eta) \, \mathrm{d}\omega(\eta).$$

This is obvious with Corollary 3.4.18, because

$$\begin{aligned}\int_R \overline{\mathscr{K}^N(\xi,\eta)} \left[\mathscr{I}_\eta^N u(\eta)\right] \, \mathrm{d}\omega(\eta) &\overset{1.}{=} \int_R \overline{\left[\mathscr{I}_\eta^N \mathscr{K}^N(\xi,\eta)\right]} u(\eta) \, \mathrm{d}\omega(\eta) \\ &\overset{2.}{=} \int_R \overline{\left[\overline{\mathscr{I}_\xi^N} \mathscr{K}^N(\xi,\eta)\right]} u(\eta) \, \mathrm{d}\omega(\eta)\end{aligned}$$

$$\overset{3.}{=} \int_R \mathcal{I}_\xi^N \, \overline{\mathscr{K}^N(\xi,\eta)} u(\eta) \, \mathrm{d}\omega(\eta)$$
$$\overset{4.}{=} \mathcal{I}_\xi^N \int_R \overline{\mathscr{K}^N(\xi,\eta)} u(\eta) \, \mathrm{d}\omega(\eta).$$

Before we prove these equations, we first have to look at some properties of sums.

Remark 8.2.5. *For sums, we have the following properties*

$$\sum_{i=a}^{L} \sum_{k=i}^{L} b_{i,k} = \sum_{k=a}^{L} \sum_{i=a}^{k} b_{i,k}, \tag{8.3}$$

$$\sum_{n=|N|}^{L} \sum_{j=-n}^{n} b_{n,j} = \sum_{j=-L}^{L} \sum_{n=n_j}^{L} b_{n,j}, \tag{8.4}$$

$$\sum_{n=n_j}^{L} \sum_{k=n_j}^{n-1} b_{n,k} = \sum_{k=n_j}^{L} \sum_{n=k+1}^{L} b_{n,k}, \tag{8.5}$$

where $L, a \in \mathbb{N}_0$, $N, j \in \mathbb{Z}$, and $n_j = \max\{|N|, |j|\}$ with $L \geq a$, $L \geq |N|$, and $L \geq |j|$.

Proof. The proof of the sum relations is straight forward.

- The first property (8.3) is proven by induction.
  Base case: Let $L = a$.
  $$\sum_{i=a}^{a} \sum_{k=i}^{a} b_{i,k} = b_{a,a} = \sum_{k=a}^{a} \sum_{i=a}^{k} b_{i,k}.$$
  Induction hypothesis: (8.3) is satisfied for L, $L \in \mathbb{N}_0$, $L \geq a$.
  Induction step: The induction step $L \to L+1$ is given with help of the induction hypothesis by
  $$\begin{aligned}\sum_{i=a}^{L+1} \sum_{k=i}^{L+1} b_{i,k} &= \sum_{i=a}^{L} \sum_{k=i}^{L+1} b_{i,k} + \sum_{k=L+1}^{L+1} b_{L+1,k} \\ &= \sum_{i=a}^{L} \sum_{k=i}^{L} b_{i,k} + \sum_{i=a}^{L} b_{i,L+1} + b_{L+1,L+1} \\ &= \sum_{k=a}^{L} \sum_{i=a}^{k} b_{i,k} + \sum_{i=a}^{L+1} b_{i,L+1} \\ &= \sum_{k=a}^{L+1} \sum_{i=a}^{k} b_{i,k}.\end{aligned}$$

- For the second property (8.4), we use the definition of n_j and property (8.3).

$$\begin{aligned}\sum_{j=-L}^{L} \sum_{n=n_j}^{L} b_{n,j} &= \sum_{j=-L}^{-|N|} \sum_{n=|j|}^{L} b_{n,j} + \sum_{j=-|N|+1}^{|N|-1} \sum_{n=|N|}^{L} b_{n,j} + \sum_{j=|N|}^{L} \sum_{n=|j|}^{L} b_{n,j} \\ &= \sum_{j=|N|}^{L} \sum_{n=j}^{L} b_{n,-j} + \sum_{n=|N|}^{L} \sum_{j=-|N|+1}^{|N|-1} b_{n,j} + \sum_{j=|N|}^{L} \sum_{n=j}^{L} b_{n,j}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=|N|}^{L} \sum_{j=|N|}^{n} b_{n,-j} + \sum_{n=|N|}^{L} \sum_{j=-|N|+1}^{|N|-1} b_{n,j} + \sum_{n=|N|}^{L} \sum_{j=|N|}^{n} b_{n,j} \\
&= \sum_{n=|N|}^{L} \left(\sum_{j=-n}^{-|N|} b_{n,j} + \sum_{j=-|N|+1}^{|N|-1} b_{n,j} + \sum_{j=|N|}^{n} b_{n,j} \right) \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} b_{n,j}.
\end{aligned}
$$

- We get the third property (8.5) with help of property (8.3) by

$$
\begin{aligned}
\sum_{n=n_j}^{L} \sum_{k=n_j}^{n-1} b_{n,k} &= \sum_{n=n_j}^{L} \left(\sum_{k=n_j}^{L} b_{n,k} - \sum_{k=n}^{L} b_{n,k} \right) \\
&= \sum_{k=n_j}^{L} \sum_{n=n_j}^{L} b_{n,k} - \sum_{n=n_j}^{L} \sum_{k=n}^{L} b_{n,k} \\
&= \sum_{k=n_j}^{L} \sum_{n=n_j}^{L} b_{n,k} - \sum_{k=n_j}^{L} \sum_{n=n_j}^{k} b_{n,k} \\
&= \sum_{k=n_j}^{L} \sum_{n=k+1}^{L} b_{n,k}.
\end{aligned}
$$

□

Corollary 8.2.6. *From the previous remark, we can conclude that*

$$
\sum_{n=|N|}^{L} \sum_{j=-n}^{n} \sum_{k=n_j}^{n-1} b_{n,j,k} = \sum_{k=|N|}^{L} \sum_{j=-k}^{k} \sum_{n=k+1}^{L} b_{n,j,k},
$$

where $n_j = \max\{|N|, |j|\}$*,* $L \in \mathbb{N}_0$*, and* $N \in \mathbb{Z}$ *with* $L \geq |N|$*.*

Proof. With the previous remark, we get

$$
\begin{aligned}
\sum_{n=|N|}^{L} \sum_{j=-n}^{n} \sum_{k=n_j}^{n-1} b_{n,j,k} &\overset{(8.4)}{=} \sum_{j=-L}^{L} \sum_{n=n_j}^{L} \sum_{k=n_j}^{n-1} b_{n,j,k} \\
&\overset{(8.5)}{=} \sum_{j=-L}^{L} \sum_{k=n_j}^{L} \sum_{n=k+1}^{L} b_{n,j,k} \\
&\overset{(8.4)}{=} \sum_{k=|N|}^{L} \sum_{j=-k}^{k} \sum_{n=k+1}^{L} b_{n,j,k}.
\end{aligned}
$$

□

Now, we can prove Theorem 8.2.2.

Proof. Here, we need previously described properties of Chapter 3. Let $N \in \mathbb{Z}$.

1. Let $\xi = \xi(t,\varphi) \in \Omega$. Then, the left-hand side of the first condition from Remark 8.2.4 can be written for two functions u_1 and $u_2 \in \mathrm{X}^2(\Omega_0)$ by

$$\begin{aligned}\int_R \overline{u_1(\xi)} \left[\mathscr{I}_\xi^N u_2(\xi)\right] \, \mathrm{d}\omega(\xi) = &\int_R (b-t)\overline{u_1(\xi)}\Delta_\xi^{*,N} u_2(\xi) \, \mathrm{d}\omega(\xi) \\ &+ \int_R \left(t^2-1\right) \overline{u_1(\xi)}\partial_t u_2(\xi) \, \mathrm{d}\omega(\xi) \\ &- \int_R L(L+2)t\overline{u_1(\xi)}u_2(\xi) \, \mathrm{d}\omega(\xi). \end{aligned} \tag{8.6}$$

The integrals exist, because from $u_1, u_2 \in \mathrm{X}^2(\Omega_0)$ we know that u_1, u_2, $\left(t^2-1\right)\partial_t u_2$, and $\Delta^{*,N} u_2$ are bounded on Ω. Then, $\mathscr{I}^N u_2$ is bounded on Ω. So, we integrate over products of bounded functions and hence, over bounded functions. The same holds true for the right-hand side of the first condition from Remark 8.2.4 with interchanged roles of u_1 and u_2. Note that in Theorem 8.2.2, we use $\mathscr{K}^N$ instead of u_1. Then, the integrals also exist, because we know from Corollary 3.4.18 that the spin-weighted spherical harmonics are in $X^2(\Omega_0)$ and therefore, $\mathscr{K}^N$ consists of products and sums of bounded functions, which are also bounded.

So, the first integral in (8.6) leads with Green's second surface identity for the spin-weighted Beltrami operator (see Theorem 8.2.1) to

$$\begin{aligned}&\int_R (b-t)\overline{u_1(\xi)}\Delta_\xi^{*,N} u_2(\xi) \, \mathrm{d}\omega(\xi) \\ &= \int_R \overline{\Delta_\xi^{*,N}} \left((b-t)\overline{u_1(\xi)}\right) u_2(\xi) \, \mathrm{d}\omega(\xi) \\ &\quad - \int_0^{2\pi} \left[\left(1-t^2\right)\left((b-t)\overline{u_1(\xi)}\partial_t u_2(\xi) - u_2(\xi)\partial_t\left((b-t)\overline{u_1(\xi)}\right)\right)\right]_{t=b} \, \mathrm{d}\varphi.\end{aligned}$$

Because

$$\overline{\Delta_\xi^{*,N}} = \Delta_\xi^* - \frac{N^2 + 2iNt\partial_\varphi}{1-t^2}$$

and $b-t$ is independent of φ, we look at the term

$$\begin{aligned}&\Delta_\xi^*\left((b-t)\overline{u_1(\xi)}\right) \\ &= \partial_t\left(\left(1-t^2\right)\partial_t(b-t)\overline{u_1(\xi)}\right) + (b-t)\frac{1}{1-t^2}\,\partial_\varphi^2\overline{u_1(\xi)} \\ &= \partial_t\left(\left(1-t^2\right)\left(-\overline{u_1(\xi)} + (b-t)\partial_t\overline{u_1(\xi)}\right)\right) + (b-t)\frac{1}{1-t^2}\,\partial_\varphi^2\overline{u_1(\xi)} \\ &= \partial_t\left(\left(t^2-1\right)\overline{u_1(\xi)}\right) - \left(1-t^2\right)\partial_t\overline{u_1(\xi)} + (b-t)\partial_t\left(\left(1-t^2\right)\partial_t\overline{u_1(\xi)}\right) \\ &\qquad + (b-t)\frac{1}{1-t^2}\,\partial_\varphi^2\overline{u_1(\xi)} \\ &= (b-t)\Delta_\xi^*\overline{u_1(\xi)} + \partial_t\left(\left(t^2-1\right)\overline{u_1(\xi)}\right) + \left(t^2-1\right)\partial_t\overline{u_1(\xi)}\end{aligned}$$

and get

$$\overline{\Delta_\xi^{*,N}}\left((b-t)\overline{u_1(\xi)}\right) = (b-t)\overline{\Delta_\xi^{*,N} u_1(\xi)} + \partial_t\left(\left(t^2-1\right)\overline{u_1(\xi)}\right) + \left(t^2-1\right)\partial_t\overline{u_1(\xi)}.$$

Furthermore, we obtain

$$\begin{aligned}
&\left[\left(1-t^2\right)\left((b-t)\overline{u_1(\xi)}\partial_t u_2(\xi) - u_2(\xi)\partial_t\left((b-t)\overline{u_1(\xi)}\right)\right)\right]_{t=b} \\
&= \underbrace{\left[\left(1-t^2\right)(b-t)\overline{u_1(\xi)}\partial_t u_2(\xi)\right]_{t=b}}_{=0} - \underbrace{\left[\left(1-t^2\right)(b-t)u_2(\xi)\partial_t\overline{u_1(\xi)}\right]_{t=b}}_{=0} \\
&\quad + \left[\left(1-t^2\right)\overline{u_1(\xi)}u_2(\xi)\right]_{t=b} \\
&= \left[\left(1-t^2\right)\overline{u_1(\xi)}u_2(\xi)\right]_{t=b}.
\end{aligned}$$

Altogether, we get for the first integral in (8.6)

$$\begin{aligned}
&\int_R (b-t)\overline{u_1(\xi)}\Delta_\xi^{*,N}u_2(\xi)\ \mathrm{d}\omega(\xi) \\
&= \int_R (b-t)u_2(\xi)\overline{\Delta_\xi^{*,N}u_1(\xi)}\ \mathrm{d}\omega(\xi) + \int_R u_2(\xi)\partial_t\left(\left(t^2-1\right)\overline{u_1(\xi)}\right)\ \mathrm{d}\omega(\xi) \\
&\quad + \int_R \left(t^2-1\right)u_2(\xi)\partial_t\overline{u_1(\xi)}\ \mathrm{d}\omega(\xi) - \int_0^{2\pi}\left[\left(1-t^2\right)\overline{u_1(\xi)}u_2(\xi)\right]_{t=b}\ \mathrm{d}\varphi.
\end{aligned}$$

The second integral in (8.6) leads with integration by parts to

$$\begin{aligned}
&\int_R \left(t^2-1\right)\overline{u_1(\xi)}\partial_t u_2(\xi)\ \mathrm{d}\omega(\xi) \\
&= \int_0^{2\pi}\int_b^1 \left(t^2-1\right)\overline{u_1(\xi)}\partial_t u_2(\xi)\ \mathrm{d}t\ \mathrm{d}\varphi \\
&= \int_0^{2\pi}\left[\left(t^2-1\right)\overline{u_1(\xi)}u_2(\xi)\right]_{t=b}^{t=1}\ \mathrm{d}\varphi - \int_0^{2\pi}\int_b^1 \partial_t\left(\left(t^2-1\right)\overline{u_1(\xi)}\right)u_2(\xi)\ \mathrm{d}t\ \mathrm{d}\varphi \qquad (8.7) \\
&= \int_0^{2\pi}\left[\left(1-t^2\right)\overline{u_1(\xi)}u_2(\xi)\right]_{t=b}\ \mathrm{d}\varphi - \int_0^{2\pi}\int_b^1 \partial_t\left(\left(t^2-1\right)\overline{u_1(\xi)}\right)u_2(\xi)\ \mathrm{d}t\ \mathrm{d}\varphi. \qquad (8.8)
\end{aligned}$$

Altogether, we obtain

$$\begin{aligned}
&\int_R \overline{u_1(\xi)}\left[\mathscr{I}_\xi^N u_2(\xi)\right]\ \mathrm{d}\omega(\xi) \\
&= \int_R (b-t)\overline{\Delta_\xi^{*,N}u_1(\xi)}u_2(\xi)\ \mathrm{d}\omega(\xi) + \int_R \left(t^2-1\right)\left(\partial_t\overline{u_1(\xi)}\right)u_2(\xi)\ \mathrm{d}\omega(\xi) \\
&\quad - \int_R L(L+2)t\overline{u_1(\xi)}u_2(\xi)\ \mathrm{d}\omega(\xi) \\
&= \int_R \overline{\left[\mathscr{I}_\xi^N u_1(\xi)\right]}u_2(\xi)\ \mathrm{d}\omega(\xi).
\end{aligned}$$

2. The second equation from Remark 8.2.4 can be proven for $\xi = \xi(t_1,\varphi_1) \in \Omega$ and $\eta = \eta(t_2,\varphi_2) \in \Omega$ with Corollary 3.3.7 and Theorem 3.3.1 by

$$\begin{aligned}
&\left(\overline{\mathscr{I}_\xi^N} - \mathscr{I}_\eta^N\right)\mathscr{K}^N(\xi,\eta) \\
&= \Big((b-t_1)\overline{\Delta_\xi^{*,N}} + \left(t_1^2-1\right)\partial_{t_1} - L(L+2)t_1 - (b-t_2)\Delta_\eta^{*,N} - \left(t_2^2-1\right)\partial_{t_2}
\end{aligned}$$

$$+L(L+2)t_2\Big) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$= (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n+1) - L(L+2))\overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$+ \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \left(t_1^2 - 1\right) \partial_{t_1}\ \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$- \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \left(t_2^2 - 1\right) \overline{{}_NY_{n,j}(\xi)}\partial_{t_2}\ {}_NY_{n,j}(\eta)$$

$$\stackrel{(3.4)}{=} (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n+1) - L(L+2))\overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$+ \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \Bigg(\left(nt_1 + \frac{Nj}{n}\right) \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$- (2n+1)\alpha_{n,j}^N\ \overline{{}_NY_{n-1,j}(\xi)}\ {}_NY_{n,j}(\eta)\Bigg)$$

$$- \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \Bigg(\left(nt_2 + \frac{Nj}{n}\right) \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$+ (2n+1)\alpha_{n,j}^N\ \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n-1,j}(\eta)\Bigg)$$

$$= (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n+2) - L(L+2))\overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$+ \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (2n+1)\alpha_{n,j}^N \left(\overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n-1,j}(\eta) - \overline{{}_NY_{n-1,j}(\xi)}\ {}_NY_{n,j}(\eta)\right).$$

Note that we use here $\overline{\Delta_\xi^{*,N}}\ \overline{{}_NY_{n,j}(\xi)} = \overline{\Delta_\xi^{*,N}\ {}_NY_{n,j}(\xi)}$.

Further, with Theorem 3.3.5 and with Corollary 8.2.6, we get

$$\left(\overline{\mathscr{I}_\xi^N} - \mathscr{I}_\eta^N\right) \mathscr{K}^N(\xi,\eta)$$

$$= (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n+2) - L(L+2))\overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$+ (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (2n+1) \sum_{k=n_j}^{n-1} \overline{{}_NY_{k,j}(\xi)}\ {}_NY_{k,j}(\eta)$$

$$= (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} (n(n+2) - L(L+2))\overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n,j}(\eta)$$

$$+ (t_1 - t_2) \sum_{k=|N|}^{L} \sum_{j=-k}^{k} \overline{{}_NY_{k,j}(\xi)}\ {}_NY_{k,j}(\eta) \sum_{n=k+1}^{L} (2n+1)$$

$$= (t_1 - t_2) \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_N Y_{n,j}(\xi)}\ {}_N Y_{n,j}(\eta) \Bigg[n(n+2) - L(L+2) + \underbrace{\sum_{k=n+1}^{L} (2k+1)}_{\substack{= (L+1)^2 - (n+1)^2 \\ = L(L+2) - n(n+2)}} \Bigg]$$

$$= 0.$$

This is equivalent to the proposition

$$\overline{\mathscr{I}_\xi^N} \mathscr{K}^N(\xi, \eta) = \mathscr{I}_\eta^N \mathscr{K}^N(\xi, \eta).$$

3. The third equation from Remark 8.2.4 can be proven with the property that

$$\overline{\mathscr{K}^N(\xi, \eta)} = \mathscr{K}^N(\eta, \xi).$$

With

$$\mathscr{I}_\xi^N = \partial_t \left((b - t) \left(1 - t^2 \right) \partial_t \right) - \left(\frac{N^2(b-t)}{1-t^2} + L(L+2)t \right) + \frac{b-t}{1-t^2} \left(\partial_\varphi^2 + 2iNt\partial_\varphi \right)$$

and with Theorem 3.4.9, we get for the right-hand side

$$\begin{aligned}
&\mathscr{I}_\xi^N\ \overline{\mathscr{K}^N(\xi, \eta)} \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_N Y_{n,j}(\eta)}\ \mathscr{I}_\xi^N\ {}_N Y_{n,j}(\xi) \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_N Y_{n,j}(\eta)} \Bigg(\partial_t \left((b - t) \left(1 - t^2 \right) \partial_t \right)\ {}_N Y_{n,j}(\xi) \\
&\quad - \left(\frac{N^2(b-t)}{1-t^2} + L(L+2)t \right)\ {}_N Y_{n,j}(\xi) + \frac{b-t}{1-t^2} \left((ij)^2 + 2iNt(ij) \right)\ {}_N Y_{n,j}(\xi) \Bigg) \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_N Y_{n,j}(\eta)} \Bigg(\partial_t \left((b - t) \left(1 - t^2 \right) \partial_t \right)\ {}_N Y_{n,j}(\xi) \\
&\quad - \left(\frac{N^2(b-t)}{1-t^2} + L(L+2)t \right)\ {}_N Y_{n,j}(\xi) + \frac{b-t}{1-t^2} \left(-j^2 - 2jNt \right)\ {}_N Y_{n,j}(\xi) \Bigg).
\end{aligned}$$

For the left-hand side in the third condition from Remark 8.2.4, we obtain with

$$\begin{aligned}
&\overline{\mathscr{I}_\xi^N} \mathscr{K}^N(\xi, \eta) \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} {}_N Y_{n,j}(\eta)\ \overline{\mathscr{I}_\xi^N}\ \overline{{}_N Y_{n,j}(\xi)} \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} {}_N Y_{n,j}(\eta) \Bigg(\partial_t \left((b - t) \left(1 - t^2 \right) \partial_t \right)\ \overline{{}_N Y_{n,j}(\xi)} \\
&\quad - \left(\frac{N^2(b-t)}{1-t^2} + L(L+2)t \right)\ \overline{{}_N Y_{n,j}(\xi)} + \frac{b-t}{1-t^2} \left((-ij)^2 - 2iNt(-ij) \right)\ \overline{{}_N Y_{n,j}(\xi)} \Bigg)
\end{aligned}$$

$$= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} {}_NY_{n,j}(\eta) \Bigg(\partial_t \left((b-t) \left(1-t^2\right) \partial_t \right) \overline{{}_NY_{n,j}(\xi)}$$
$$- \left(\frac{N^2(b-t)}{1-t^2} + L(L+2)t \right) \overline{{}_NY_{n,j}(\xi)} + \frac{b-t}{1-t^2} \left(-j^2 - 2jNt\right) \overline{{}_NY_{n,j}(\xi)} \Bigg)$$

that

$$\overline{\overline{\mathscr{I}_\xi^N \mathscr{K}^N(\xi,\eta)}}$$
$$= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_NY_{n,j}(\eta)} \Bigg(\partial_t \left((b-t) \left(1-t^2\right) \partial_t \right) {}_NY_{n,j}(\xi)$$
$$- \left(\frac{N^2(b-t)}{1-t^2} + L(L+2)t \right) {}_NY_{n,j}(\xi) + \frac{b-t}{1-t^2} \left(-j^2 - 2jNt\right) {}_NY_{n,j}(\xi) \Bigg).$$

So, the left- and the right-hand side in the third condition from Remark 8.2.4 are equal. Then,

$$\overline{\overline{\mathscr{I}_\xi^N \mathscr{K}^N(\xi,\eta)}} = \mathscr{I}_\xi^N \, \overline{\mathscr{K}^N(\xi,\eta)}.$$

4. For any $u \in \mathrm{X}^2(\Omega_0)$ and for $\xi \in \Omega$, we get

$$\begin{aligned}
\mathscr{I}_\xi^N \int_R \mathscr{K}^N(\xi,\eta) u(\eta) \, \mathrm{d}\omega(\eta) &= \mathscr{I}_\xi^N \int_R \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_NY_{n,j}(\xi)} \, {}_NY_{n,j}(\eta) u(\eta) \, \mathrm{d}\omega(\eta) \\
&= \mathscr{I}_\xi^N \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \overline{{}_NY_{n,j}(\xi)} \int_R {}_NY_{n,j}(\eta) u(\eta) \, \mathrm{d}\omega(\eta) \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \left(\mathscr{I}_\xi^N \, \overline{{}_NY_{n,j}(\xi)} \right) \int_R {}_NY_{n,j}(\eta) u(\eta) \, \mathrm{d}\omega(\eta) \\
&= \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \int_R \mathscr{I}_\xi^N \, \overline{{}_NY_{n,j}(\xi)} \, {}_NY_{n,j}(\eta) u(\eta) \, \mathrm{d}\omega(\eta) \\
&= \int_R \mathscr{I}_\xi^N \mathscr{K}^N(\xi,\eta) u(\eta) \, \mathrm{d}\omega(\eta).
\end{aligned}$$

So, obviously, the integration and the differential operator can be interchanged.

□

Theorem 8.2.7. *The commuting relation also holds true for an integral over the unit sphere. This means that for $\xi \in \Omega$*

$$\begin{aligned}
\int_\Omega \overline{\mathscr{K}^N(\xi,\eta)} \left[\mathscr{I}_\eta^N u(\eta) \right] \, \mathrm{d}\omega(\eta) &= \int_\Omega \left[\mathscr{I}_\xi^N \overline{\mathscr{K}^N(\xi,\eta)} \right] u(\eta) \, \mathrm{d}\omega(\eta) \\
&= \mathscr{I}_\xi^N \int_\Omega \overline{\mathscr{K}^N(\xi,\eta)} u(\eta) \, \mathrm{d}\omega(\eta).
\end{aligned}$$

Proof. The proof is analogous to the proof of Theorem 8.2.2. For the second and third condition of Remark 8.2.4, the integral does not appear and for the fourth condition in the integral, we can obviously replace R by Ω. So, we only have to look at the first condition of Remark 8.2.4 with an integral over Ω. Here, we use instead of the Theorem 8.2.1 Green's second surface identity for the spin-weighted Beltrami operator from Theorem 3.4.27. Furthermore, in (8.7), respectively (8.8), we obtain for the term

$$\int_0^{2\pi} \left[\left(t^2 - 1\right) \overline{u_1(\xi)} u_2(\xi) \right]_{t=-1}^{t=1} \, \mathrm{d}\varphi = 0.$$

Then, we get the proposition. □

8.3 Computation of the Slepian Functions on the Spherical Cap

Because the matrices K, k, and $\boldsymbol{k}$ and therefore, the matrix K^N, $N \in \mathbb{Z}$, are supposed to be ill-conditioned, we need the commuting operator from the previous chapter to implement the Slepian functions on the spherical cap. Implementing this operator $\mathscr{I}^N$ is difficult, but instead we can formulate a commuting matrix I^N with this operator. Then, we can calculate the eigenvectors, the eigenfunctions, and the eigenvalues [82]. Consequently, we can compute the scalar, the vector, and the tensor Slepian functions.

We know already that $\mathscr{I}_\xi^N$ and $\mathscr{K}^N$ commute for all $N \in \mathbb{Z}$. Now, we look at the eigenfunctions of the commuting operator $\mathscr{I}^N$. Later we will see that $\mathscr{I}_\xi^N$ and $\mathscr{K}^N$ have the same eigenfunctions $\mathscr{G}_\alpha^N$. Therefore, we use this previously in the notation. This means that for $\xi \in \Omega$ and $N \in \mathbb{Z}$

$$\mathscr{I}_\xi^N \mathscr{G}_\alpha^N(\xi) = \chi_\alpha \mathscr{G}_\alpha^N(\xi),$$
$$\int_R \overline{\mathscr{K}^N(\xi,\eta)} \mathscr{G}_\alpha^N(\eta) \, \mathrm{d}\omega(\eta) = \lambda^N \mathscr{G}_\alpha^N(\xi),$$

where λ^N and χ_α are not necessarily equal. This means that the kernel function and its commuting differential operator have the same eigenfunctions, but do not need to have the same eigenvalues.

Furthermore,

$$\mathscr{I}_\xi^N \mathscr{G}_\alpha^N(\xi) = \chi_\alpha \mathscr{G}_\alpha^N(\xi)$$

leads to

$$\sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \left(G_{n',j'}^N\right)_\alpha \mathscr{I}_\xi^N \; {}_NY_{n',j'}(\xi) = \chi_\alpha \sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \left(G_{n',j'}^N\right)_\alpha \; {}_NY_{n',j'}(\xi).$$

Upon multiplying by $\overline{{}_NY_{n,j}(\xi)}$, $n = |N|, \ldots, L$, $j = -n, \ldots, n$, we obtain

$$\sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \left(G_{n',j'}^N\right)_\alpha \overline{{}_NY_{n,j}(\xi)} \left(\mathscr{I}_\xi^N \; {}_NY_{n',j'}(\xi)\right) = \chi_\alpha \sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \left(G_{n',j'}^N\right)_\alpha \overline{{}_NY_{n,j}(\xi)} \; {}_NY_{n',j'}(\xi).$$

Integration over the unit sphere Ω and interchanging of sum and integral leads with Theorem 3.4.21 to

$$\sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \left(G_{n',j'}^N\right)_\alpha \int_\Omega \overline{{}_NY_{n,j}(\xi)} \left(\mathscr{I}_\xi^N \; {}_NY_{n',j'}(\xi)\right) \, \mathrm{d}\omega(\xi)$$

$$= \chi_\alpha \sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \left(G^N_{n',j'}\right)_\alpha \underbrace{\int_\Omega \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\ \mathrm{d}\omega(\xi)}_{=\delta_{n,n'}\delta_{j,j'}}.$$

This yields

$$\sum_{n'=|N|}^{L} \sum_{j'=-n'}^{n'} \left(G^N_{n',j'}\right)_\alpha I^N_{nj,n'j'} = \chi_\alpha \left(G^N_{n,j}\right)_\alpha$$

for all $n = |N|, \ldots, L$ and all $j = -n, \ldots, n$ and particularly

$$I^N G^N_\alpha = \chi_\alpha G^N_\alpha,$$

where

$$I^N_{nj,n'j'} := \int_\Omega \overline{{}_NY_{n,j}(\xi)} \left(\mathscr{I}^N_\xi\ {}_NY_{n',j'}(\xi)\right)\ \mathrm{d}\omega(\xi),$$

$$I^N := \begin{pmatrix} I^N_{|N|,-|N|,|N|,-|N|} & \cdots & I^N_{|N|,-|N|,LL} \\ \vdots & \ddots & \vdots \\ I^N_{LL,|N|,-|N|} & \cdots & I^N_{LL,LL} \end{pmatrix}.$$

We know already

$$K^N G^N_\alpha = \lambda^N G^N_\alpha,$$

where

$$K^N_{nj,n'j'} = \int_R \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\ \mathrm{d}\omega(\xi),$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$ and from Theorem 8.2.2 that $\mathscr{I}^N_\xi$ and $\mathscr{K}^N$ commute for all $N \in \mathbb{Z}$. Then, we get the following corollary.

Corollary 8.3.1. *K^N and I^N also commute for all $N \in \mathbb{Z}$, this means that*

$$K^N I^N = I^N K^N.$$

So, they have the same eigenvectors G^N_α, if I^N has a simple spectrum of eigenvalues.

Proof. Let $N \in \mathbb{Z}$, $n, l = |N|, \ldots, L$, $j = -n, \ldots, n$, and $m = -l, \ldots, l$. We start with the commutation of the matrices and then show that they have the same eigenvectors.

- The left-hand side gives with Theorem 8.2.7 and with Theorem 3.4.21

$$\begin{aligned}
&\left(K^N I^N\right)_{nj,lm} \\
&= \sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} K^N_{nj,n'j'} I^N_{n'j',lm} \\
&= \sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} \int_R \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\ \mathrm{d}\omega(\xi) \int_\Omega \overline{{}_NY_{n',j'}(\eta)} \left(\mathscr{I}^N_\eta\ {}_NY_{l,m}(\eta)\right)\ \mathrm{d}\omega(\eta) \\
&= \int_R \overline{{}_NY_{n,j}(\xi)} \int_\Omega \underbrace{\sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} \overline{{}_NY_{n',j'}(\eta)}\ {}_NY_{n',j'}(\xi)}_{=\mathscr{K}^N(\eta,\xi)=\overline{\mathscr{K}^N(\xi,\eta)}} \left(\mathscr{I}^N_\eta\ {}_NY_{l,m}(\eta)\right)\ \mathrm{d}\omega(\eta)\ \mathrm{d}\omega(\xi)
\end{aligned}$$

$$= \int_R \overline{{}_NY_{n,j}(\xi)} \int_\Omega \left[\mathscr{I}_\xi^N \overline{\mathscr{K}^N(\xi,\eta)} \right] \ {}_NY_{l,m}(\eta)\ \mathrm{d}\omega(\eta)\ \mathrm{d}\omega(\xi)$$
$$= \int_R \overline{{}_NY_{n,j}(\xi)}\ \mathscr{I}_\xi^N \sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} {}_NY_{n',j'}(\xi) \underbrace{\int_\Omega \overline{{}_NY_{n',j'}(\eta)}\ {}_NY_{l,m}(\eta)\ \mathrm{d}\omega(\eta)}_{=\delta_{n',l}\delta_{j',m}}\ \mathrm{d}\omega(\xi)$$
$$= \int_R \overline{{}_NY_{n,j}(\xi)} \left(\mathscr{I}_\xi^N\ {}_NY_{l,m}(\xi) \right)\ \mathrm{d}\omega(\xi)$$

and the right-hand side with Theorem 8.2.2 and with Theorem 3.4.21

$$\left(I^N K^N\right)_{nj,lm}$$
$$= \sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} I^N_{nj,n'j'} K^N_{n'j',lm}$$
$$= \sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} \int_\Omega \overline{{}_NY_{n,j}(\xi)} \left(\mathscr{I}_\xi^N\ {}_NY_{n',j'}(\xi) \right)\ \mathrm{d}\omega(\xi) \int_R \overline{{}_NY_{n',j'}(\eta)}\ {}_NY_{l,m}(\eta)\ \mathrm{d}\omega(\eta)$$
$$= \int_\Omega \int_R \overline{{}_NY_{n,j}(\xi)} \mathscr{I}_\xi^N \underbrace{\sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} {}_NY_{n',j'}(\xi)\ \overline{{}_NY_{n',j'}(\eta)}}_{=\overline{\mathscr{K}^N(\xi,\eta)}}\ {}_NY_{l,m}(\eta)\ \mathrm{d}\omega(\eta)\ \mathrm{d}\omega(\xi)$$
$$= \int_\Omega \overline{{}_NY_{n,j}(\xi)} \int_R \overline{\mathscr{K}^N(\xi,\eta)} \left(\mathscr{I}_\eta^N\ {}_NY_{l,m}(\eta) \right)\ \mathrm{d}\omega(\eta)\ \mathrm{d}\omega(\xi)$$
$$= \int_R \sum_{n'=|N|}^{L} \sum_{j'=-n}^{n} \underbrace{\int_\Omega \overline{{}_NY_{n,j}(\xi)}\ {}_NY_{n',j'}(\xi)\ \mathrm{d}\omega(\xi)}_{=\delta_{n,n'}\delta_{j,j'}} \overline{{}_NY_{n',j'}(\eta)} \left(\mathscr{I}_\eta^N {}_NY_{l,m}(\eta) \right)\ \mathrm{d}\omega(\eta)$$
$$= \int_R \overline{{}_NY_{n,j}(\eta)} \left(\mathscr{I}_\eta^N\ {}_NY_{l,m}(\eta) \right)\ \mathrm{d}\omega(\eta).$$

The interchanging of the integrals is possible, because of Fubini's theorem. So, both sides are equal and we get the proposition.

- The second part of the corollary is a well-known theorem of linear algebra [41, page 63 f.]. It claims that if K^N and I^N commute, then they have the same eigenvectors G^N_α, if all eigenvalues χ_α are simple eigenvalues of I^N to the eigenvector G^N_α. This can be proved by the following. We know that

$$I^N G^N_\alpha = \chi_\alpha G^N_\alpha.$$

Then, with

$$I^N K^N G^N_\alpha = K^N I^N G^N_\alpha = \chi_\alpha K^N G^N_\alpha,$$

we know that $K^N G^N_\alpha$ is an eigenvector of I^N to the eigenvalue χ_α. If I^N has a simple spectrum, this new eigenvector must be a multiple of the old one. This means that

$$K^N G^N_\alpha = \lambda^N_\alpha G^N_\alpha.$$

So, K^N and I^N have the same eigenvectors G^N_α, if I^N has a simple spectrum.

□

Note that because $I^N, K^N \in \mathbb{R}^{[(L+1)^2-N^2]\times[(L+1)^2-N^2]}$ (because $n = |N|, \ldots, L$, $j = -n, \ldots, n$), we obtain $(L+1)^2 - N^2$ orthogonal eigenvectors G_α^N, orthogonal eigenfunctions $\mathscr{G}_\alpha^N$, and eigenvalues λ_α^N, where $\alpha = 1, \ldots, (L+1)^2 - N^2$. With the Gram-Schmidt algorithm, we can orthonormalize the eigenvectors and eigenfunctions.

If K^N and I^N have the same eigenvectors, we can calculate this G_α^N by

$$I^N G_\alpha^N = \chi_\alpha G_\alpha^N.$$

Therefore, we first calculate the matrix I^N, where

$$I_{nj,n'j'}^N = \int_\Omega \overline{{}_N Y_{n,j}(\xi)} \left(\mathscr{I}_\xi^N \; {}_N Y_{n',j'}(\xi)\right) \, \mathrm{d}\omega(\xi)$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n$.

Lemma 8.3.2. *We obtain for*

$$I_{nj,n'j'}^N = \int_\Omega \overline{{}_N Y_{n,j}(\xi)} \left(\mathscr{I}_\xi^N \; {}_N Y_{n',j'}(\xi)\right) \, \mathrm{d}\omega(\xi)$$

the following result.

$$\begin{aligned}
I_{nj,nj}^N &= -\left[n(n+1)b + Nj\left(1 - \frac{L(L+2)+1}{n(n+1)}\right)\right], \\
I_{nj,n+1,j}^N &= \left[(n+1)^2 - 1 - L(L+2)\right] \alpha_{n+1,j}^N \\
&= \left[n(n+2) - L(L+2)\right] \frac{\sqrt{(n+1-N)(n+1+N)}}{n+1} \sqrt{\frac{(n+1-j)(n+1+j)}{(2n+1)(2n+3)}}, \\
I_{n+1,j,nj}^N &= \left[n(n+2) - L(L+2)\right] \alpha_{n+1,j}^N = I_{nj,n+1,j}^N, \\
I_{nj,n'j'}^N &= 0, \qquad \textit{else}
\end{aligned}$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n$. So, I^N is a symmetric tridiagonal matrix.

Proof. For $\xi \in \Omega$, we take a look at $\mathscr{I}_\xi^N \; {}_N Y_{n,j}(\xi)$ and use Theorem 3.3.1. Then, we get

$$\begin{aligned}
&\mathscr{I}_\xi^N \; {}_N Y_{n,j}(\xi) \\
&= \left[(b-t)\Delta_\xi^{*,N} + \left(t^2 - 1\right)\partial_t - L(L+2)t\right] \; {}_N Y_{n,j}(\xi) \\
&= -n(n+1)(b-t) \; {}_N Y_{n,j}(\xi) + \left[\left(t^2 - 1\right)\partial_t - L(L+2)t\right] \; {}_N Y_{n,j}(\xi) \\
&\overset{(3.5)}{=} -n(n+1)b \; {}_N Y_{n,j}(\xi) + \left[n(n+1) - L(L+2)\right] t \; {}_N Y_{n,j}(\xi) \\
&\qquad - \left[(n+1)t + \frac{Nj}{n+1}\right] \; {}_N Y_{n,j}(\xi) + (2n+1)\alpha_{n+1,j}^N \; {}_N Y_{n+1,j}(\xi) \\
&= -\left[n(n+1)b + \frac{Nj}{n+1}\right] \; {}_N Y_{n,j}(\xi) + (2n+1)\alpha_{n+1,j}^N \; {}_N Y_{n+1,j}(\xi) \\
&\qquad + \left[(n-1)(n+1) - L(L+2)\right] t \; {}_N Y_{n,j}(\xi) \\
&\overset{(3.6)}{=} -\left[n(n+1)b + \frac{Nj}{n+1}\right] \; {}_N Y_{n,j}(\xi) + \left[(n-1)(n+1) - L(L+2)\right] \alpha_{n,j}^N \; {}_N Y_{n-1,j}(\xi) \\
&\qquad + \left[(n-1)(n+1) - L(L+2) + 2n + 1\right] \alpha_{n+1,j}^N \; {}_N Y_{n+1,j}(\xi)
\end{aligned}$$

$$
\begin{aligned}
&- [(n-1)(n+1) - L(L+2)] \ \frac{Nj}{n(n+1)} \ {}_N Y_{n,j}(\xi) \\
=& - \left[n(n+1)b + \frac{Nj}{n(n+1)} \left(n + n^2 - 1 - L(L+2) \right) \right] \ {}_N Y_{n,j}(\xi) \\
&+ \left[n^2 - 1 - L(L+2) \right] \alpha^N_{n,j} \ {}_N Y_{n-1,j}(\xi) + [n(n+2) - L(L+2)] \, \alpha^N_{n+1,j} \ {}_N Y_{n+1,j}(\xi) \\
=& - \left[n(n+1)b + \frac{Nj}{n(n+1)} (n(n+1) - 1 - L(L+2)) \right] \ {}_N Y_{n,j}(\xi) \\
&+ \left[n^2 - 1 - L(L+2) \right] \alpha^N_{n,j} \ {}_N Y_{n-1,j}(\xi) + [n(n+2) - L(L+2)] \, \alpha^N_{n+1,j} \ {}_N Y_{n+1,j}(\xi).
\end{aligned}
$$

With the property of the spin-weighted spherical harmonics of Theorem 3.4.21,

$$
\int_\Omega {}_N Y_{n,j}(\xi) \ \overline{{}_N Y_{n',j'}(\xi)} \ \mathrm{d}\omega(\xi) = \delta_{n,n'} \delta_{j,j'},
$$

we obtain

$$
\begin{aligned}
I^N_{nj,nj} &= - \left[n(n+1)b + Nj \left(1 - \frac{L(L+2)+1}{n(n+1)} \right) \right], \\
I^N_{nj,n+1,j} &= \left[(n+1)^2 - 1 - L(L+2) \right] \alpha^N_{n+1,j} \\
&= [n(n+2) - L(L+2)] \ \frac{\sqrt{(n+1-N)(n+1+N)}}{n+1} \ \sqrt{\frac{(n+1-j)(n+1+j)}{(2n+1)(2n+3)}}, \\
I^N_{n+1,j,nj} &= [n(n+2) - L(L+2)] \, \alpha^N_{n+1,j} = I^N_{nj,n+1,j}, \\
I^N_{nj,n'j'} &= 0, \qquad \text{else}
\end{aligned}
$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n$. It is obvious that I^N is a symmetric tridiagonal matrix. □

Now, we can compute the matrix I^N.

Corollary 8.3.3. *The commuting matrix I^N has a simple spectrum for all $N \in \mathbb{Z}$.*

Proof. From linear algebra, it is well known that a real, symmetric tridiagonal matrix has real eigenvalues and all the eigenvalues are simple, if all off-diagonal elements are nonzero [25].
Let $N \in \mathbb{Z}$, $n = |N|, \ldots, L$ and $j = -n, \ldots, n$. It is obvious that I^N is a real, symmetric tridiagonal matrix. So, we have to show that all off-diagonal elements of I^N are nonzero. We know that

$$
I^N_{n+1,j,nj} = [n(n+2) - L(L+2)] \, \alpha^N_{n+1,j} = I^N_{nj,n+1,j},
$$

where

$$
\alpha^N_{n+1,j} = \frac{\sqrt{n+1-N}\sqrt{n+1+N}}{n+1} \ \sqrt{\frac{(n+1-j)(n+1+j)}{(2n+1)(2n+3)}}.
$$

The off-diagonal elements can be zero, if

- $\alpha^N_{n+1,j} = 0$. For $n = 0$, we also get that $N = 0$ and $j = 0$ and therefore, $\alpha^N_{n+1,j} \neq 0$. For $n > 0$, the coefficients $\alpha^N_{n+1,j} = 0$ is equivalent to one of the following equations

$$
\left| \begin{aligned} n+1-N &= 0 \\ n+1+N &= 0 \\ n+1-j &= 0 \\ n+1+j &= 0 \end{aligned} \right|.
$$

For $n > 0$, this leads to

$$\left| \begin{matrix} n+1 = |N| \\ n+1 = |j| \end{matrix} \right| .$$

Because $n = |N|, \ldots, L$ and $j = -n, \ldots, n$, so $n \geq |N|$ and $|j| \leq n$, we get

$$\left| \begin{matrix} n = |N| - 1 \geq |N| \\ n+1 = |j| \leq n \end{matrix} \right| .$$

Then, $\alpha^N_{n+1,j} \neq 0$ for all $N \in \mathbb{Z}$, all $n = |N|, \ldots, L$, and all $j = -n, \ldots, n$.

- $n(n+2) - L(L+2) = 0$. This is the case for $n = L$ since $n \geq 0$, but $I^N_{Lj,L+1,j} = I^N_{L+1,j,Lj}$ does not exist.

So, the off-diagonal elements of I^N are nonzero and therefore, we get the proposition that I^N has a simple spectrum. □

Now, we know that K^N and I^N have the same eigenvectors and we can easily calculate these eigenvectors $G^N_\alpha = \left(\left(G^N_{n,j} \right)_\alpha \right)_{n,j}$ by solving

$$I^N G^N_\alpha = \chi_\alpha G^N_\alpha .$$

To obtain the eigenfunctions $\mathscr{G}^N_\alpha$ of the eigenvalue problems

$$\mathscr{I}^N \mathscr{G}^N_\alpha(\xi) = \chi_\alpha \mathscr{G}^N_\alpha(\xi)$$

and

$$\int_R \mathscr{K}^N(\xi, \eta) \mathscr{G}^N_\alpha(\xi) \, \mathrm{d}\omega(\xi) = \lambda^N_\alpha \mathscr{G}^N_\alpha(\eta),$$

we have to compute

$$\mathscr{G}^N_\alpha(\xi) = \sum_{n=|N|}^{L} \sum_{j=-n}^{n} \left(G^N_{n,j} \right)_\alpha \; {}_N Y_{n,j}(\xi).$$

We get the eigenvalues by

$$\lambda^N_\alpha = \overline{G^N_\alpha}^T K^N G^N_\alpha = \int_R |\mathscr{G}^N_\alpha(\xi)|^2 \, \mathrm{d}\omega(\xi).$$

So, we have to calculate the matrix elements

$$K^N_{nj,n'j'} = \int_R \overline{{}_N Y_{n,j}(\xi)} \; {}_N Y_{n',j'}(\xi) \, \mathrm{d}\omega(\xi)$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n$.

Now, we can calculate the matrix elements of the kernel matrix K^N.

Theorem 8.3.4. *The kernel matrix for the spherical cap is given by*

$$K^N_{nj,n'j'} = \frac{(-1)^{N+j}}{2} \sqrt{(2n+1)(2n'+1)} \sum_{k=|n-n'|}^{n+n'} \begin{pmatrix} n & k & n' \\ j & 0 & -j \end{pmatrix} \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \times \left[P_{k-1}(b) - P_{k+1}(b) \right] \delta_{j,j'}$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$.

Proof. From Theorem 8.1.2, we know that we have to look at the integral

$$\int_b^1 X_{k,j-j'}(t) \int_0^{2\pi} e^{-i(j-j')\varphi} \, \mathrm{d}\varphi \, \mathrm{d}t = 2\pi\delta_{j,j'} \int_b^1 \underbrace{X_{k,0}(t)}_{=\sqrt{\frac{2k+1}{4\pi}}\, P_k(t)} \, \mathrm{d}t,$$

where we use Definition 2.4.37. We get with Corollary 2.4.9

$$\begin{aligned}\int_b^1 X_{k,j-j'}(t) \int_0^{2\pi} e^{-i(j-j')\varphi} \, \mathrm{d}\varphi \, \mathrm{d}t &= 2\pi\delta_{j,j'} \sqrt{\frac{2k+1}{4\pi}} \, \frac{1}{2k+1} \left[P_{k-1}(b) - P_{k+1}(b)\right] \\ &= \sqrt{\frac{\pi}{(2k+1)}} \left[P_{k-1}(b) - P_{k+1}(b)\right] \delta_{j,j'}.\end{aligned}$$

With Theorem 8.1.2, we obtain the proposition

$$\begin{aligned}K^N_{nj,n'j'} &= (-1)^{N+j} \sum_{k=|n-n'|}^{n+n'} \sqrt{\frac{(2n+1)(2n'+1)(2k+1)}{4\pi}} \begin{pmatrix} n & k & n' \\ j & j'-j & -j' \end{pmatrix} \\ &\quad \times \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \int_R X_{k,j-j'}(t) e^{-i(j-j')\varphi} \, \mathrm{d}\omega(\xi(t,\varphi)) \\ &= \frac{(-1)^{N+j}}{2} \sqrt{(2n+1)(2n'+1)} \sum_{k=|n-n'|}^{n+n'} \begin{pmatrix} n & k & n' \\ j & 0 & -j \end{pmatrix} \\ &\quad \times \begin{pmatrix} n & k & n' \\ -N & 0 & N \end{pmatrix} \left[P_{k-1}(b) - P_{k+1}(b)\right] \delta_{j,j'}\end{aligned}$$

for all $N \in \mathbb{Z}$, all $n, n' = |N|, \ldots, L$, all $j = -n, \ldots, n$, and all $j' = -n', \ldots, n'$. □

All in all, we can compute the orthonormal eigenvectors $G^{\pm N}_\alpha$, the orthonormal eigenfunctions $\mathscr{G}^{\pm N}_\alpha$, and the eigenvalues $\lambda^{\pm N}_\alpha$ for $\alpha = 1, \ldots, (L+1)^2 - N^2$. So, we can calculate the scalar, vector, and tensor Slepian functions on the spherical cap with the results from Chapter 4, Chapter 5.1.3, and Chapter 6.1.5.

8.4 The Shannon Number of the Slepian Functions on the Spherical Cap

From Chapter 4, Chapter 5, and Chapter 6, we know that the Shannon number is given by

$$\begin{aligned}S_{\text{scalar}} &= (L+1)^2 \, \frac{A}{4\pi} \\ S_{\text{vector}} &= \left[3(L+1)^2 - 2\right] \, \frac{A}{4\pi} \\ S_{\text{tensor}} &= \left[9(L+1)^2 - 12\right] \, \frac{A}{4\pi}.\end{aligned}$$

So, we need the area A_{cap} of the spherical cap defined by $b = \cos\theta \leq t \leq 1$ to calculate the Shannon number of the three cases.

We see from Figure 8.2 that $\cos\theta = \frac{c}{r}$. Then, $c = r\cos\theta = rb$ and so, $h = r - c = r(1-b)$.

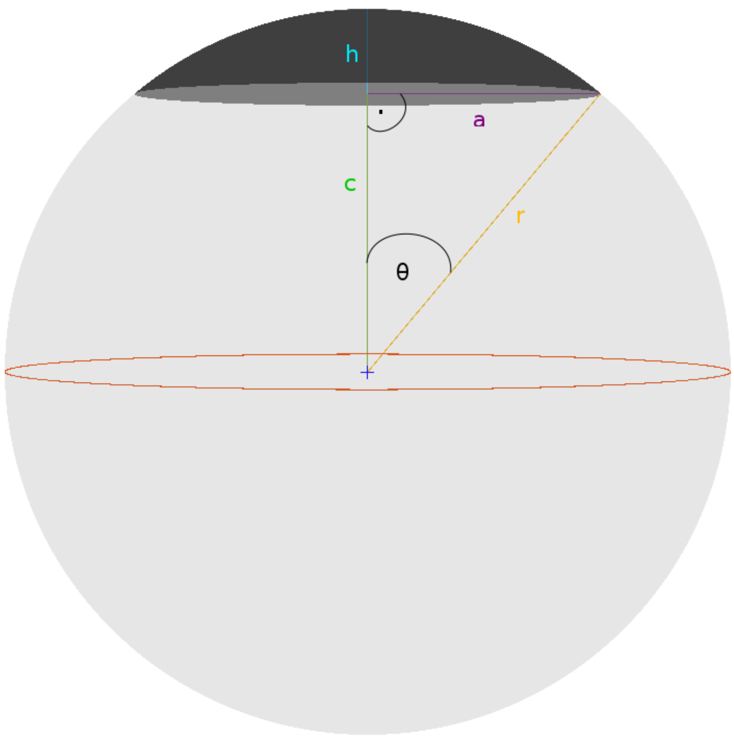


Figure 8.2: Transverse section of the sphere of radius r with the spherical cap of the angle θ.

Therefore, the area is given by

$$A_{\text{cap}} = 2\pi r h = 2\pi r^2(1-b),$$

where $r = 1$ for the unit sphere Ω.

Then, we get

$$\frac{A_{\text{cap}}}{4\pi} = \frac{1-b}{2}$$

and therefore,

$$\begin{aligned} S_{\text{scalar}} &= (L+1)^2 \, \frac{1-b}{2}, \\ S_{\text{vector}} &= [3(L+1)^2 - 2] \, \frac{1-b}{2}, \\ S_{\text{tensor}} &= [9(L+1)^2 - 12] \, \frac{1-b}{2}. \end{aligned}$$

8.5 Implementation of Tensor Slepian Functions on the Spherical Cap

In this section, we present the pointwise Euclidean norm of selected tensor Slepian functions on the spherical cap. All numerical calculations were done with `MATLAB R2015b`. The calculation of the Legendre polynomials, of the associated Legendre functions, of the spherical harmonics, and of its derivatives are based on the algorithms of [23].

From the previous section, we know how to calculate the spin-weighted Slepian functions and their eigenvalues and eigenvectors. Additionally, we know from Chapter 6.1.5 how to calculate the tensor Slepian functions on the spherical cap. In our example, we use a spherical cap with radius 40°. This is equal to $\theta = \frac{2}{9}\pi$. For the bandlimit, we choose $L = 18$. This choice of parameters is dedicated to the choice of parameters in [78]. Then, the whole number of tensor Slepian functions is $9(L+1)^2 - 12 = 3237$ and for our spherical cap, we get the Shannon number approximately by $S \approx 379$.

In Figure 8.3, we see the distribution of the sorted eigenvalues for the previously described example. The red plus marks the location of the Shannon number. We see that the eigenvalues are mostly near to 1 or near to zero and only a small number is in between. The largest eigenvalue is $\lambda_1 = 0.999999999403780$, so it is approximately 1. A problem is that there are small negative eigenvalues (the smallest eigenvalue is $\lambda_{3237} = -4.3082 \cdot 10^{-16}$; basically zero with respect to the machine accuracy). We assume that they are numerical artifacts. Numerical experiments have shown that they depend on the bandwidth and on the region size. We examine that this effect occurs for increasing bandwidth and decreasing region size. We do not worry about these small negative eigenvalues, because they are so small that we can say that they are approximately zero. Furthermore, the aim of the Slepian functions is to construct a localized basis system. For this purpose, we have previously recognized that we only use the functions up to the Shannon number and skip the rest. The basis functions to the negative eigenvalues are less concentrated in the region of interest R and therefore, they are not interesting for us.

The results we get for our example show that the tensor Slepian functions are sorted as expected. In Figure 8.4 to Figure 8.10, we see the pointwise Euclidean norm of the tensor Slepian functions for a selection of α with $\alpha = 1, \dots, 3237$ plotted on the Driscoll-Healy grid (see Appendix B.4). The first tensor Slepian functions are concentrated in the region of interest, the spherical cap located at the North pole and marked by the inner of the red circle (see Figure 8.4 to Figure 8.6). The ones near to the Shannon number are concentrated around the boundary of the spherical cap (see Figure 8.7 and Figure 8.8) and the last ones, the less concentrated functions, are localized in the opposite cap. This means that the less concentrated tensor Slepian functions of the spherical cap at the North pole with θ are the best concentrated ones of the spherical cap at the South pole with radius $\pi - \theta$ (see Figure 8.9 and Figure 8.10).

It is obvious that the Euclidean norm of some tensor Slepian functions are equal, because we know from Chapter 6.1.5 that they are based on the spin-weighted Slepian functions. These are three times the spin-weighted Slepian functions of spin weight zero, twice those of spin weight 1 and -1 and once those of spin weight 2 and -2. We know that these spin-weighted Slepian functions are multiplied by a tensor given by the tensor product of two orthonormal unit vectors on the sphere. So, the norm of this tensor is always one. Therefore, we take this into account, for example when we choose the best concentrated tensor Slepian functions, because the norm of the first nine looks equal.

Figure 8.8 displays that the Shannon number is, as expected, a good estimate for significant eigenvalues. The tensor Slepian function for the Shannon number is concentrated at the boundary of the spherical cap. However, the concentration only occurs in the inner of the spherical cap. A suggestion in [60] is to use a number smaller than the Shannon number as a better estimate. Therefore, by looking at the tensor Slepian functions, we select the functions of Figure 8.7. These functions are concentrated at the boundary of the spherical cap that are almost contained in the inner area of the spherical cap.

The second row in Figure 8.10 shows two different mapping projections for the norm of the tensor Slepian function for $\alpha = 3237$. This is the least concentrated tensor Slepian function and is concentrated at the South pole. We use these two map projections, because this concentration at the South pole is poorly recognizable in the otherwise used projection. However, this projection is for all other α the better choice.

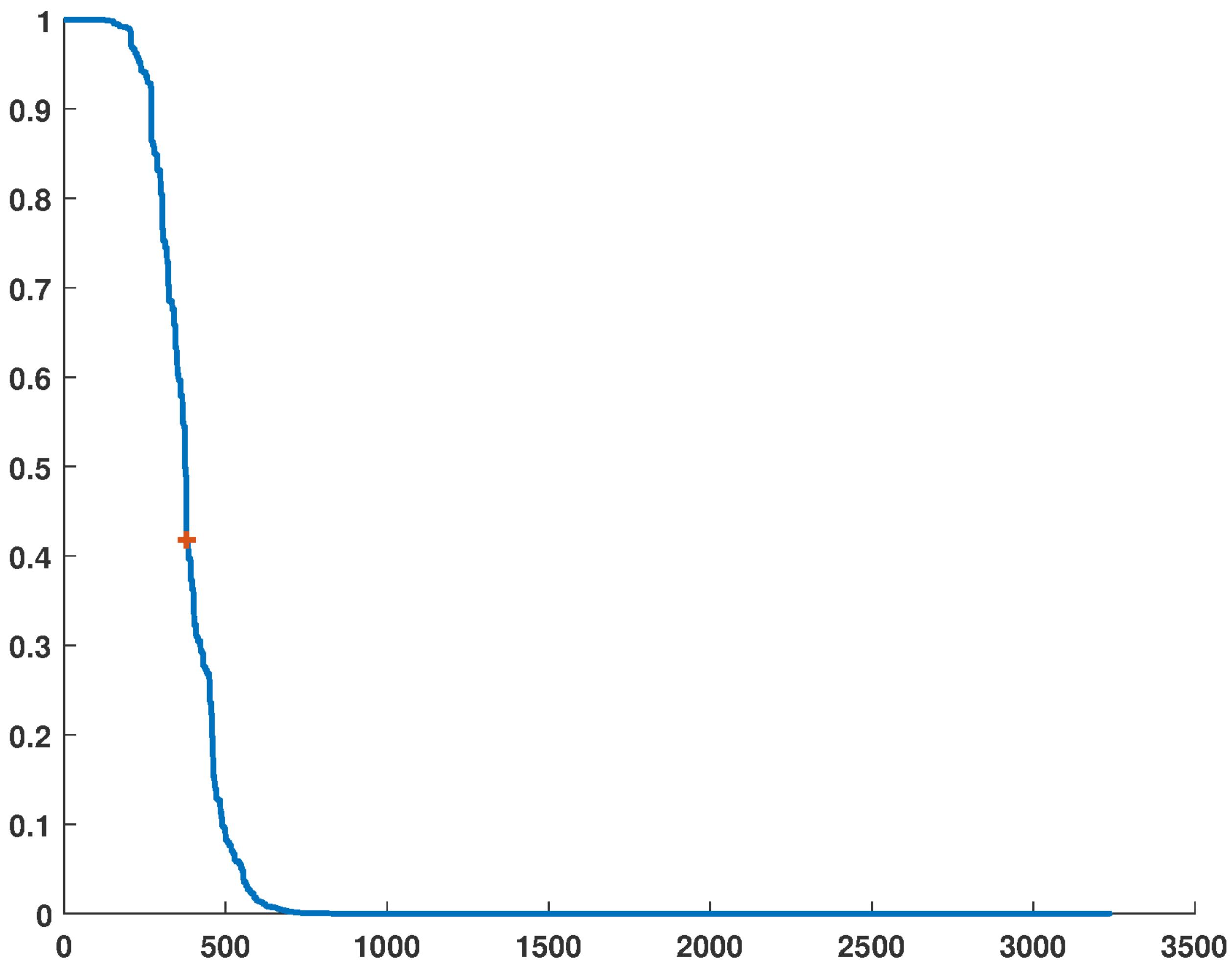


Figure 8.3: Distribution of the sorted eigenvalues of the tensor Slepian functions with bandlimit $L = 18$ for a spherical cap with radius $40°$. Altogether, we have $9(L+1)^2 - 12 = 3237$ functions and the Shannon number, marked by the red $+$, is given by $S \approx 379$.

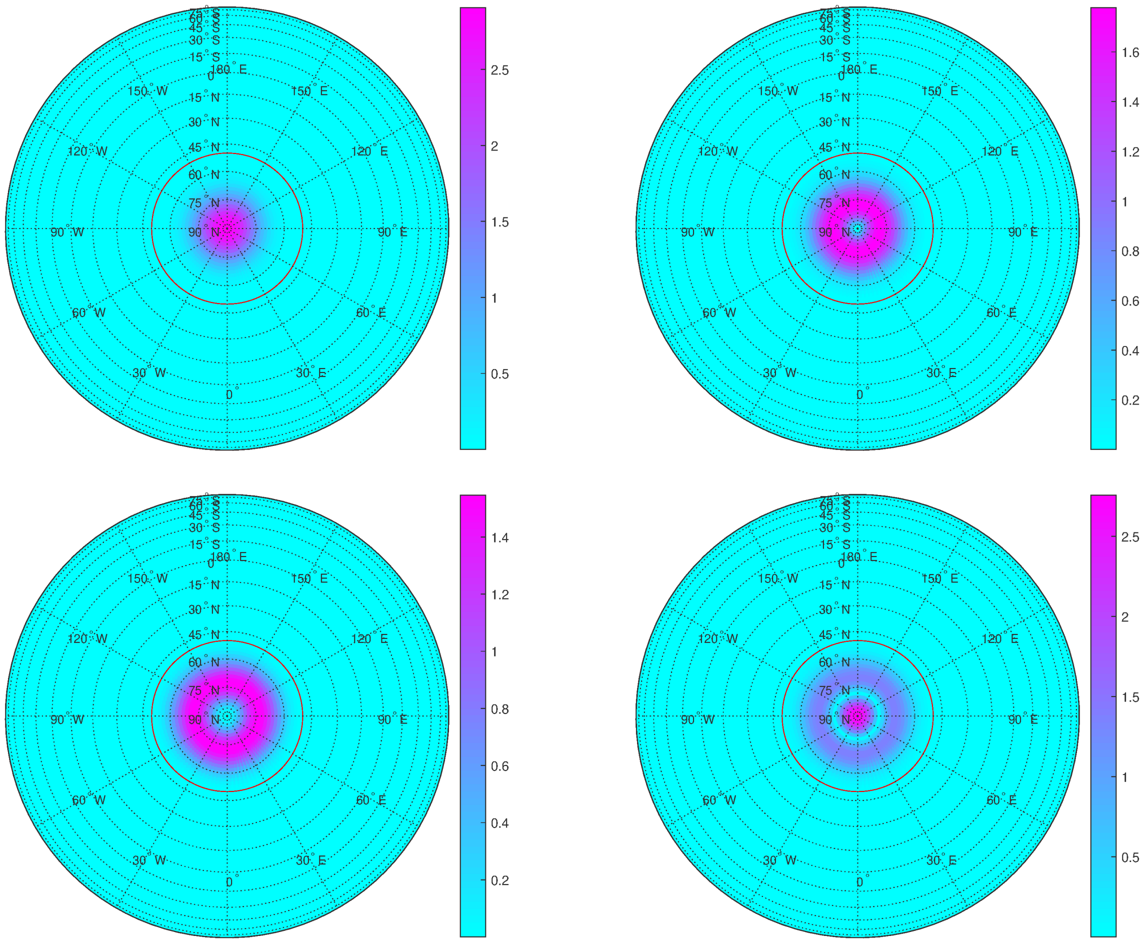


Figure 8.4: The norm of the best concentrated tensor Slepian functions with bandlimit $L = 18$ in a spherical cap with radius $40°$ at the North pole. The cap is marked as the inner of the red circle. Here, $\alpha = 1$ (top-left), $\alpha = 10$ (top-right), $\alpha = 28$ (bottom-left) and $\alpha = 46$ (bottom-right).

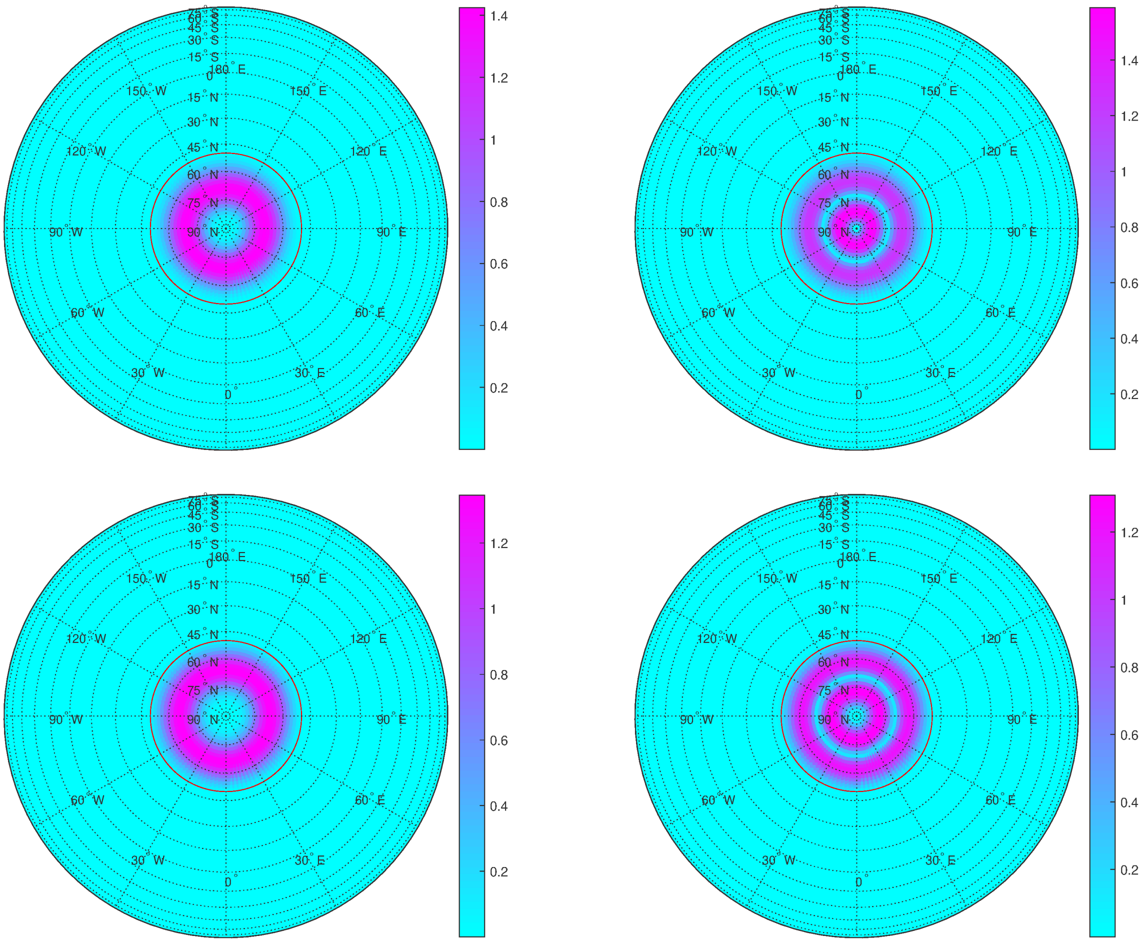


Figure 8.5: The norm of the best concentrated tensor Slepian functions with bandlimit $L = 18$ in a spherical cap with radius $40°$ at the North pole. The cap is marked as the inner of the red circle. Here, $\alpha = 55$ (top-left), $\alpha = 73$ (top-right), $\alpha = 91$ (bottom-left) and $\alpha = 109$ (bottom-right).

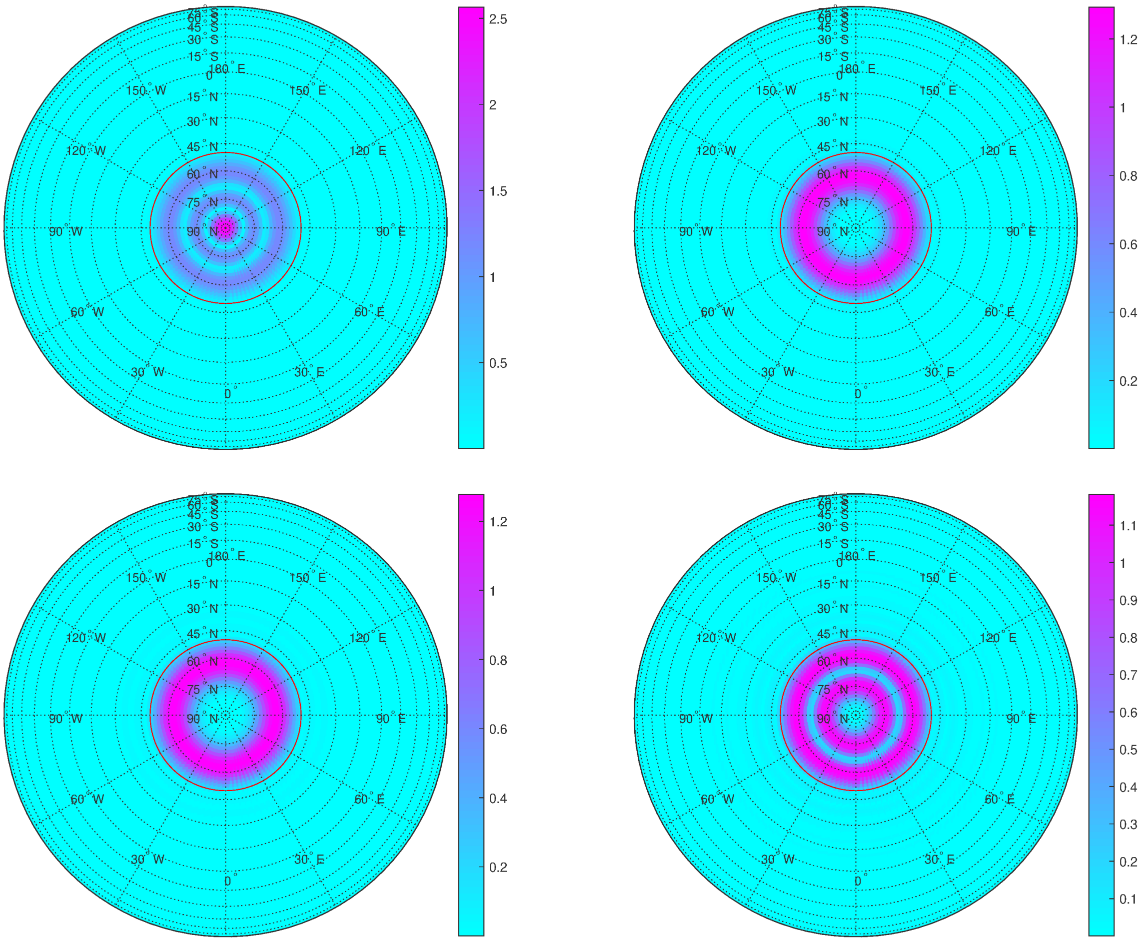


Figure 8.6: The norm of the best concentrated tensor Slepian functions with bandlimit $L = 18$ in a spherical cap with radius $40°$ at the North pole. The cap is marked as the inner of the red circle. Here, $\alpha = 127$ (top-left), $\alpha = 136$ (top-right), $\alpha = 148$ (bottom-left) and $\alpha = 154$ (bottom-right).

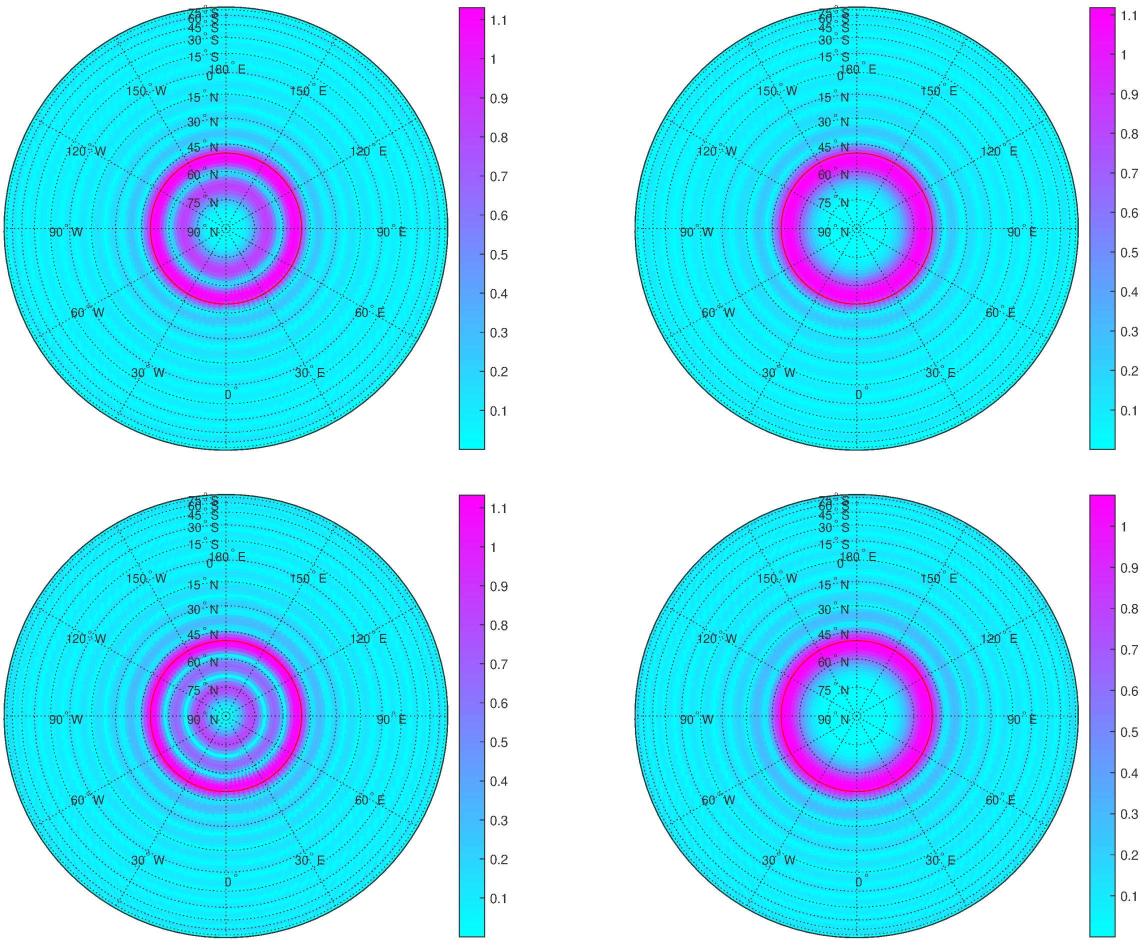


Figure 8.7: The norm of the tensor Slepian functions near to the Shannon number with bandlimit $L = 18$ in a spherical cap with radius $40°$ at the North pole. The cap is marked as the inner of the red circle. Here, $\alpha = 300$ (top-left), $\alpha = 304$ (top-right), $\alpha = 324$ (bottom-left) and $\alpha = 342$ (bottom-right).

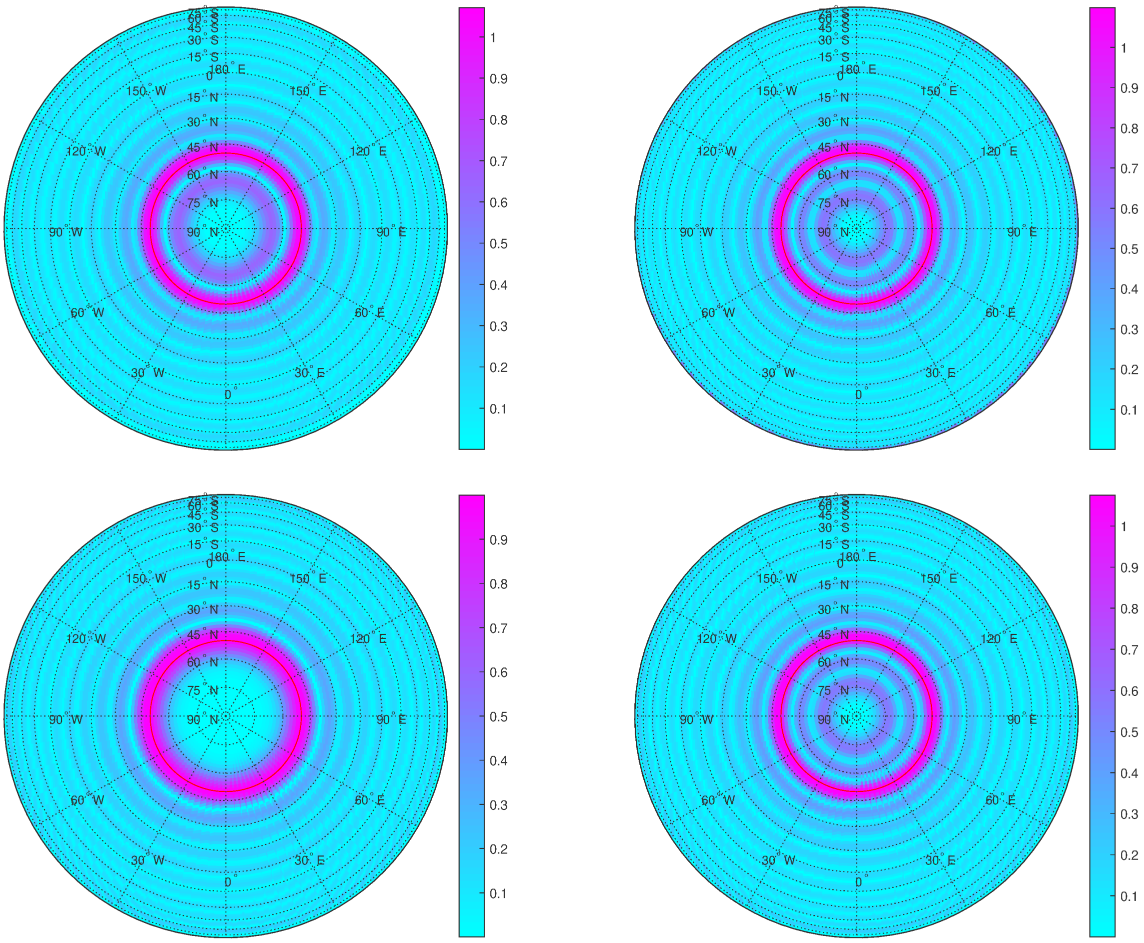


Figure 8.8: The norm of the tensor Slepian functions near to the Shannon number with bandlimit $L = 18$ in a spherical cap with radius $40°$ at the North pole. The cap is marked as the inner of the red circle. Here, $\alpha = 376$ (top-left), $\alpha = 379$ (top-right), $\alpha = 391$ (bottom-left) and $\alpha = 393$ (bottom-right).

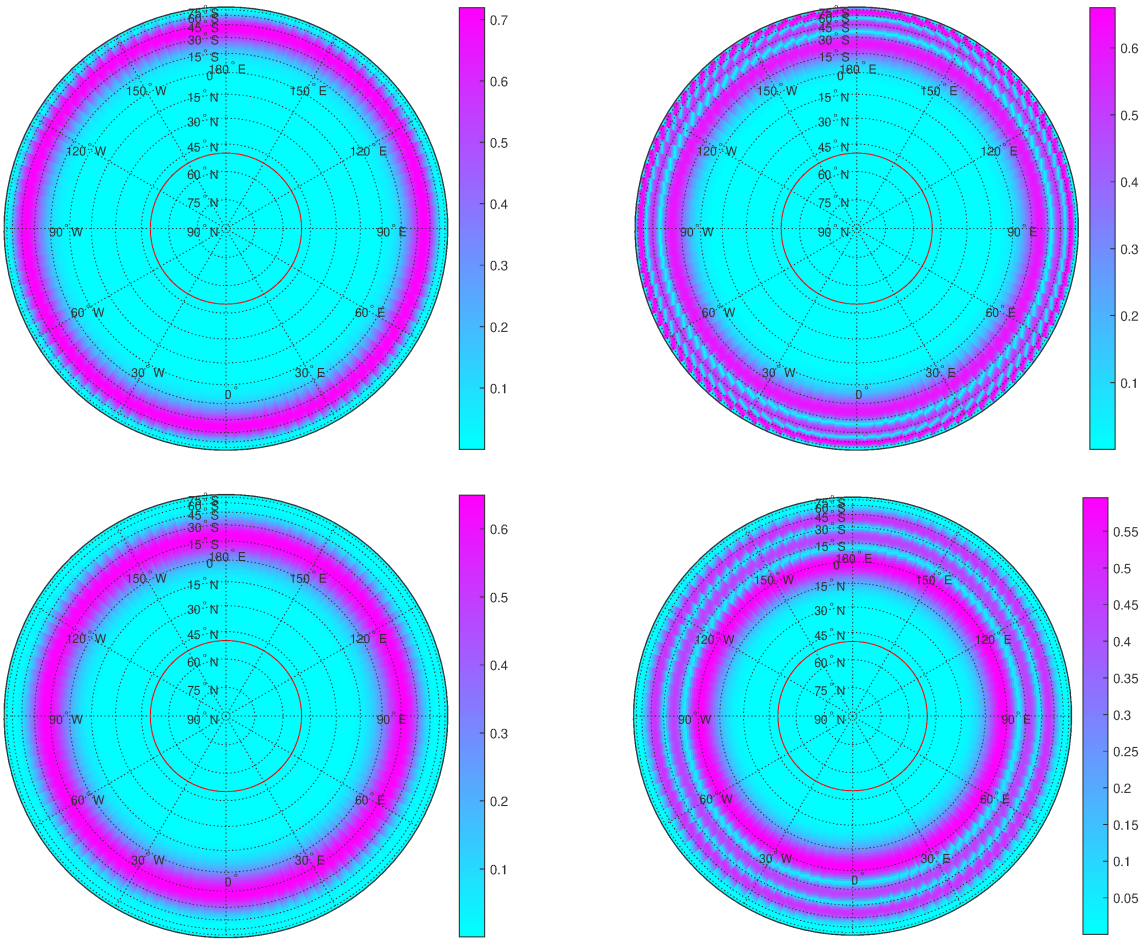


Figure 8.9: The norm of the less concentrated tensor Slepian functions with bandlimit $L = 18$ in a spherical cap with radius 40° at the North pole. The cap is marked as the inner of the red circle. Here, $\alpha = 3206$ (top-left), $\alpha = 3209$ (top-right), $\alpha = 3214$ (bottom-left) and $\alpha = 3225$ (bottom-right).

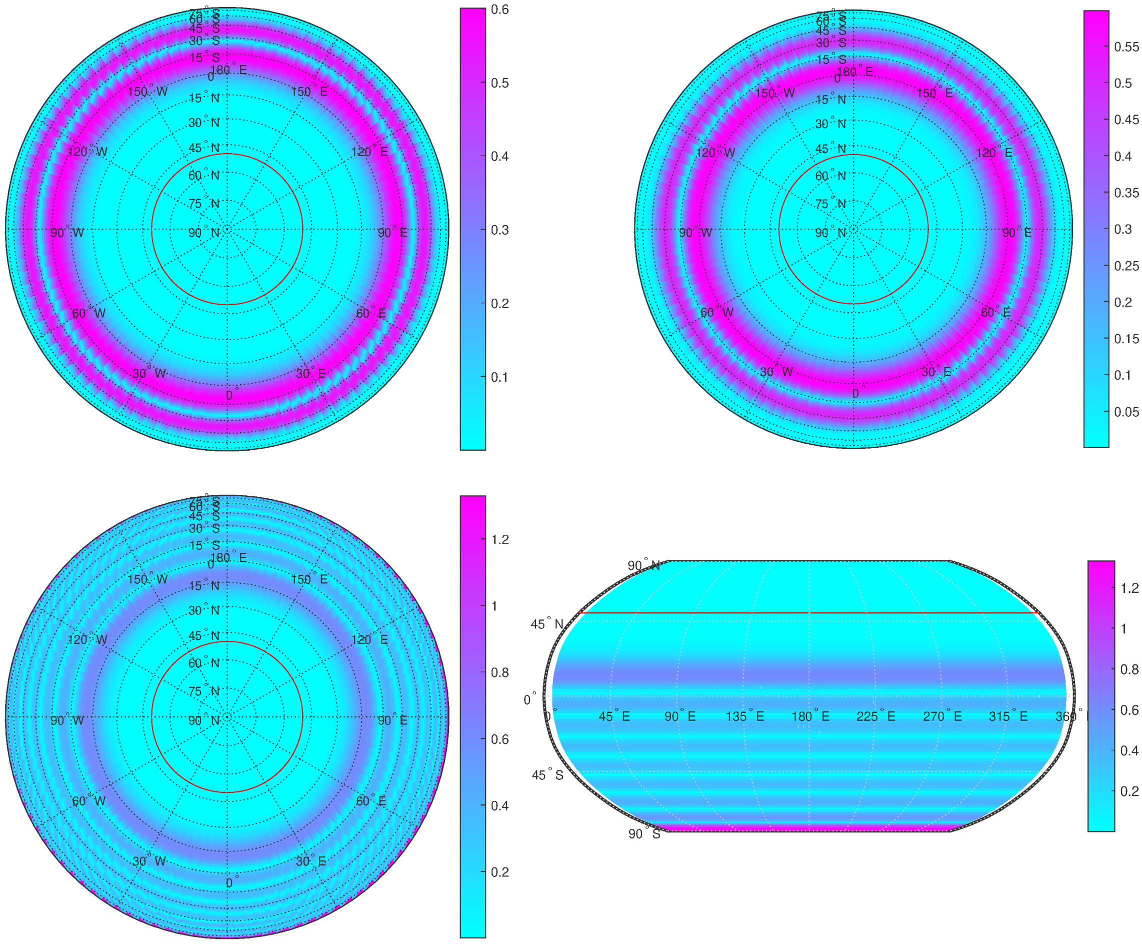


Figure 8.10: The norm of the less concentrated tensor Slepian functions with bandlimit $L = 18$ in a spherical cap with radius 40° at the North pole. The cap is marked as the inner of the red circle. Here, $\alpha = 3228$ (top-left), $\alpha = 3233$ (top-right), $\alpha = 3237$ for the same map projection (bottom-left) and for another map projection (bottom-right).

Chapter 9

CMB Polarization on the Spherical Cap

The following content is described in [1, 42, 45, 46, 53, 62, 76, 97, 99, 102]. The CMB (cosmic microwave background) anisotropies are used for studies of the early universe and for classical cosmology, in details for the study of geometric structure, energy content, and evolution of the universe. Furthermore, the CMB anisotropies determine weather density perturbations.

An important fact is that the CMB anisotropies can completely be studied in terms of temperature and polarization anisotropies. The temperature anisotropies are scalar and the polarization anisotropies are tensor fields. Moreover, the polarization is measured and described by scalar fields, the Stokes parameters Q and U.

From practical experiments, we do not get the CMB anisotropies on the full sphere due to various reasons. For ground and balloon based experiments, we get only measurements on very small partions of the sky. For satellite based experiments like COBE (Cosmic Background Explorer) and WMAP (Wilkinson Microwave Anisotropy Probe), data from the galactic region is strongly determined by different types of galactic emission. All in all, data from CMB anisotropies are only given on parts of the unit sphere. They are mostly given on the hemisphere, which can be described as a spherical cap. So, using Slepian functions on spherical caps to analyze CMB anisotropies is a valuable method.

Because the temperature anisotropies are scalar fields, they can be written in terms of the spherical harmonics by

$$T = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} a_{n,j} Y_{n,j},$$

where

$$a_{n,j} = \int_{\Omega} T(\xi) \overline{Y_{n,j}(\xi)} \, \mathrm{d}\omega(\xi).$$

So, the construction of the Slepian functions to analyze the temperature anisotropies is analogous to the construction of the scalar Slepian functions (see Chapter 4). This was previously done by [1] for the spherical cap and the spherical double cap. Moreover, the temperature anisotropies are the best probes of the early universe and their analysis is the most powerful test for determining cosmological parameters but we get more information by the analysis of the polarization.

The polarization helps to distinguish the gravitational potential. Moreover, it can be uniquely separated into an electric (E-mode) and a magnetic (B-mode) component, where E fields and B fields are orthogonal to each other on the unit sphere. The E-modes are produced by electron scattering. The B-modes are more complicated. These modes are produced by weak lensing of the electric polarization and from cosmic inflation, determined by the density of primordial gravitational waves.
Furthermore, detecting and excluding the magnetic component of the polarization has a fundamental significance in cosmology. So, the separability of the electric and magnetic component of the polarization is a property both needed and important. We know already that measurements of the polarization is only given on parts of the unit sphere, mostly on the hemisphere. So, we need this separation into electric and magnetic component not only on the unit sphere, but also on spherical caps. However, in the previously stated sources, a unique separation into electric and magnetic component on parts of the sphere is, for some reason, not given any longer.

So, finding a set of functions that is orthogonal on the spherical cap is useful for the analysis of the CMB anisotropies for both the temperature (see [1] and Chapter 4) as well as the polarization.

In this chapter, we will use the method of tensor Slepian functions from Chapter 6 in general, and in particular on the spherical cap from Chapter 8, for the CMB polarization, because this is a tensor valued field. First, we start with the definition of the CMB polarization. Then, we construct the tensor Slepian functions for the CMB polarization and in the end, we will perform multiple numerical experiments for synthetically generated polarization data.

9.1 Definition of the CMB Polarization

We start with the definition of the CMB polarization and its basis functions.

Definition 9.1.1. *The CMB (cosmic microwave background) polarization $\boldsymbol{p} \in \mathbf{l}^2(\Omega)$ is a tensor field. It is separated into an electric (E-mode) component $\boldsymbol{e} \in \mathbf{l}^2(\Omega)$ and a magnetic (B-mode) component $\boldsymbol{b} \in \mathbf{l}^2(\Omega)$ [1, 13, 44, 46, 53]. So,*

$$\boldsymbol{p} = \boldsymbol{e} + \boldsymbol{b}.$$

The basis functions for the CMB polarization, the electric and magnetic tensor spherical harmonics, are given for $n \geq 2$ and $j = -n, \ldots, n$ for the electric component by [1, 13, 62, 79, 90, 92]

$$\boldsymbol{y}^{\mathrm{E}}_{n,j} := \boldsymbol{y}^{(2,3)}_{n,j}$$

and for the magnetic component by

$$\boldsymbol{y}^{\mathrm{B}}_{n,j} := \boldsymbol{y}^{(3,2)}_{n,j}.$$

Remark 9.1.2. *The majority of authors (see [1, 13, 44, 62, 79, 90, 92]) use the notation $Y^{\mathrm{E}}_{n,j}$ and $Y^{\mathrm{B}}_{n,j}$ for the electric and magnetic tensor spherical harmonics. In [46], instead $Y^{\mathrm{G}}_{n,j}$ for "gradient" and $Y^{\mathrm{C}}_{n,j}$ for "curl" is used, because the electric component is called the curl-free part of the polarization and the magnetic component the curl part of the polarization [1, 44] (see also [46, 53, 62, 99]).*

Lemma 9.1.3. *We obtain the electric and magnetic tensor spherical harmonics for $\xi \in \Omega$*

$$\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) = \frac{1}{\sqrt{2}}\left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) + {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right),$$
$$\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) = \frac{-i}{\sqrt{2}}\left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) - {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right).$$

So, the term and therefore, the polarization has spin weight ± 2.

Proof. From Theorem 3.9.3, we know that for $\xi \in \Omega$

$$\frac{-1}{\sqrt{2}}\left(-\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) \pm i\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)\right) = {}_{\pm 2}Y_{n,j}(\xi)\left(\tau_\pm \otimes \tau_\pm\right).$$

Therefore, we get

$$\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) - i\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) = \sqrt{2}\;{}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+), \tag{9.1}$$
$$\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) + i\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) = \sqrt{2}\;{}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-). \tag{9.2}$$

Then, subtracting these equations leads to

$$2i\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) = -\sqrt{2}\;{}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+) + \sqrt{2}\;{}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-)$$

and consequently,

$$\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) = \frac{-i}{\sqrt{2}}\left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) - {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right).$$

By adding the two equations, (9.1) and (9.2), we get

$$\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) = \frac{1}{\sqrt{2}}\left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) + {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right).$$

□

Corollary 9.1.4. *So, it is obvious that we get for the CMB polarization*

$$\boldsymbol{p} \in \mathbf{l}^2_{\mathrm{CMB}}(\Omega) := \mathbf{l}^2_{(2,3)}(\Omega) \oplus \mathbf{l}^2_{(3,2)}(\Omega).$$

Definition 9.1.5. *The components of a by L bandlimited polarization $\boldsymbol{p} = \boldsymbol{e} + \boldsymbol{b} \in \mathbf{l}^2_{\mathrm{CMB}}(\Omega)$ are given by*

$$\boldsymbol{e} = \sum_{n=2}^{L}\sum_{j=-n}^{n} \boldsymbol{e}_{n,j}\boldsymbol{y}^{\mathrm{E}}_{n,j},$$
$$\boldsymbol{b} = \sum_{n=2}^{L}\sum_{j=-n}^{n} \boldsymbol{b}_{n,j}\boldsymbol{y}^{\mathrm{B}}_{n,j}$$

and we define the dimension $d := (L+1)^2 - 4$, where the polarization $\boldsymbol{p}$ consists of $2d$ basis functions.

9.2 Tensor Slepian Functions for CMB Polarization on Spherical Caps

Now, we want to calculate the Slepian functions for the CMB polarization. The resulting kernel matrix of the CMB polarization for the Slepian eigenvalue problem is *not* separated into an electric and a magnetic component. Consequently, the same holds true for the Slepian functions, because they are mutually dependend, too. However, we can use the spin-weighted spherical harmonics in the same way as in Chapter 6. Then, we have to solve the concentration problem for spin weight 2 and -2. Previously, we have solved this for the spherical cap (see Chapter 8).

So, first we look at the needed properties of the basis functions. With (see Lemma 6.1.11)

$$(\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\pm \otimes \tau_\pm)} = 1, \qquad (\tau_\pm \otimes \tau_\pm) : \overline{(\tau_\mp \otimes \tau_\mp)} = 0,$$

we obtain for $\xi \in \Omega$ [79]

$$\begin{aligned}
&\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)} \\
&= \frac{1}{2}\ \left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) + {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right) : \left(\overline{{}_{-2}Y_{n',j'}(\xi)}\ \overline{(\tau_- \otimes \tau_-)} + \overline{{}_{2}Y_{n',j'}(\xi)}\ \overline{(\tau_+ \otimes \tau_+)}\right) \\
&= \frac{1}{2}\ \left({}_{-2}Y_{n,j}(\xi)\ \overline{{}_{-2}Y_{n',j'}(\xi)} + {}_{2}Y_{n,j}(\xi)\ \overline{{}_{2}Y_{n',j'}(\xi)}\right), \\
&\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)} \\
&= \frac{1}{2}\ \left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) - {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right) : \left(\overline{{}_{-2}Y_{n',j'}(\xi)}\ \overline{(\tau_- \otimes \tau_-)} - \overline{{}_{2}Y_{n',j'}(\xi)}\ \overline{(\tau_+ \otimes \tau_+)}\right) \\
&= \frac{1}{2}\ \left({}_{-2}Y_{n,j}(\xi)\ \overline{{}_{-2}Y_{n',j'}(\xi)} + {}_{2}Y_{n,j}(\xi)\ \overline{{}_{2}Y_{n',j'}(\xi)}\right), \\
&\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)} \\
&= \frac{i}{2}\ \left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) + {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right) : \left(\overline{{}_{-2}Y_{n',j'}(\xi)}\ \overline{(\tau_- \otimes \tau_-)} - \overline{{}_{2}Y_{n',j'}(\xi)}\ \overline{(\tau_+ \otimes \tau_+)}\right) \\
&= \frac{i}{2}\ \left({}_{-2}Y_{n,j}(\xi)\ \overline{{}_{-2}Y_{n',j'}(\xi)} - {}_{2}Y_{n,j}(\xi)\ \overline{{}_{2}Y_{n',j'}(\xi)}\right), \\
&\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)} \\
&= \frac{-i}{2}\ \left({}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) - {}_{2}Y_{n,j}(\xi)(\tau_+ \otimes \tau_+)\right) : \left(\overline{{}_{-2}Y_{n',j'}(\xi)}\ \overline{(\tau_- \otimes \tau_-)} + \overline{{}_{2}Y_{n',j'}(\xi)}\ \overline{(\tau_+ \otimes \tau_+)}\right) \\
&= \frac{-i}{2}\ \left({}_{-2}Y_{n,j}(\xi)\ \overline{{}_{-2}Y_{n',j'}(\xi)} - {}_{2}Y_{n,j}(\xi)\ \overline{{}_{2}Y_{n',j'}(\xi)}\right).
\end{aligned}$$

With Theorem 3.4.21,

$$\int_\Omega {}_{N}Y_{n,j}(\xi)\ \overline{{}_{N}Y_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) = \delta_{n,n'}\delta_{j,j'},$$

we get [79]

$$\begin{aligned}
\int_\Omega \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) &= \int_\Omega \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) \\
&= \frac{1}{2}\int_\Omega \left({}_{-2}Y_{n,j}(\xi)\ \overline{{}_{-2}Y_{n',j'}(\xi)} + {}_{2}Y_{n,j}(\xi)\ \overline{{}_{2}Y_{n',j'}(\xi)}\right)\ \mathrm{d}\omega(\xi)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\ (\delta_{n,n'}\delta_{j,j'} + \delta_{n,n'}\delta_{j,j'}) \\
&= \delta_{n,n'}\delta_{j,j'}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) &= -\int_\Omega \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) \\
&= \frac{i}{2}\int_\Omega \left({}_{-2}Y_{n,j}(\xi)\ \overline{{}_{-2}Y_{n',j'}(\xi)} - {}_{2}Y_{n,j}(\xi)\ \overline{{}_{2}Y_{n',j'}(\xi)}\right)\ \mathrm{d}\omega(\xi) \\
&= \frac{i}{2}\ (\delta_{n,n'}\delta_{j,j'} - \delta_{n,n'}\delta_{j,j'}) \\
&= 0.
\end{aligned}
$$

Next, we take a look at the CMB concentration problem.

Problem 9.2.1. *We want to calculate the by L bandlimited polarization $\boldsymbol{p}$ optimally concentrated within the region $R \subset \Omega$. Then, the concentration problem is given by*

$$
\lambda = \frac{\int_R \boldsymbol{p}(\xi) : \overline{\boldsymbol{p}(\xi)}\ \mathrm{d}\omega(\xi)}{\int_\Omega \boldsymbol{p}(\xi) : \overline{\boldsymbol{p}(\xi)}\ \mathrm{d}\omega(\xi)} = \max,
$$

where

$$
\boldsymbol{p}(\xi) = \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\boldsymbol{e}_{n,j}\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) + \boldsymbol{b}_{n,j}\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)\right).
$$

Then, we get

$$
\begin{aligned}
&\int_R \boldsymbol{p}(\xi) : \overline{\boldsymbol{p}(\xi)}\ \mathrm{d}\omega(\xi) \\
&= \sum_{n=2}^{L}\sum_{j=-n}^{n}\sum_{n'=2}^{L}\sum_{j'=-n'}^{n'} \int_R \left(\boldsymbol{e}_{n,j}\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) + \boldsymbol{b}_{n,j}\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)\right) : \left(\overline{\boldsymbol{e}_{n',j'}}\overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)} + \overline{\boldsymbol{b}_{n',j'}}\overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)}\right)\ \mathrm{d}\omega(\xi) \\
&= \sum_{n=2}^{L}\sum_{j=-n}^{n}\sum_{n'=2}^{L}\sum_{j'=-n'}^{n'} \Bigg(\boldsymbol{e}_{n,j}\overline{\boldsymbol{e}_{n',j'}} \int_R \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) + \boldsymbol{e}_{n,j}\overline{\boldsymbol{b}_{n',j'}} \int_R \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) \\
&\quad + \boldsymbol{b}_{n,j}\overline{\boldsymbol{e}_{n',j'}} \int_R \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) + \boldsymbol{b}_{n,j}\overline{\boldsymbol{b}_{n',j'}} \int_R \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi)\Bigg)
\end{aligned}
$$

and analogously,

$$
\begin{aligned}
&\int_\Omega \boldsymbol{p}(\xi) : \overline{\boldsymbol{p}(\xi)}\ \mathrm{d}\omega(\xi) \\
&= \sum_{n=2}^{L}\sum_{j=-n}^{n}\sum_{n'=2}^{L}\sum_{j'=-n'}^{n'} \Bigg(\boldsymbol{e}_{n,j}\overline{\boldsymbol{e}_{n',j'}} \int_\Omega \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) + \boldsymbol{e}_{n,j}\overline{\boldsymbol{b}_{n',j'}} \int_\Omega \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) \\
&\quad + \boldsymbol{b}_{n,j}\overline{\boldsymbol{e}_{n',j'}} \int_\Omega \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi) + \boldsymbol{b}_{n,j}\overline{\boldsymbol{b}_{n',j'}} \int_\Omega \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) : \overline{\boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi)}\ \mathrm{d}\omega(\xi)\Bigg) \\
&= \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\boldsymbol{e}_{n,j}\overline{\boldsymbol{e}_{n,j}} + \boldsymbol{b}_{n,j}\overline{\boldsymbol{b}_{n,j}}\right).
\end{aligned}
$$

Problem 9.2.2. *So, the concentration problem leads to the eigenvalue problem*

$$\lambda = \frac{\overline{\boldsymbol{g}}^{\mathrm{T}} \boldsymbol{k} \boldsymbol{g}}{\overline{\boldsymbol{g}}^{\mathrm{T}} \boldsymbol{g}} = \max,$$

where

$$\boldsymbol{k} := \begin{pmatrix} \boldsymbol{k}^{\mathrm{E}} & \boldsymbol{k}^{\mathrm{EB}} \\ \overline{\left(\boldsymbol{k}^{\mathrm{EB}}\right)}^{\mathrm{T}} & \boldsymbol{k}^{\mathrm{B}} \end{pmatrix}, \qquad \boldsymbol{g} := (\boldsymbol{e}_{2,-2}, \ldots, \boldsymbol{e}_{L,L}, \boldsymbol{b}_{2,-2}, \ldots, \boldsymbol{b}_{L,L})^{\mathrm{T}},$$

$$\boldsymbol{k}^{\mathrm{E}}_{nj,n'j'} := \int_R \overline{\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi)} : \boldsymbol{y}^{\mathrm{E}}_{n',j'}(\xi) \; \mathrm{d}\omega(\xi),$$
$$\boldsymbol{k}^{\mathrm{B}}_{nj,n'j'} := \int_R \overline{\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)} : \boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi) \; \mathrm{d}\omega(\xi),$$

and

$$\boldsymbol{k}^{\mathrm{EB}}_{nj,n'j'} := \int_R \overline{\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi)} : \boldsymbol{y}^{\mathrm{B}}_{n',j'}(\xi) \; \mathrm{d}\omega(\xi).$$

Now, we use the spin-weighted spherical harmonics in the same way like in Chapter 6. So, for $\xi \in \Omega$ with

$$\frac{-1}{\sqrt{2}} \left(-\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) \pm i\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)\right) = {}_{\pm 2}Y_{n,j}(\xi)(\tau_\pm \otimes \tau_\pm) \tag{9.3}$$

and with

$$\int_M \left({}_{\pm 2}Y_{n,j}(\xi)(\tau_\pm \otimes \tau_\pm)\right) : \left(\overline{{}_{\pm 2}Y_{n',j'}(\xi)(\tau_\pm \otimes \tau_\pm)}\right) \; \mathrm{d}\omega(\xi) = \int_M {}_{\pm 2}Y_{n,j}(\xi)\overline{{}_{\pm 2}Y_{n',j'}(\xi)} \; \mathrm{d}\omega(\xi)$$
$$= \begin{cases} \delta_{n,n'}\delta_{j,j'}, & \text{for } M = \Omega \\ K^{\pm 2}_{nj,n'j'}, & \text{for } M = R \end{cases},$$
$$\int_M \left({}_{\pm 2}Y_{n,j}(\xi)(\tau_\pm \otimes \tau_\pm)\right) : \left(\overline{{}_{\mp 2}Y_{n',j'}(\xi)(\tau_\mp \otimes \tau_\mp)}\right) \; \mathrm{d}\omega(\xi) = 0$$

for $M \in \{\Omega, R\}$, we obtain the following problem.

Problem 9.2.3. *We get from the concentration problem the eigenvalue problem*

$$\boldsymbol{k}\boldsymbol{g} = \lambda \boldsymbol{g},$$

where

$$\boldsymbol{k} := \begin{pmatrix} K^2 & 0 \\ 0 & K^{-2} \end{pmatrix}, \qquad \boldsymbol{g} := \left(G^2_{2,-2}, \ldots, G^2_{L,L}, G^{-2}_{2,-2}, \ldots, G^{-2}_{L,L}\right)^{\mathrm{T}}.$$

So, we have to solve the eigenvalue problems for spin weight 2 *and* -2, *this means that*

$$K^{\pm 2} G^{\pm 2} = \lambda G^{\pm 2}.$$

This was previously discussed and solved in Chapter 6.1.4. With these results, we can formulate the vector of the Slepian coefficients by

$$\boldsymbol{g}_\alpha = \begin{cases} \begin{pmatrix} G^2_\alpha \\ 0 \end{pmatrix}, & \alpha = 1, \ldots, d \\ \begin{pmatrix} 0 \\ G^{-2}_{\alpha-d} \end{pmatrix}, & \alpha = d+1, \ldots, 2d \end{cases}$$

and the Slepian functions for the CMB concentration problem for $\xi \in \Omega$ by

$$\begin{aligned}\boldsymbol{g}_\alpha(\xi) &= \begin{cases} \mathscr{G}^2_\alpha(\xi)(\tau_+ \otimes \tau_+), & \alpha = 1, \ldots, d \\ \mathscr{G}^{-2}_{\alpha-d}(\xi)(\tau_- \otimes \tau_-), & \alpha = d+1, \ldots, 2d \end{cases} \\ &= \begin{cases} \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(G^2_{n,j}\right)_\alpha \, {}_2Y_{n,j}(\xi)(\tau_+ \otimes \tau_+), & \alpha = 1, \ldots, d \\ \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(G^{-2}_{n,j}\right)_{\alpha-d} \, {}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-), & \alpha = d+1, \ldots, 2d \end{cases}.\end{aligned} \tag{9.4}$$

Now, we can write the polarization in the basis of the electric and magnetic spherical harmonics and in the basis of the spin-weighted spherical harmonics. This means that for $\xi \in \Omega$

$$\begin{aligned}\boldsymbol{p}(\xi) &= \boldsymbol{e}(\xi) + \boldsymbol{b}(\xi) \\ &= \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{e}_{n,j} \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) + \boldsymbol{b}_{n,j} \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)\right) \\ &= \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(G^2_{n,j} \, {}_2Y_{n,j}(\xi)(\tau_+ \otimes \tau_+) + G^{-2}_{n,j} \, {}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-)\right).\end{aligned}$$

Furthermore, we can write the polarization in the basis of the spin-weighted Slepian functions for $\xi \in \Omega$ by

$$\begin{aligned}\boldsymbol{p}(\xi) &= \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \boldsymbol{g}_\alpha(\xi) \\ &= \sum_{\alpha=1}^{d} \boldsymbol{p}_\alpha \mathscr{G}^2_\alpha(\xi)(\tau_+ \otimes \tau_+) + \sum_{\alpha=d+1}^{2d} \boldsymbol{p}_\alpha \mathscr{G}^{-2}_{\alpha-d}(\xi)(\tau_- \otimes \tau_-).\end{aligned} \tag{9.5}$$

Next, we want to formulate the polarization in the basis of electric and magnetic tensor Slepian functions.

Theorem 9.2.4. *Now, we can formulate the polarization in the basis of the CMB Slepian functions with separation into electric and magnetic component. This means that*

$$\begin{aligned}\boldsymbol{p} &= \sum_{\alpha=1}^{2d} \boldsymbol{p}^{\mathrm{CMB}}_\alpha \boldsymbol{g}^{\mathrm{CMB}}_\alpha \\ &= \sum_{\alpha=1}^{2d} \left(\boldsymbol{p}_\alpha \boldsymbol{g}^{\mathrm{E}}_\alpha + \boldsymbol{p}_\alpha \boldsymbol{g}^{\mathrm{B}}_\alpha\right),\end{aligned}$$

where the electric and magnetic Slepian functions are given by

$$\begin{aligned}\boldsymbol{g}^{\mathrm{E}}_\alpha &= \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{g}^{\mathrm{E}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{E}}_{n,j}, \\ \boldsymbol{g}^{\mathrm{B}}_\alpha &= \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{g}^{\mathrm{B}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{B}}_{n,j}\end{aligned}$$

and the coefficients of their eigenvectors by

$$\left(\boldsymbol{g}^{\mathrm{E}}_{n,j}\right)_\alpha = \frac{1}{\sqrt{2}} \begin{cases} \left(G^{2}_{n,j}\right)_\alpha, & \alpha = 1,\dots,d \\ \left(G^{-2}_{n,j}\right)_{\alpha-d}, & \alpha = d+1,\dots,2d \end{cases},$$

$$\left(\boldsymbol{g}^{\mathrm{B}}_{n,j}\right)_\alpha = \frac{-i}{\sqrt{2}} \begin{cases} \left(G^{2}_{n,j}\right)_\alpha, & \alpha = 1,\dots,d \\ -\left(G^{-2}_{n,j}\right)_{\alpha-d}, & \alpha = d+1,\dots,2d \end{cases}$$

for all $\alpha = 1,\dots,2d$*, all* $n = 2,\dots,L$*, and all* $j = -n,\dots,n$*.*

Proof. To prove the theorem, we use the previous identities (9.3), (9.4), and (9.5). Then, we obtain for $\xi \in \Omega$

$$\begin{aligned}
\boldsymbol{p}(\xi) &= \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \boldsymbol{\mathscr{g}}_\alpha(\xi) \\
&= \sum_{\alpha=1}^{d} \boldsymbol{p}_\alpha \mathscr{G}^{2}_\alpha(\xi)(\tau_+ \otimes \tau_+) + \sum_{\alpha=d+1}^{2d} \boldsymbol{p}_\alpha \mathscr{G}^{-2}_{\alpha-d}(\xi)(\tau_- \otimes \tau_-) \\
&= \sum_{\alpha=1}^{d} \boldsymbol{p}_\alpha \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(G^{2}_{n,j}\right)_\alpha \ {}_2Y_{n,j}(\xi)(\tau_+ \otimes \tau_+) \\
&\quad + \sum_{\alpha=d+1}^{2d} \boldsymbol{p}_\alpha \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(G^{-2}_{n,j}\right)_{\alpha-d} \ {}_{-2}Y_{n,j}(\xi)(\tau_- \otimes \tau_-) \\
&= \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\sum_{\alpha=1}^{d} \boldsymbol{p}_\alpha \left(G^{2}_{n,j}\right)_\alpha \frac{-1}{\sqrt{2}} \left(-\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) + i\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)\right) \right. \\
&\qquad\qquad \left. + \sum_{\alpha=d+1}^{2d} \boldsymbol{p}_\alpha \left(G^{-2}_{n,j}\right)_{\alpha-d} \frac{-1}{\sqrt{2}} \left(-\boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) - i\boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi)\right) \right) \\
&= \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\frac{1}{\sqrt{2}} \left(\sum_{\alpha=1}^{d} \boldsymbol{p}_\alpha \left(G^{2}_{n,j}\right)_\alpha + \sum_{\alpha=d+1}^{2d} \boldsymbol{p}_\alpha \left(G^{-2}_{n,j}\right)_{\alpha-d} \right) \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) \right. \\
&\qquad\qquad \left. - \frac{i}{\sqrt{2}} \left(\sum_{\alpha=1}^{d} \boldsymbol{p}_\alpha \left(G^{2}_{n,j}\right)_\alpha - \sum_{\alpha=d+1}^{2d} \boldsymbol{p}_\alpha \left(G^{-2}_{n,j}\right)_{\alpha-d} \right) \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) \right).
\end{aligned}$$

Next, we see that

$$\begin{aligned}
\boldsymbol{p} &= \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\boldsymbol{g}^{\mathrm{E}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{E}}_{n,j} + \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\boldsymbol{g}^{\mathrm{B}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{B}}_{n,j} \\
&= \sum_{\alpha=1}^{2d} \left(\boldsymbol{p}_\alpha \boldsymbol{\mathscr{g}}^{\mathrm{E}}_\alpha + \boldsymbol{p}_\alpha \boldsymbol{\mathscr{g}}^{\mathrm{B}}_\alpha \right)
\end{aligned}$$

with

$$\boldsymbol{\mathscr{g}}^{\mathrm{E}}_\alpha = \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\boldsymbol{g}^{\mathrm{E}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{E}}_{n,j},$$

$$\boldsymbol{\mathscr{g}}^{\mathrm{B}}_\alpha = \sum_{n=2}^{L}\sum_{j=-n}^{n} \left(\boldsymbol{g}^{\mathrm{B}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{B}}_{n,j}$$

and

$$\left(\boldsymbol{g}_{n,j}^{\mathrm{E}}\right)_\alpha = \frac{1}{\sqrt{2}} \begin{cases} \left(G_{n,j}^{2}\right)_\alpha, & \alpha = 1, \ldots, d \\ \left(G_{n,j}^{-2}\right)_{\alpha-d}, & \alpha = d+1, \ldots, 2d \end{cases},$$

$$\left(\boldsymbol{g}_{n,j}^{\mathrm{B}}\right)_\alpha = \frac{-i}{\sqrt{2}} \begin{cases} \left(G_{n,j}^{2}\right)_\alpha, & \alpha = 1, \ldots, d \\ -\left(G_{n,j}^{-2}\right)_{\alpha-d}, & \alpha = d+1, \ldots, 2d \end{cases}$$

for all $\alpha = 1, \ldots, 2d$, all $n = 2, \ldots, L$, and all $j = -n, \ldots, n$. □

Then, we know that the CMB tensor Slepian functions can be separated into an electric and a magnetic component by

$$\boldsymbol{\mathcal{g}}_\alpha = \boldsymbol{\mathcal{g}}_\alpha^{\mathrm{CMB}} = \boldsymbol{\mathcal{g}}_\alpha^{\mathrm{E}} + \boldsymbol{\mathcal{g}}_\alpha^{\mathrm{B}}$$

for $\alpha = 1, \ldots, 2d$. Note that, for this reason, we also have $2d$ functions per electric and per magnetic tensor Slepian functions. They cannot generate a basis, because this is twice as much as electric and magnetic tensor spherical harmonics basis functions.

So, the electric component is given by

$$\begin{aligned} \boldsymbol{e} &= \sum_{n=2}^{L} \sum_{j=-n}^{n} \boldsymbol{e}_{n,j} \boldsymbol{y}_{n,j}^{\mathrm{E}} \\ &= \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \boldsymbol{\mathcal{g}}_\alpha^{\mathrm{E}} \\ &= \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{g}_{n,j}^{\mathrm{E}}\right)_\alpha \boldsymbol{y}_{n,j}^{\mathrm{E}} \end{aligned}$$

with the coefficients

$$\boldsymbol{e}_{n,j} = \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \left(\boldsymbol{g}_{n,j}^{\mathrm{E}}\right)_\alpha .$$

The magnetic component is given by

$$\begin{aligned} \boldsymbol{b} &= \sum_{n=2}^{L} \sum_{j=-n}^{n} \boldsymbol{b}_{n,j} \boldsymbol{y}_{n,j}^{\mathrm{B}} \\ &= \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \boldsymbol{\mathcal{g}}_\alpha^{\mathrm{B}} \\ &= \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{g}_{n,j}^{\mathrm{B}}\right)_\alpha \boldsymbol{y}_{n,j}^{\mathrm{B}} \end{aligned}$$

with the coefficients

$$\boldsymbol{b}_{n,j} = \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \left(\boldsymbol{g}_{n,j}^{\mathrm{B}}\right)_\alpha .$$

Now, we look at properties of the CMB tensor Slepian functions.

Theorem 9.2.5. *The CMB tensor spherical harmonics form a complete orthonormal basis* *of* $\mathrm{l}^2_{\mathrm{CMB}}(\Omega)$. *Consequently, for* $\boldsymbol{f} \in \mathrm{l}^2_{\mathrm{CMB}}(\Omega)$, *we obtain*

$$\lim_{L\to\infty} \left\| \boldsymbol{f} - \sum_{\mathrm{M}\in\{\mathrm{E},\mathrm{B}\}} \sum_{n=2}^{L} \sum_{j=-n}^{n} \langle \boldsymbol{f}, \boldsymbol{y}^{\mathrm{M}}_{n,j} \rangle_{\mathrm{l}^2(\Omega)} \boldsymbol{y}^{\mathrm{M}}_{n,j} \right\|_{\mathrm{l}^2(\Omega)} = 0.$$

This means that every function $\boldsymbol{f} \in \mathrm{l}^2_{\mathrm{CMB}}(\Omega)$ *can be written uniquely in the* $\mathrm{l}^2(\Omega)$*-sense in terms of a Fourier series by*

$$\boldsymbol{f} = \sum_{\mathrm{M}\in\{\mathrm{E},\mathrm{B}\}} \sum_{n=2}^{\infty} \sum_{j=-n}^{n} \langle \boldsymbol{f}, \boldsymbol{y}^{\mathrm{M}}_{n,j} \rangle_{\mathrm{l}^2(\Omega)} \boldsymbol{y}^{\mathrm{M}}_{n,j}.$$

Furthermore, for every function $\boldsymbol{f}, \boldsymbol{g} \in \mathrm{l}^2_{\mathrm{CMB}}(\Omega)$, *the Parseval identity is fulfilled such that*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{\mathrm{l}^2(\Omega)} = \sum_{\mathrm{M}\in\{\mathrm{E},\mathrm{B}\}} \sum_{n=2}^{\infty} \sum_{j=-n}^{n} \langle \boldsymbol{f}, \boldsymbol{y}^{\mathrm{M}}_{n,j} \rangle_{\mathrm{l}^2(\Omega)} \overline{\langle \boldsymbol{g}, \boldsymbol{y}^{\mathrm{M}}_{n,j} \rangle_{\mathrm{l}^2(\Omega)}}.$$

The proof holds true, because of Theorem 2.6.12.

Theorem 9.2.6. *We also get the following properties for the CMB tensor Slepian functions. The Slepian functions are orthonormal on the unit sphere and orthogonal on the region of interest R. This means that*

$$\langle \boldsymbol{g}_\alpha, \boldsymbol{g}_\beta \rangle_{\mathrm{l}^2(\Omega)} = \delta_{\alpha,\beta}, \tag{9.6}$$

$$\langle \boldsymbol{g}_\alpha, \boldsymbol{g}_\beta \rangle_{\mathrm{l}^2(R)} = \lambda_\alpha \delta_{\alpha,\beta} \tag{9.7}$$

for all $\alpha, \beta = 1, \ldots, 2d$. *Furthermore, we get for the eigenvectors*

$$\sum_{n=2}^{L} \sum_{j=-n}^{n} \left(G^i_{n,j}\right)_\alpha \overline{\left(G^i_{n,j}\right)_\beta} = \delta_{\alpha,\beta},$$

$$\sum_{n=2}^{L} \sum_{j=-n}^{n} \sum_{n'=2}^{L} \sum_{j'=-n'}^{n'} \left(G^i_{n,j}\right)_\alpha \overline{K^i_{nj,n'j'}} \overline{\left(G^i_{n',j'}\right)_\beta} = \lambda_\alpha \delta_{\alpha,\beta}$$

for all $\alpha, \beta = 1, \ldots, d$ *and all* $i \in \{-2, 2\}$.

Theorem 9.2.7. *The CMB tensor Slepian functions* $\{\boldsymbol{g}_\alpha\}_{\alpha=1,\ldots,2d}$ *form a complete orthonormal basis system of the by L bandlimited subset of* $\mathrm{l}^2_{\mathrm{CMB}}(\Omega)$. *Therefore, we can write every by L bandlimited tensor field* $\boldsymbol{f} \in \mathrm{l}^2_{\mathrm{CMB}}(\Omega)$ *in the basis of the CMB tensor spherical harmonics and in the basis of the CMB tensor Slepian functions. This means that for* $\xi \in \Omega$

$$\begin{aligned} \boldsymbol{f}(\xi) &= \sum_{\mathrm{M}\in\{\mathrm{E},\mathrm{B}\}} \sum_{n=2}^{L} \sum_{j=-n}^{n} \underbrace{\langle \boldsymbol{f}, \boldsymbol{y}^{\mathrm{M}}_{n,j} \rangle_{\mathrm{l}^2(\Omega)}}_{=:\boldsymbol{f}^{\mathrm{M}}_{n,j}} \boldsymbol{y}^{\mathrm{M}}_{n,j}(\xi) \\ &= \sum_{\alpha=1}^{2d} \underbrace{\langle \boldsymbol{f}, \boldsymbol{g}_\alpha \rangle_{\mathrm{l}^2(\Omega)}}_{=:\boldsymbol{f}_\alpha} \boldsymbol{g}_\alpha(\xi). \end{aligned}$$

Theorem 9.2.8. *Additionally, we get for* $\xi \in \Omega$ *with*

$$\boldsymbol{y}^{\pm 2}_{n,j}(\xi) := {}_{\pm 2}Y_{n,j}(\xi) \left(\tau_\pm \otimes \tau_\pm\right)$$

the properties

$$\boldsymbol{y}^i_{n,j}(\xi) = \sum_{\alpha=1}^{d} \overline{\left(G^i_{n,j}\right)_\alpha} \boldsymbol{g}_\alpha(\xi),$$

$$\sum_{\alpha=1}^{2d} \left(\boldsymbol{g}_{n,j}\right)_\alpha \overline{\left(\boldsymbol{g}_{n',j'}\right)_\alpha} = \delta_{n,n'}\delta_{j,j'}$$

for all $i, i' \in \{-2, 2\}$, *all* $n, n' = 2, \ldots, L$, *all* $j = -n, \ldots, n$, *and all* $j' = -n', \ldots, n'$.

Theorem 9.2.9. *The CMB tensor Slepian functions also fulfill the following properties*

$$\sum_{\alpha=1}^{2d} \lambda_\alpha \left(\boldsymbol{g}_{n,j}\right)_\alpha \overline{\left(\boldsymbol{g}_{n',j'}\right)_\alpha} = \boldsymbol{k}_{nj,n'j'},$$

$$\sum_{\alpha=1}^{2d} \lambda_\alpha \boldsymbol{g}_\alpha(\xi)\overline{\boldsymbol{g}_\alpha(\eta)} = \sum_{i\in\{-2,2\}} \sum_{n=2}^{L} \sum_{j=-n}^{n} \sum_{i'\in\{-2,2\}} \sum_{n'=2}^{L} \sum_{j'=-n'}^{n'} \boldsymbol{y}^i_{n,j}(\xi) K^i_{nj,n'j'} \delta_{i,i'} \overline{\boldsymbol{y}^{i'}_{n',j'}(\eta)}$$

for all $n, n' = 2, \ldots, L$, *all* $j = -n, \ldots, n$, *and* $j' = -n', \ldots, n'$ *and for* $\xi, \eta \in \Omega$.

The proofs are analogous to the proofs of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, and Theorem 4.2.4.

Next, we calculate the polarization coefficients.

Theorem 9.2.10. *The polarization coefficients in the basis of the Slepian functions can be calculated by*

$$\boldsymbol{p}_\alpha = \frac{1}{\lambda_\alpha} \left\langle \boldsymbol{p}, \boldsymbol{g}_\alpha \right\rangle_{\mathrm{l}^2(R)}.$$

Proof. With (9.5), we see that

$$\begin{aligned}
\left\langle \boldsymbol{p}, \boldsymbol{g}_\alpha \right\rangle_{\mathrm{l}^2(R)} &= \int_R \boldsymbol{p}(\xi) : \overline{\boldsymbol{g}_\alpha(\xi)}\, \mathrm{d}\omega(\xi) \\
&= \sum_{\beta=1}^{2d} \boldsymbol{p}_\beta \int_R \boldsymbol{g}_\beta(\xi) : \overline{\boldsymbol{g}_\alpha(\xi)}\, \mathrm{d}\omega(\xi) \\
&= \sum_{\beta=1}^{2d} \boldsymbol{p}_\beta \left\langle \boldsymbol{g}_\beta, \boldsymbol{g}_\alpha \right\rangle_{\mathrm{l}^2(R)}.
\end{aligned}$$

With property (9.7) of the Slepian functions, we obtain

$$\begin{aligned}
\left\langle \boldsymbol{p}, \boldsymbol{g}_\alpha \right\rangle_{\mathrm{l}^2(R)} &= \sum_{\beta=1}^{2d} \boldsymbol{p}_\beta \lambda_\beta \delta_{\beta,\alpha} \\
&= \lambda_\alpha \boldsymbol{p}_\alpha.
\end{aligned}$$

So, we get the proposition. □

Finally, we remark the following important facts.

Remark 9.2.11. *Note that the electric and magnetic tensor Slepian functions are not orthogonal on the unit sphere and on* R, *because there exists twice the amount of electric* *or magnetic tensor Slepian functions compared to the electric or magnetic tensor spherical harmonics. However, we do not need it, because by utilizing the orthogonality of the spin-weighted Slepian functions of spin weight* 2 *and* −2*, we have the desired properties to calculate the polarization coefficients in the basis of the tensor Slepian functions. These coefficients are also the polarization coefficients in the basis of the electric and magnetic tensor Slepian functions.*

Remark 9.2.12. *The spectral concentration of spacelimited polarization works like for spacelimited tensor fields from Chapter 7.3, where we use solely the spin weight* 2 *and* −2 *case again.*

9.3 Application and Numerical Experiments

In this section, we want to approximate a given polarization with help of the CMB tensor Slepian functions. Here, we present our first numerical results. As before, all numerical calculations are performed with `MATLAB R2015b`.

We previously explained that in practical experiments, the CMB polarization can only be given on spherical caps in most cases. This is why we construct synthetic examples of given polarizations on the spherical cap in the following section.

We know from Theorem 9.2.10 that we can calculate the polarization coefficients for the Slepian functions by

$$\boldsymbol{p}_\alpha = \frac{1}{\lambda_\alpha} \langle \boldsymbol{p}, \boldsymbol{g}_\alpha \rangle_{l^2(R)}.$$

The integration of the inner product is calculated by two Gauß quadratures. It consists of a Gauß-Tschebyscheff and a Gauß-Legendre quadrature as described in Appendix A. Here, we use the `MATLAB` codes of Greg von Winckel [95] for the Gauß-Legendre quadrature weights and nodes and the `MATLAB` codes of Christian Gerhards for the combination of Gauß-Tschebyscheff and Gauß-Legendre quadrature on the spherical cap. The point grid of the integration over the spherical cap is described in Appendix B.1.

With the polarization coefficients for the Slepian functions, we know that we can reconstruct the polarization for $\xi \in \Omega$ by

$$\boldsymbol{p}(\xi) = \sum_{\alpha=1}^{2d} \boldsymbol{p}_\alpha \boldsymbol{g}_\alpha(\xi).$$

For comparison, we can calculate the polarization coefficients for the electric and magnetic spherical harmonics according to the least squares method, if the polarization is given on a point grid. Here, we have to solve

$$\boldsymbol{p}(\xi) = \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\boldsymbol{e}_{n,j} \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi) + \boldsymbol{b}_{n,j} \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi) \right).$$

The point grids used are described in Appendix B. These are the integration grid, the HEALPix grid, and the Reuter grid. For the spherical cap, we cut the HEALPix and the Reuter grid accordingly to the size of the cap. Although the HEALPix grid is a special grid of cosmology, we primarily show the main results on the integration grid for reasons of comparability.

So, we reconstruct the polarization by Slepian functions, where we calculate the polarization coefficients by integration on the integration grid. For comparison, we reconstruct the polarization by electric and magnetic tensor spherical harmonics. Here, we calculate the polarization coefficients by the least squares method mostly on the integration grid, but also on both the HEALPix grid and the Reuter grid. Furthermore, for the Slepian method, we also calculate the coefficients by the least squares method.

For the application, we construct a synthetic polarization on the appropriate point grid. Here, we induce the given polarization by

$$\boldsymbol{p}(\xi_k) = \sum_{n=2}^{L} \sum_{j=-n}^{n} \left(\tilde{\boldsymbol{e}}_{n,j} \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi_k) + \tilde{\boldsymbol{b}}_{n,j} \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi_k) \right) + \sum_{\alpha=1}^{S} \tilde{\boldsymbol{p}}_\alpha \boldsymbol{g}_\alpha(\xi_k)$$

for $\xi_k \in \Omega$, $k = 1, \ldots, M$, points on the grid. We choose this construction such that there is no preference either to the electric and magnetic tensor spherical harmonics or to the Slepian functions. The coefficients $\tilde{\boldsymbol{e}}_{n,j}$, $\tilde{\boldsymbol{b}}_{n,j}$ and $\tilde{\boldsymbol{p}}_\alpha$ are generated randomly with the `MATLAB` function `rand`, decreased by the degree of the tensor spherical harmonics. This means that we calculate with `rand` the coefficients $\varepsilon_{n,j}$ and $\beta_{n,j}$ and then, calculate the electric and magnetic tensor spherical harmonic coefficients by

$$\tilde{\boldsymbol{e}}_{n,j} = \frac{1 + \varepsilon_{n,j}}{n}, \qquad \tilde{\boldsymbol{b}}_{n,j} = \frac{1 + \beta_{n,j}}{n}.$$

The coefficients $\tilde{\boldsymbol{p}}_\alpha$ are calculated only with `rand`.

Moreover, we can separate the polarization into its electric and its magnetic component. So, we know on the one hand that

$$\boldsymbol{e}(\xi) = \sum_{\alpha=1}^{2d} \left(\boldsymbol{g}^{\mathrm{E}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi), \qquad \boldsymbol{b}(\xi) = \sum_{\alpha=1}^{2d} \left(\boldsymbol{g}^{\mathrm{B}}_{n,j}\right)_\alpha \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi),$$

where

$$\left(\boldsymbol{g}^{\mathrm{E}}_{n,j}\right)_\alpha = \frac{1}{\sqrt{2}} \begin{cases} \left(G^{2}_{n,j}\right)_\alpha, & \alpha = 1, \ldots, d \\ \left(G^{-2}_{n,j}\right)_{\alpha-d}, & \alpha = d+1, \ldots, 2d \end{cases},$$
$$\left(\boldsymbol{g}^{\mathrm{B}}_{n,j}\right)_\alpha = \frac{-i}{\sqrt{2}} \begin{cases} \left(G^{2}_{n,j}\right)_\alpha, & \alpha = 1, \ldots, d \\ -\left(G^{-2}_{n,j}\right)_{\alpha-d}, & \alpha = d+1, \ldots, 2d \end{cases}.$$

On the other hand, we get

$$\boldsymbol{e}(\xi) = \sum_{n=2}^{L} \sum_{j=-n}^{n} \boldsymbol{e}_{n,j} \boldsymbol{y}^{\mathrm{E}}_{n,j}(\xi), \qquad \boldsymbol{b}(\xi) = \sum_{n=2}^{L} \sum_{j=-n}^{n} \boldsymbol{b}_{n,j} \boldsymbol{y}^{\mathrm{B}}_{n,j}(\xi).$$

In Figure 9.1, we see the norm of the given polarization for different sizes of spherical caps. Here, we show a given polarization for bandlimit $L = 18$ on the spherical cap with radius 40° at the North pole and on the hemisphere, which is the spherical cap with radius 90° at the North pole.
In the following figures, Figure 9.2 to Figure 9.5, we show the norm of the difference of the calculated polarization to the given polarizations. Furthermore, we plot all differences on

the spherical cap (left columns) and on the whole sphere (right columns).
In Figure 9.2, we show the polarization for bandlimit $L = 18$ on a spherical cap with radius $40°$ at the North pole by tensor Slepian functions with coefficients calculated by the least squares method on the integration grid. All tensor Slepian functions are included. In Figure 9.3, we see the reconstruction of the polarization by tensor Slepian functions with coefficients calculated by spherical integration. Here, we represent the method by including all tensor Slepian functions (top), all tensor Slepian functions with an eigenvalue larger than 10^{-2}, and all tensor Slepian functions up to the Shannon number. Figure 9.4 delivers the according results for the reconstruction of the polarization by the electric and magnetic tensor spherical harmonics with coefficients calculated by the least squares method. Here, the results on the integration grid (top), on the HEALPix grid (center), and on the Reuter grid (bottom) are represented.
Figure 9.5 shows the results for the case of bandlimit $L = 18$ on the hemisphere. We compare again the reconstruction of the polarization by tensor Slepian functions with coefficients calculated by spherical integration, where we include all Slepian functions (top) and all functions with an eigenvalue larger than 10^{-2} (center), to the reconstruction of the polarization by the electric and magnetic tensor spherical harmonics with coefficients calculated by the least squares method on the integration grid (bottom).

We can affirm that the method of tensor Slepian functions with coefficients calculated by integration yields robust results. A graphic representation of the results for a spherical cap of radius $40°$ and for the hemisphere is displayed in Figure 9.3 and Figure 9.5 and yields comparable results.
We compare the method of tensor Slepian functions (coefficients calculated by integration) with the construction of the polarization by the electric and magnetic spherical harmonics (coefficients calculated by the least squares method on different grids). On the integration grid, this comparison is displayed at the top of Figure 9.4 and at the bottom of Figure 9.5. Here, we see that the results of our method are comparable to the conventional method inside the cap, yet slightly inferior.
One reason for this could be the numerical error caused by the Gauß quadrature, as the spin-weighted spherical harmonics are no polynomials and thus are not exactly integrated. So, improving the integration method for getting more accurate results with the tensor Slepian functions is a possibility for future investigations. Here, we have to construct a integration method, which is exact for the spin-weighted spherical harmonics. This is not trivial. Some connected problems are listed in Appendix A.1.3.

We suppose that the slight inferiority of our results can be eliminated by using the Shannon number for calculating the polarization. Furthermore, we suppose to get better results outside the spherical cap. So, we include not all tensor Slepian functions for the reconstruction of the polarization. This means that for $\xi \in \Omega$

$$\boldsymbol{p}(\xi) = \sum_{\alpha=1}^{S} \boldsymbol{p}_\alpha \boldsymbol{g}_\alpha(\xi).$$

Our numerical experiments have shown that the Shannon number is too small an estimate for the number of significant eigenvalues (see the bottom of Figure 9.3). Therefore, we enlarged the number by including all eigenfunctions with an eigenvalue larger than 10^{-2} (see the center of Figure 9.3). For the example in Figure 9.3, there are for $L = 18$ on the whole $d = 2\left((L+1)^2 - 4\right) = 714$ Slepian functions (top). The Shannon number is given by $S \approx 84$ (bottom). Our estimate is given by 137 (center).

We see that for including all tensor Slepian functions the error of the approximation inside the spherical cap is numerically zero, but outside there are huge errors. With help of the Shannon number, we get better results outside the cap, but the error inside the cap is increasing. For including all tensor Slepian functions with an eigenvalue larger than 10^{-2}, we obtain better results than for using the Shannon number. Indeed, the error inside the spherical cap is still distinctly larger than those of using all tensor Slepian functions and those of reconstructing the polarization by the electric and magnetic tensor spherical harmonics (see Figure 9.4). However, for that case, we get better results on the whole sphere and particularly, outside the cap than all comparable cases for a sufficiently large bandlimit. We also obtain similar results for the example shown in Figure 9.5.

The reconstruction of the polarization by the method of tensor Slepian functions with coefficients calculated by the least squares method on the integration grid is given in Figure 9.2. We show this case with including all tensor Slepian functions. Here, we see that the error at the boundary of the spherical cap is large. However, on the whole sphere, we get similar results as for the reconstruction by the electric and magnetic tensor spherical harmonics.

All in all, our proposed new method yields robust results compared to the standard methods. However, further investigations need to be made. The first, and arguably most important point, is an even better estimation of some kind of Shannon number. Secondly, the calculation of polarization coefficients, by integration and also by the least squares method, is prone to error. Here, we can develop a integration method for the spin-weighted spherical harmonics. A third area for improvements is the regularization of the Slepian reconstruction by a smoothing filter. One possible approach might be the convolution with an approximate identity such as the cp-scaling function. Fourthly, our numerical experiments can be extended to either a different set of synthetic polarization data, for example for localized ones like the Abel-Poisson kernel, or to polarization data observed in real world. Lastly, the spherical harmonics, and therefore, the electric and magnetic tensor spherical harmonics, can produce artifacts for larger degrees in the case of regional data, especially with irregular point grids. So, it is interesting to see how the Slepian functions behave under such conditions. We already know that also the tensor Slepian functions produced artifacts in our experiments. However, there is a more sophisticated way for the Slepian functions to remove them. The integration on irregular point grids will be another challenge therein.

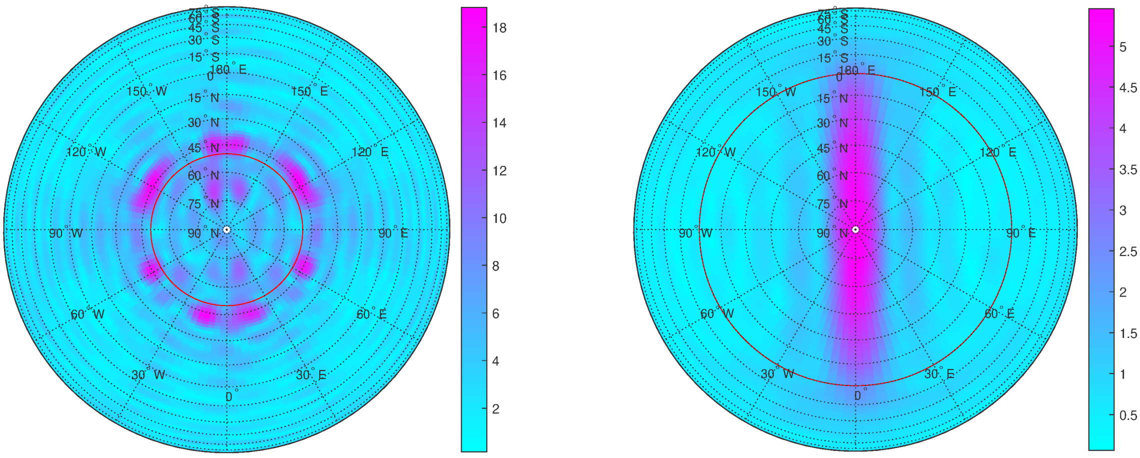


Figure 9.1: The norm of the *given polarization* in a spherical cap at the North pole. Here, for bandlimit $L = 18$ in a spherical cap with radius $40°$ (left) and in the hemisphere, a spherical cap with radius $90°$ (right). The red circle denotes the boundary of the spherical cap.

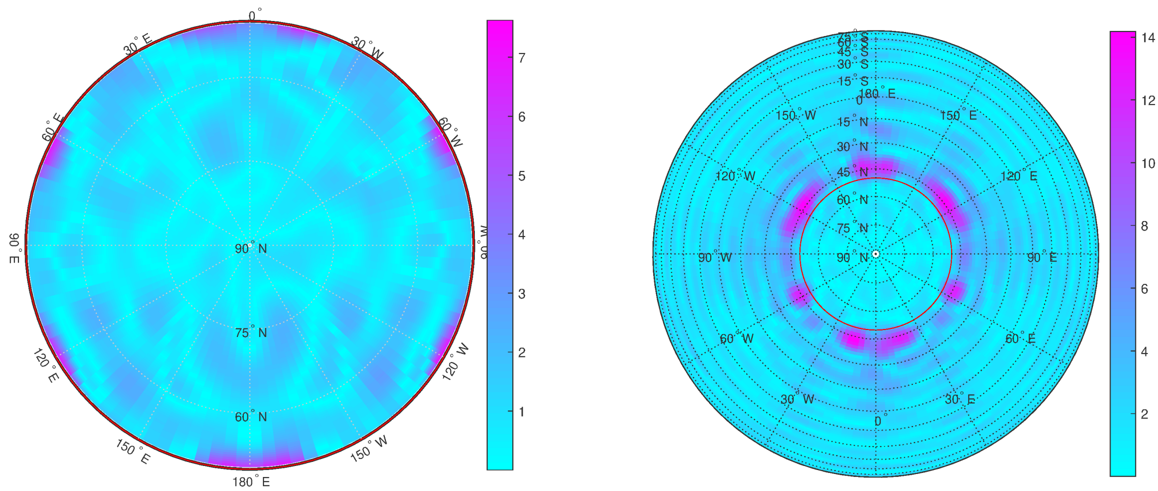


Figure 9.2: The norm of the difference of the calculated polarization to the given polarization with bandlimit $L = 18$ in a spherical cap with radius $40°$ at the North pole. Here, the polarization is calculated by *tensor Slepian functions with coefficients from the least squares method on the integration grid* with 10100 points on the spherical cap (left) and on the whole sphere (right). The red circle denotes the boundary of the spherical cap.

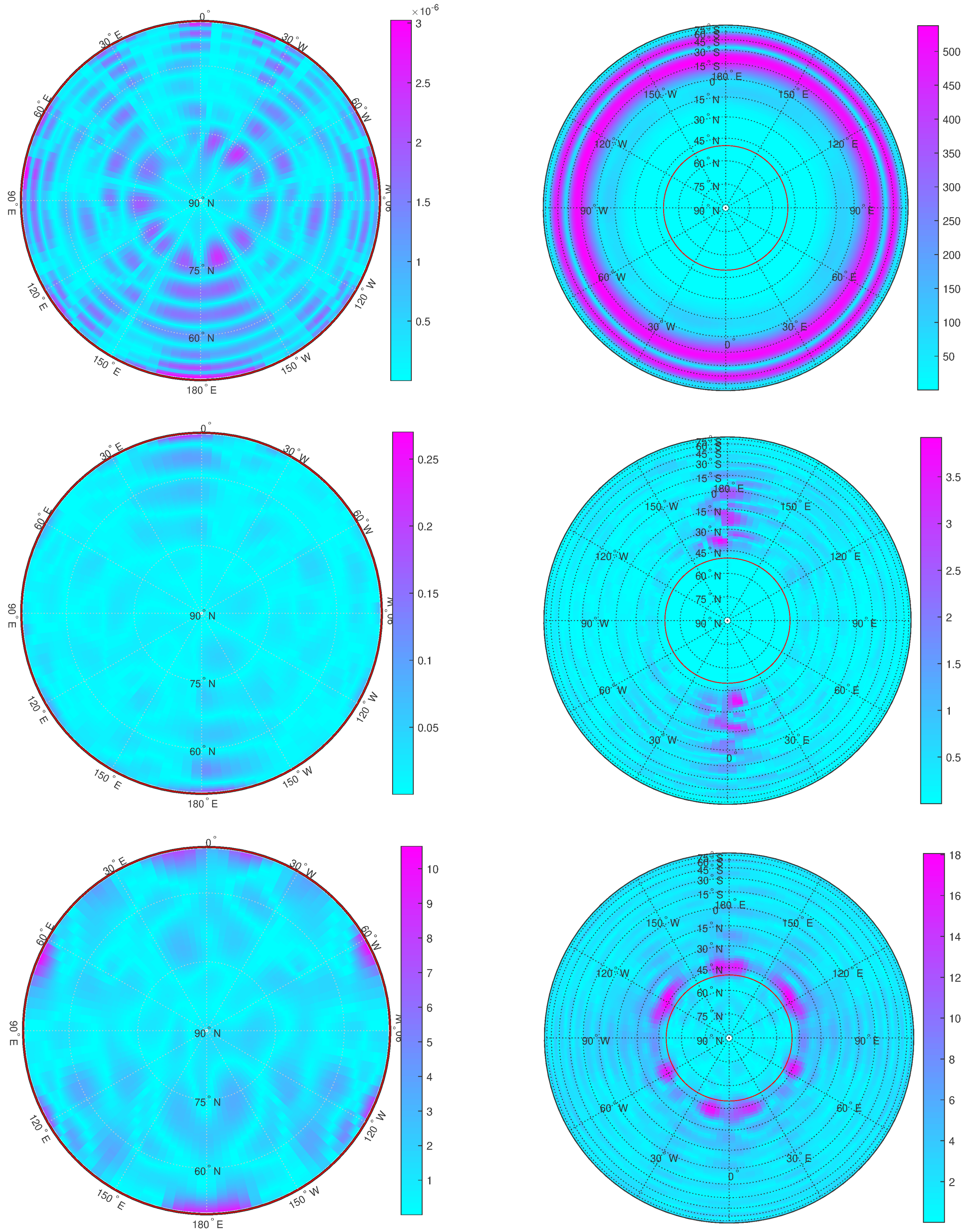


Figure 9.3: The norm of the difference of the calculated polarization to the given polarization with bandlimit $L = 18$ in a spherical cap with radius 40° at the North pole. Here, the polarization is calculated by *tensor Slepian functions with coefficients from integration* with 10100 points plotted on the spherical cap (left) and on the whole sphere (right). At the top, *all tensor Slepian functions* are used, in the center we use *all tensor Slepian functions with eigenvalues larger than* 10^{-2}, and at the bottom, *all tensor Slepian functions up to the Shannon number* are used.

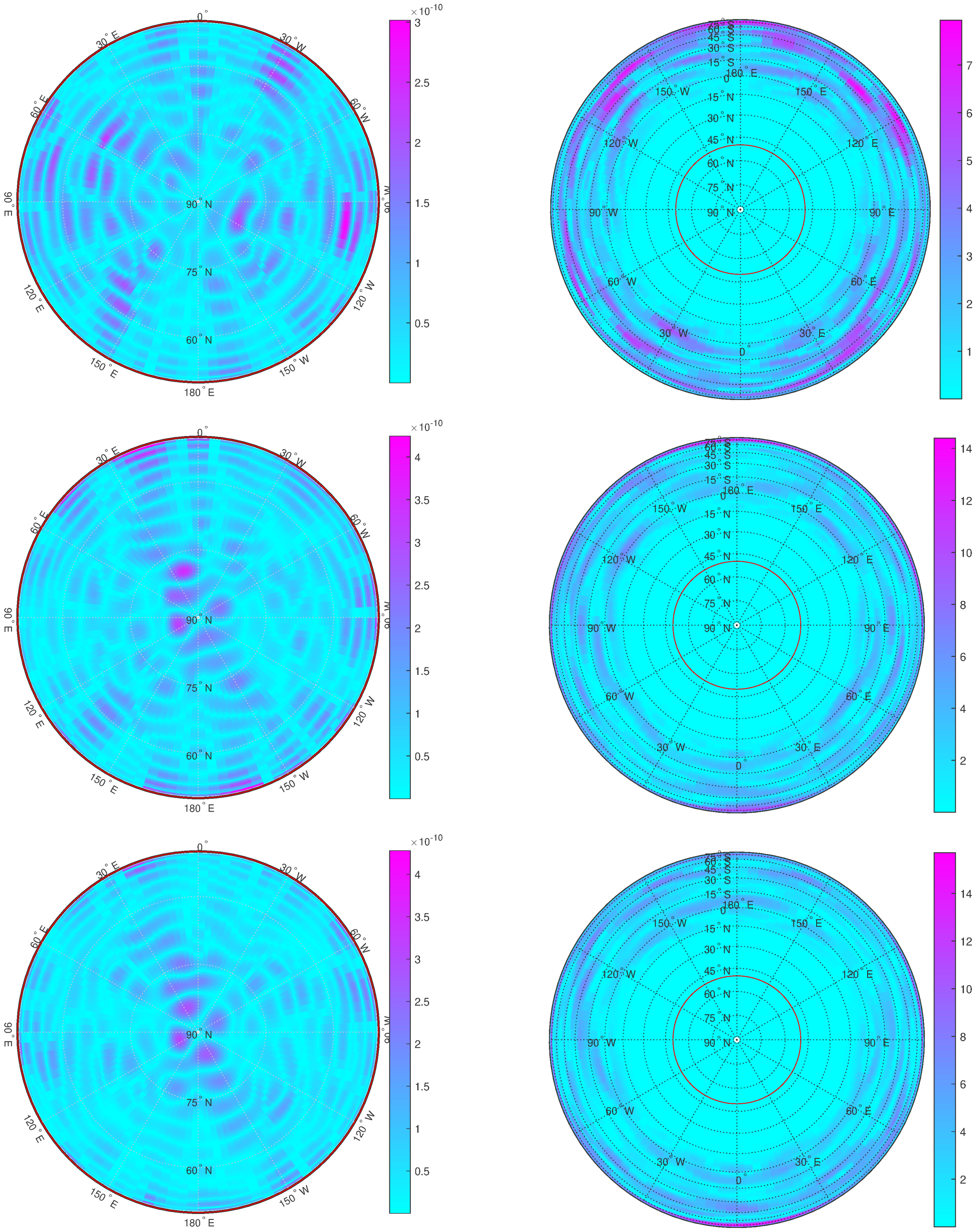


Figure 9.4: The norm of the difference of the calculated polarization to the given polarization with bandlimit $L = 18$ in a spherical cap with radius $40°$ at the North pole. Here, the polarization is calculated by *electric and magnetic spherical harmonics on the integration grid* with 10100 points plotted on the spherical cap (top-left) and on the whole sphere (top-right), *on the HEALPix grid* with 10224 points on the spherical cap (center-left) and on the whole sphere (center-right), and *on the Reuter grid* with 10281 points on the spherical cap (bottom-left) and on the whole sphere (bottom-right). The red circle denotes the boundary of the spherical cap.

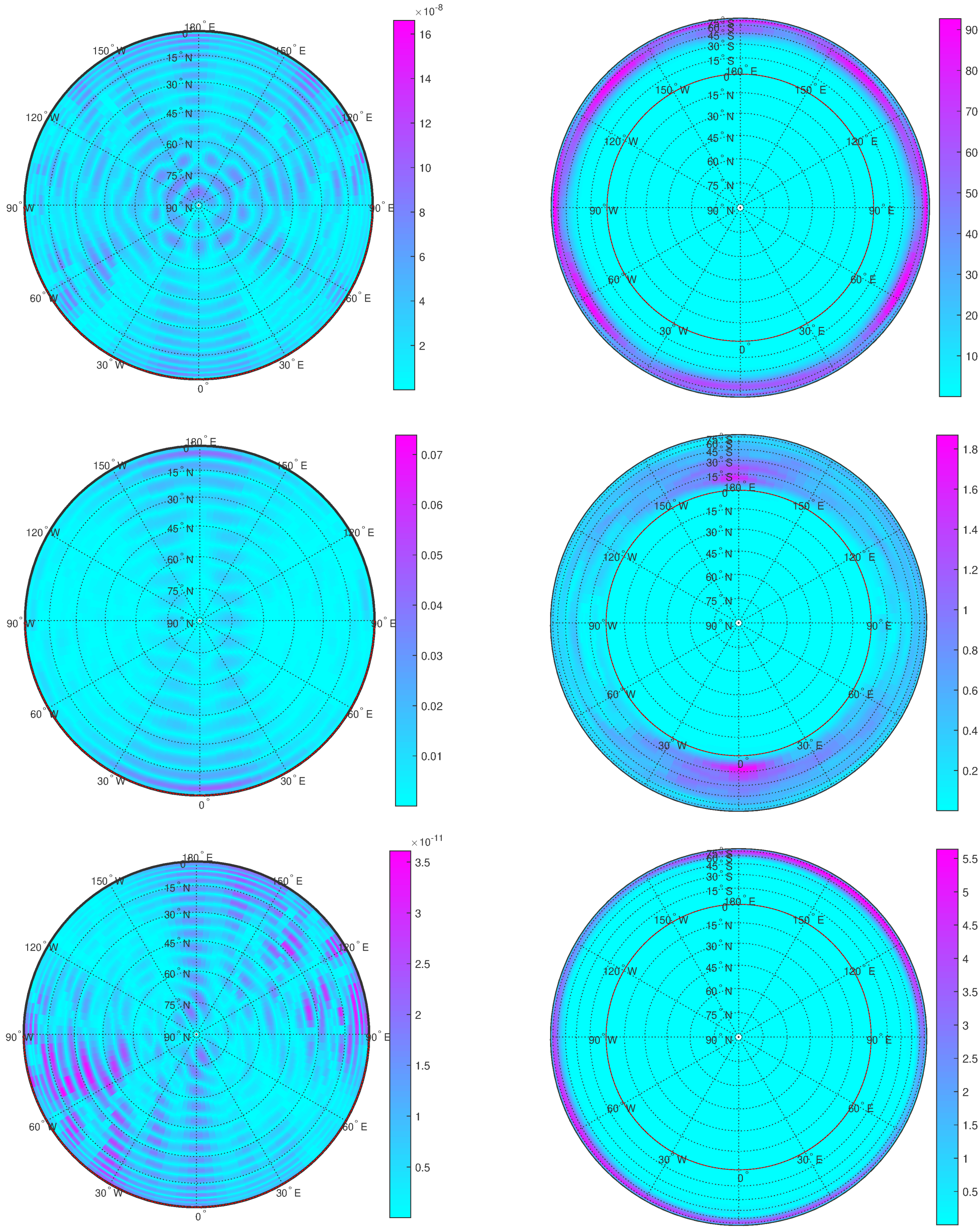


Figure 9.5: The norm of the difference of the calculated polarization to the given polarization with bandlimit $L = 18$ in the *hemisphere* (a spherical cap with radius $90°$ at the North pole). Here, the polarization is calculated by *tensor Slepian functions with coefficients from integration* with 10100 points plotted on the spherical cap (top-left) and on the whole sphere (top-right). Moreover, the polarization is calculated by *tensor Slepian functions for all eigenvalues larger than* 10^{-2} *with coefficients from integration* with 10100 points plotted on the spherical cap (center-left) and on the whole sphere (center-right). Furthermore, the polarization is calculated by *electric and magnetic spherical harmonics on the integration grid* with 10100 points on the spherical cap (bottom-left) and on the whole sphere (bottom-right). The red circle denotes the boundary of the spherical cap.

Chapter 10

Conclusions

10.1 Summary

This thesis can be divided into three main subjects. First, we had the mathematical theory of the spin-weighted spherical harmonics. Second, we developed the tensor Slepian functions by calculating the more general spin-weighted Slepian functions. Here, we constructed a commuting operator for the spherical cap as region of interest. Finally, we applied the method of tensor Slepian functions to the CMB polarization.

In short, we described a unified mathematical theory of the spin-weighted spherical harmonics of Newman and Penrose [63]. Here, we not only collected and proved known properties from the mathematical point of view, but also showed new properties, including recursion relations, the Christoffel-Darboux formula, and Green's second surface identity for the spin-weighted Beltrami operator $\Delta^{*,N}$. We also proved the uniqueness of the spin-weighted spherical harmonics as the eigenfunctions of the spin-weighted Beltrami operator.
Furthermore, we mutually connected all definitions and formulations of the spin-weighted spherical harmonics known to date. As an example, the spin-weighted spherical harmonics in this thesis are formulated by the spin raising and lowering operator $\eth$ and $\overline{\eth}$, by the Wigner D-function, by the Jacobi polynomials, and in a finite series expansion.
Moreover, we introduced mathematical notations like the set of the $(*, N)$-harmonic functions of spin weight N and degree n, denoted by Harm_n^N, which is spanned by the spin-weighted spherical harmonics. These notations enabled us to connect the theory of spin-weighted spherical harmonics closely to the theory of the well-known spherical harmonics.
We not only related the theory of the two kinds of scalar spherical harmonics to each other, but also related the vector spherical harmonics of Hill [40] and the tensor spherical harmonics of Freeden, Gervens, and Schreiner [27] to the spin-weighted spherical harmonics multiplied by a spherical unit vector respectively a tensor product of spherical unit vectors.

We used these relations to design the Slepian functions on the sphere. Here, we recapitulated the scalar and the vector Slepian functions from [78, 82] and from [43, 67]. Furthermore, we constructed the tensor Slepian functions on the sphere.
For all these cases, we have to solve an eigenvalue problem respectively an integral equation. With the relation of the scalar, vector, and tensor spherical harmonics to the spin-weighted spherical harmonics, all cases (scalar, vectorial, and tensorial) can be reduced to a single, more general spin-weighted case. For this, we have to solve the Slepian eigenvalue problem for a fixed spin weight N. We need spin weight zero for the scalar Slepian functions. Additionally, we need to add for the vector Slepian functions spin weight ± 1 and further, we need to add for the tensor Slepian functions spin weight ± 2.

We reduced the complexity, meaning that we only have to find the spin-weighted Slepian functions for a general and fixed spin weight N by solving the spin-weighted eigenvalue problem $K^N G^N = \lambda G^N$. Then, we can directly construct the scalar, the vector, and the tensor Slepian functions.
Furthermore, we formulated the Shannon number for those cases as an estimate for the number of significant eigenvalues. This means that we get from the Shannon number the number of well concentrated Slepian functions within the region of interest.

For the spherical cap, we also designed a commuting operator $\mathscr{I}^N$ to the spin-weighted kernel function $\mathscr{K}^N$. From the eigenfunction equation of the commuting operator, we formulated the alternative eigenvalue problem $I^N G^N = \chi G^N$. Here, the tridiagonal matrix I^N commutes with the kernel matrix K^N. Furthermore, these two matrices have the same eigenvectors. These eigenvectors deliver us the spin-weighted spherical harmonic coefficients of the spin-weighted Slepian functions.
So, we showed a stable way for solving the spin-weighted Slepian eigenvalue problem for the spherical cap and therefore, to calculate the spin-weighted, the scalar, the vector, and the tensor Slepian functions on the spherical cap.

We used this method to implement the tensor Slepian functions. Here, we displayed that the method works as expected. The resulting first Slepian functions are well concentrated within the spherical cap. The Slepian functions near to the Shannon number are concentrated on the boundary of the spherical cap. And the last Slepian functions are concentrated outside the spherical cap up to the complementary spherical cap.

Moreover, we used the introduced method of the tensor Slepian functions to construct the tensor Slepian functions for the CMB (cosmic microwave background) polarization. The CMB polarization is a tensor field. It can be separated into an electric and a magnetic component, which is given in the basis of the electric and magnetic tensor spherical harmonics. We connected these basis functions to the tensor spherical harmonics of Freeden, Gervens, and Schreiner and therefore, used the here described method to construct tensor Slepian functions for the CMB polarization. Here, we formulated the polarization in the electric and magnetic Slepian functions.
The application of the method of tensor Slepian functions to the CMB polarization is useful, since measurements of the polarization are only given on parts of the sphere. Mainly, these measurements are given on the hemisphere, which *is* a spherical cap.

We tested our method in first of their kind numerical experiments using synthetically generated polarization data. In a nutshell, the reconstruction of the given data with help of the Slepian functions delivered robust results. We developed an estimate for significant eigenvalues, for which we obtained superior results compared to the commonly used Shannon number. This newly developed estimate works best outside the spherical cap. Since these numerical experiments are the first of their kind, the main focus lied on the construction of the theoretical framework, which necessarily includes first numerical tests on feasibility. But all in all, there are points of improvements and open questions for future researches left.

10.2 Outlook

There are still open questions that make future research necessary.

We were endeavored to write down and show the complete theory of the spin-weighted spherical harmonics in a unified mathematical way. However, the spin-weighted spherical harmonics have an enormous spectrum of versatility, viability, and properties, so that there needs more work to be done upon arriving at a complete and unified theory.

Furthermore, the application and implementation of the tensor Slepian functions on arbitrary regions as in [81] for the scalar Slepian functions remains an open topic. Moreover, the construction of a commuting operator for the spherical double cap and belt for the vector and tensor Slepian functions is nowhere to be found at the present moment.
The application of the tensor Slepian functions to real data like data, for example from the satellite mission GOCE, is another possibility for future researches.
Furthermore, we have to improve the estimate of the number of significant eigenvalues. Moreover, our method of calculating the polarization coefficients was sufficient, but certainly leaves room for further improvements. This can be achieved through regularization of the Slepian reconstruction by a smoothing filter like the convolution with the cp-function. After implementing the suggested improvements, a re-conduction of the experiment within the same framework for testing the magnitude of possible improvements is advisable. Furthermore, an extension to other synthetic polarization data like localized ones, for example the Abel-Poisson kernel, and/or the implementation of the CMB tensor Slepian functions to real polarization measurements needs yet to be done.

Further, the theory of the Slepian functions can be extended to tensors of higher order. Here, we expect that additional spin weights will be added. In Chapter 8, we have already paved the way for this, because it contains the theory of the Slepian functions for an arbitrary spin weight N.

Moreover, one can think about using the method of [19, 20] to construct a spatially localized tensor basis on the sphere instead of using Slepian functions.

With regard to the researches of the Geomathematics Group of Siegen, the construction of a dictionary out of tensorial functions, like the tensor Slepian functions for different regions, for tensor valued inverse problems in analogy to [89] can be contemplated.

Appendix A

Integration on the Sphere

In this appendix, we want to calculate the integral

$$\int_R F(\xi)\ \mathrm{d}\omega(\xi)$$

over the region of interest R, where $F \in \mathrm{L}^2(\Omega)$. We have to calculate such an integral for the CMB Slepian polarization coefficients $\boldsymbol{p}_\alpha$. For the spin-weighted kernel matrix for arbitrary regions K^N, we also have to calculate such an integral, but we do not address this directly in this thesis. The calculation of this integral can be done according to quadrature methods. We deal with these methods in the next section. But first, we start with well-known quadrature methods.

A.1 Gauß Quadrature

Here, we look at general properties and definitions of the Gauß quadrature.

Definition A.1.1. *We define the weighted scalar product on (a, b) by [38]*

$$\langle F, G\rangle_w := \int_a^b F(x)\overline{G(x)}w(x)\ \mathrm{d}x,$$

where $F, G \in \mathrm{L}^2(\Omega)$ and the weight function $w(x) > 0$ is integrable.

Note that in the following, we look only at real functions and therefore, we omit the complex conjugation of the scalar product.

Problem A.1.2. *We want to calculate integrals by a quadrature [38] such that*

$$\int_a^b F(x)w(x)\ \mathrm{d}x = \sum_{k=1}^{m} w_k F(x_k)$$

for some F, where $w(x) > 0$ is an integrable weight function for $x \in (a, b)$ and w_k are the given weights. x_k, $k = 1, \dots, m$, are the roots of orthogonal polynomials P_m of degree $m \in \mathbb{N}$ with respect to the weighted scalar product [75]. This means that for polynomials with

$$\langle P_{l_1}, P_{l_2}\rangle_w = 0$$

for $l_1 \neq l_2$, we get that $P_m(x_k) = 0$. The weights are given by

$$w_k = \int_a^b L_k(x)w(x)\ \mathrm{d}x,$$

where the Lagrange polynomials are defined by

$$L_k(x) = \prod_{j=1, j\neq k}^{m} \frac{x - x_j}{x_k - x_j}.$$

Theorem A.1.3. *The Gauß quadrature is exact for polynomials up to a polynomial degree of $2m - 1$ [38].*

Definition A.1.4. *We define the norm of the orthogonal polynomials by*

$$\gamma_n := \|P_n\|_w^2 := \langle P_n, P_n\rangle_w.$$

Lemma A.1.5. *The weights can be calculated by [38]*

$$w_k = \left(\sum_{n=0}^{m-1} \frac{P_n^2(x_k)}{\gamma_n}\right)^{-1}$$

for $k = 1, \ldots, m$, where x_k are the roots of P_m, $m \in \mathbb{N}$.

Proof. Here, we follow mainly [38]. We see from the definition that the Lagrange polynomials L_k are polynomials of degree $m - 1$. With the real orthogonal polynomials P_n, we have an orthogonal basis P_n, and therefore, we can write the Lagrange polynomials in this basis by

$$L_k = \sum_{n=0}^{m-1} c_{n,k} P_n.$$

Then, we can calculate the coefficients by

$$c_{n,k} = \frac{1}{\|P_n\|_w^2} \langle L_k, P_n\rangle_w = \frac{1}{\gamma_n} \int_a^b L_k(x)P_n(x)w(x)\ \mathrm{d}x.$$

So, we get a weighted integration over a polynomial of degree less or equal $2m-2$. Therefore, we can calculate the integral exactly by Gauß quadrature by

$$c_{n,k} = \frac{1}{\gamma_n} \sum_{j=1}^{m} w_j \underbrace{L_k(x_j)}_{=\delta_{k,j}} P_n(x_j) = \frac{w_k}{\gamma_n} P_n(x_k).$$

Next, we get for the norm of the Lagrange polynomials on the one hand

$$\begin{aligned}
\|L_k\|_w^2 &= \langle L_k, L_k\rangle_w \\
&= \left\langle \sum_{n=0}^{m-1} c_{n,k} P_n, \sum_{n'=0}^{m-1} c_{n',k} P_{n'} \right\rangle_w \\
&= \sum_{n=0}^{m-1} \sum_{n'=0}^{m-1} c_{n,k} c_{n',k} \underbrace{\langle P_n, P_{n'}\rangle_w}_{=\gamma_n \delta_{n,n'}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{m-1} \gamma_n c_{n,k}^2 \\
&= w_k^2 \sum_{n=0}^{m-1} \frac{P_n^2(x_k)}{\gamma_n}.
\end{aligned}
$$

Because L_k^2 is a polynomial of degree $2m-2$ and therefore, can be exactly integrated by Gauß quadrature, we obtain on the other hand

$$
\begin{aligned}
\|L_k\|_w^2 &= \int_a^b L_k^2(x) w(x)\ \mathrm{d}x \\
&= \sum_{j=1}^{m} w_j \underbrace{L_K^2(x_j)}_{=\delta_{j,k}} \\
&= w_k.
\end{aligned}
$$

This leads directly to the proposition

$$
w_k = \left(\sum_{n=0}^{m-1} \frac{P_n^2(x_k)}{\gamma_n} \right)^{-1}.
$$

□

A.1.1 Gauß-Tschebyscheff Quadrature

Now, we take a look at the Gauß-Tschebyscheff quadrature [38].

Definition A.1.6. *The Tschebyscheff polynomials are defined by*

$$
T_n(x) = \cos(n \arccos x)
$$

for $n \in \mathbb{N}_0$ and $x \in [-1, 1]$.

Lemma A.1.7. *The Tschebyscheff polynomials are orthogonal polynomials with*

$$
\langle T_n, T_m \rangle_w = \begin{cases} \pi, & n = m = 0 \\ \frac{\pi}{2}, & n = m > 0 \\ 0, & \textit{else} \end{cases},
$$

where $w(x) = \frac{1}{\sqrt{1-x^2}} > 0$ is integrable for $n, m \in \mathbb{N}_0$.

Proof. We calculate the scalar product and get by substitution of $t = \arccos x$

$$
\begin{aligned}
\langle T_n, T_m \rangle_w &= \int_{-1}^{1} T_n(x) T_m(x) w(x)\ \mathrm{d}x \\
&= \int_{-1}^{1} \cos(n \arccos x) \cos(m \arccos x) \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x \\
&= \int_{\pi}^{0} \cos(nt) \cos(mt) \frac{1}{\sin t} (-\sin t)\ \mathrm{d}t \\
&= \int_{0}^{\pi} \cos(nt) \cos(mt)\ \mathrm{d}t
\end{aligned}
$$

$$= \frac{1}{2} \int_0^\pi (\cos((n-m)t) + \cos((n+m)t)) \, dt.$$

Now, we have to distinguish several cases. We start with $n = m = 0$. Then, we obtain

$$\langle T_n, T_m \rangle_w = \frac{1}{2} \int_0^\pi 2 \, dt = \pi.$$

Next, we look at $n = m > 0$ and get

$$\begin{aligned}\langle T_n, T_m \rangle_w &= \frac{1}{2} \int_0^\pi (1 + \cos((n+m)t)) \, dt \\ &= \frac{1}{2} \left(\pi + \left[\frac{1}{n+m} \sin((n+m)t) \right]_{t=0}^{t=\pi} \right) \\ &= \frac{\pi}{2}.\end{aligned}$$

Else, we obtain

$$\begin{aligned}\langle T_n, T_m \rangle_w &= \frac{1}{2} \left[\frac{1}{n-m} \sin((n-m)t) + \frac{1}{n+m} \sin((n+m)t) \right]_{t=0}^{t=\pi} \\ &= 0.\end{aligned}$$

□

Lemma A.1.8. *Furthermore, it is obvious that the roots of the Tschebyscheff polynomials T_m of degree $m \in \mathbb{N}$ can be calculated by*

$$x_k = \cos\left(\frac{2k-1}{2m}\pi\right), \quad k = 1, \ldots, m.$$

Lemma A.1.9. *The Tschebyscheff weights are given by*

$$w_k = \frac{\pi}{m}, \quad k = 1, \ldots, m \in \mathbb{N}.$$

Proof. From Lemma A.1.5 and with the addition theorems for the cosine, Theorem 2.1.9, we get with the previous lemmas

$$\begin{aligned}w_k &= \left(\sum_{n=0}^{m-1} \frac{T_n^2(x_k)}{\|T_n\|_w^2} \right)^{-1} \\ &= \left(\frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{m-1} \cos^2\left(\frac{(2k-1)n}{2m}\pi \right) \right)^{-1} \\ &= \left(\frac{1}{\pi} + \frac{1}{\pi} \sum_{n=1}^{m-1} \left(1 + \cos\left(\frac{(2k-1)n}{m}\pi \right) \right) \right)^{-1} \\ &= \left(\frac{m}{\pi} + \frac{1}{\pi} \sum_{n=1}^{m-1} \cos\left(\frac{(2k-1)n}{m}\pi \right) \right)^{-1}.\end{aligned}$$

For the sum, we have to distinguish several cases. We set $x = \frac{(2k-1)}{m}\pi$. If m is even, we obtain

$$\begin{aligned}\sum_{n=1}^{m-1}\cos(nx) &= \sum_{n=1}^{\frac{m}{2}-1}\cos(nx) + \underbrace{\cos\left(\frac{m}{2}x\right)}_{=\cos\left(\frac{2k-1}{2}\pi\right)=0} + \sum_{n=\frac{m}{2}+1}^{m-1}\cos(nx)\\ &= \sum_{n=1}^{\frac{m}{2}-1}\cos(nx) + \sum_{n=1}^{\frac{m}{2}-1}\cos((m-n)x).\end{aligned}$$

With properties of the cosine, Theorem 2.1.9, we see that

$$\cos((m-n)x) = \underbrace{\cos(mx)}_{=-1}\cos(nx) + \underbrace{\sin(mx)}_{=0}\sin(nx) = -\cos(nx) \tag{A.1}$$

and therefore, we get

$$\sum_{n=1}^{m-1}\cos(nx) = \sum_{n=1}^{\frac{m}{2}-1}\cos(nx) - \sum_{n=1}^{\frac{m}{2}-1}\cos(nx) = 0.$$

If m is odd, we obtain

$$\begin{aligned}\sum_{n=1}^{m-1}\cos(nx) &= \sum_{n=1}^{\frac{m-1}{2}}\cos(nx) + \sum_{n=\frac{m-1}{2}+1}^{m-1}\cos(nx)\\ &= \sum_{n=1}^{\frac{m-1}{2}}\cos(nx) + \sum_{n=1}^{\frac{m-1}{2}}\cos((m-n)x).\end{aligned}$$

With (A.1), we get again

$$\sum_{n=1}^{m-1}\cos(nx) = \sum_{n=1}^{\frac{m-1}{2}}\cos(nx) - \sum_{n=1}^{\frac{m-1}{2}}\cos(nx) = 0.$$

Altogether, we obtain

$$\sum_{n=1}^{m-1}\cos\left(\frac{(2k-1)n}{m}\pi\right) = 0$$

and therefore,

$$w_k = \frac{\pi}{m}.$$

□

So, the Gauß-Tschebyscheff quadrature [38] is given by

$$\int_{-1}^{1} F(x)w(x)\ \mathrm{d}x = \frac{\pi}{m}\sum_{k=1}^{m} F(x_k)$$

with $w(x) = \frac{1}{\sqrt{1-x^2}}$, $x_k = \cos\left(\frac{2k-1}{2m}\pi\right)$, and $w_k = \frac{\pi}{m}$, $k = 1, \ldots, m \in \mathbb{N}$ for polynomials F of degree less or equal $2m-1$.

A.1.2 Gauß-Legendre Quadrature

Next, we take a look at the Gauß-Legendre quadrature [38].

For the Legendre polynomials, we know from Lemma 2.4.5 that

$$\int_{-1}^{1} P_n(t)P_m(t)\ \mathrm{d}t = \frac{2}{2n+1}\ \delta_{n,m}$$

for $n, m \in \mathbb{N}_0$. In particular, we get

$$\gamma_n = \int_{-1}^{1} P_n^2(t)\ \mathrm{d}t = \frac{2}{2n+1}.$$

So, they are orthogonal. The weight function is given by $w(x) = 1 > 0$. Then, the Gauß-Legendre quadrature is given for polynomials F of degree less or equal $2m-1$ by

$$\int_{-1}^{1} F(t)\ \mathrm{d}t = \sum_{k=0}^{m} w_k F(t_k),$$

where the Legendre polynomials P_m are zero for $t_1, \ldots, t_m$ and the Legendre weights are denoted by w_k, $k = 1, \ldots, m \in \mathbb{N}$.

Lemma A.1.10. *With $P_0(t) = 1$, $P_0'(t) = 0$ and $P_1(t) = t$, $P_1'(t) = 1$ and with the recursion relations*

$$P_n(t) = \frac{(2n-1)tP_{n-1}(t) - (n-1)P_{n-2}(t)}{n},$$
$$P_n'(t) = \frac{nP_{n-1}(t) - ntP_n(t)}{1-t^2}$$

for $n \in \mathbb{N}_0$, $n \geq 2$, we can calculate the roots of the Legendre polynomials P_m, $m \in \mathbb{N}$, by Newton's method [75]. This means that for a suitable starting points t_0 we can calculate the roots t_k, $k \in \mathbb{N}$, $k \geq 1$, iteratively by

$$t_k = t_{k-1} - \frac{P_n(t_{k-1})}{P_n'(t_{k-1})}.$$

Proof. This lemma is directly given by the definition of Newton's method and by the definition of the Legendre polynomials, Definition 2.4.1, and by the recursion relations of the Legendre polynomials, which we get from Remark 2.4.3 and Theorem 2.4.4 for $n \in \mathbb{N}_0$ by

$$\left(t^2-1\right) P_n'(t) = ntP_n(t) - nP_{n-1}(t),$$
$$(2n+1)tP_n(t) = nP_{n-1}(t) + (n+1)P_{n+1}(t).$$

□

Lemma A.1.11. *The leading coefficient of the Legendre polynomials of degree $n \in \mathbb{N}_0$ is given by*

$$A_n = \frac{(2n)!}{2^n (n!)^2}.$$

Proof. From Rodriguez formula and the binomial theorem, we get for $n \in \mathbb{N}_0$

$$\begin{aligned} P_n(t) &= \frac{1}{2^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left(t^2-1\right)^n \\ &= \frac{1}{2^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k t^{2n-2k} \\ &= \frac{1}{2^n n!}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}(-1)^k \frac{(2n-2k)!}{(n-2k)!}\, t^{n-2k}. \end{aligned}$$

So, the leading coefficient of the Legendre polynomials of degree n is given by

$$A_n = \frac{1}{2^n n!}\binom{n}{0}(-1)^0 \frac{(2n)!}{n!} = \frac{(2n)!}{2^n (n!)^2}.$$

□

Lemma A.1.12. *We get the following Christoffel-Darboux formula for the Legendre polynomials [10, 55, 87]*

$$(t_1-t_2)\sum_{n=0}^{L-1}(2n+1)P_n(t_1)P_n(t_2) = L\left(P_L(t_1)P_{L-1}(t_2) - P_{L-1}(t_1)P_L(t_2)\right)$$

for $t_1, t_2 \in [-1,1]$ *and* $L \in \mathbb{N}$.

Proof. With Theorem 3.4.9 for $\xi = \xi(t,\varphi) \in \Omega$, we receive

$${}_0Y_{n,0}(\xi) = \sqrt{\frac{2n+1}{4\pi}}\; P_n(t)$$

and with Theorem 3.3.5 for $\xi = \xi(t_1,\varphi_1) \in \Omega$ and $\eta = \eta(t_2,\varphi_2) \in \Omega$,

$$(t_1-t_2)\sum_{n=|j|}^{L-1}\overline{Y_{n,j}(\xi)}Y_{n,j}(\eta) = \sqrt{\frac{(L-j)(L+j)}{(2L-1)(2L+1)}}\left(\overline{Y_{L,j}(\xi)}Y_{L-1,j}(\eta) - \overline{Y_{L-1,j}(\xi)}Y_{L,j}(\eta)\right),$$

we get

$$\begin{aligned} (t_1-t_2)\sum_{n=0}^{L-1}\frac{2n+1}{4\pi}P_n(t_1)P_n(t_2) = \frac{L}{\sqrt{(2L-1)(2L+1)}}\Bigg(&\frac{\sqrt{(2L+1)(2L-1)}}{4\pi}P_L(t_1)P_{L-1}(t_2) \\ &-\frac{\sqrt{(2L+1)(2L-1)}}{4\pi}P_{L-1}(t_1)P_L(t_2)\Bigg). \end{aligned}$$

So, we obtain the Christoffel-Darboux formula for the Legendre polynomials

$$(t_1-t_2)\sum_{n=0}^{L-1}(2n+1)P_n(t_1)P_n(t_2) = L\left(P_L(t_1)P_{L-1}(t_2) - P_{L-1}(t_1)P_L(t_2)\right).$$

□

Lemma A.1.13. *Then, we get for $t \in [-1,1]$ [10]*

$$\sum_{n=0}^{L-1} \frac{P_n^2(t)}{\gamma_n} = \frac{A_{L-1}}{A_L \gamma_{L-1}} \left(P_L'(t) P_{L-1}(t) - P_{L-1}'(t) P_L(t) \right).$$

With t_k a root of P_L, this leads to

$$P_L'(t_k) P_{L-1}(t_k) = \frac{A_L \gamma_{L-1}}{A_{L-1}} \sum_{n=0}^{L-1} \frac{P_n^2(t_k)}{\gamma_n}.$$

Proof. With the definitions from above, the Christoffel-Darboux formula for the Legendre polynomials leads for $t_1, t_2 \in [-1,1]$ to

$$\sum_{n=0}^{L-1} \frac{2n+1}{2} P_n(t_1) P_n(t_2) = \frac{L}{2} \frac{P_L(t_1) P_{L-1}(t_2) - P_{L-1}(t_1) P_L(t_2)}{t_1 - t_2}.$$

Further, with

$$\begin{aligned} \frac{A_L \gamma_{L-1}}{A_{L-1}} &= \frac{(2L)!}{2^L (L!)^2} \frac{2^{L-1}((L-1)!)^2}{(2L-2)!} \frac{2}{2L-1} \\ &= \frac{2L(2L-1)}{2L^2} \frac{2}{2L-1} \\ &= \frac{2}{L}, \end{aligned} \tag{A.2}$$

we obtain

$$\sum_{n=0}^{L-1} \frac{P_n(t_1) P_n(t_2)}{\gamma_n} = \frac{A_{L-1}}{A_L \gamma_{L-1}} \frac{P_L(t_1) P_{L-1}(t_2) - P_{L-1}(t_1) P_L(t_2)}{t_1 - t_2}.$$

For $t = t_1 = t_2$, we get

$$\begin{aligned} \sum_{n=0}^{L-1} \frac{P_n^2(t)}{\gamma_n} &= \frac{A_{L-1}}{A_L \gamma_{L-1}} \lim_{t_2 \to t_1} \frac{P_L(t_1) P_{L-1}(t_2) - P_{L-1}(t_1) P_L(t_2)}{t_1 - t_2} \\ &= \frac{A_{L-1}}{A_L \gamma_{L-1}} \lim_{t_2 \to t_1} \frac{(P_L(t_1) - P_L(t_2)) P_{L-1}(t_2) - (P_{L-1}(t_1) - P_{L-1}(t_2)) P_L(t_2)}{t_1 - t_2} \\ &= \frac{A_{L-1}}{A_L \gamma_{L-1}} \left(P_L'(t) P_{L-1}(t) - P_{L-1}'(t) P_L(t) \right). \end{aligned}$$

This is the first proposition. For $t = t_k$ with t_k a root of P_L, we get directly the second one. □

Corollary A.1.14. *With the previous lemma and with Lemma A.1.5, we can calculate the Legendre weights for $k = 1, \ldots, m \in \mathbb{N}$ by [51]*

$$w_k = \left(\sum_{n=0}^{L-1} \frac{P_n^2(t_k)}{\gamma_n} \right)^{-1} = \frac{A_L \gamma_{L-1}}{A_{L-1} P_L'(t_k) P_{L-1}(t_k)}.$$

We can calculate the result of the previous corollary more accurately. With (A.2) and with the recursion relation, Lemma 2.4.4,

$$LP_{L-1}(t_k) = \left(1 - t_k^2\right) P_L'(t_k) + Lt_k \underbrace{P_L(t_k)}_{=0} = \left(1 - t_k^2\right) P_L'(t_k),$$

we get for the Legendre weights for $k = 1, \ldots, m \in \mathbb{N}$ [51]

$$w_k = \frac{2}{(1 - t_k^2) \left(P_L'(t_k)\right)^2}.$$

Alternatively, we can calculate the Legendre weights by solving the system of linear equations [75]

$$\sum_{k=1}^{m} w_k P_n(t_k) = \int_{-1}^{1} P_n(t) \, \mathrm{d}t,$$

where for $n \in \mathbb{N}_0$

$$\int_{-1}^{1} P_n(t) \, \mathrm{d}t = \int_{-1}^{1} P_0(t) P_n(t) \, \mathrm{d}t = \begin{cases} 2, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

A.1.3 Gauß Quadrature with Spin-Weighted Legendre Polynomials

The integrals we want to calculate depend on a spin weight $N \in \mathbb{Z}$. We know already that spin-weighted spherical harmonics are in general not polynomials. So, it is not possible to integrate them exactly by utilizing a common quadrature method. Therefore, we tried to develop a Gauß quadrature with spin-weighted Legendre polynomials. We will see that this is connected to a series of obstacles and difficulties. Nevertheless, we look at the Gauß quadrature with spin-weighted Legendre polynomials from Definition 3.7.1.

We know from Definition 3.7.1 that for $\xi = \xi(t, \varphi) \in \Omega$, for all $N \in \mathbb{Z}$, and all $n \in \mathbb{N}_0$, $n \geq |N|$, the spin-weighted Legendre polynomials are given by

$${}_N P_n(t) = \frac{(-1)^n}{2^n \sqrt{(n+N)!(n-N)!}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left[(1-t)^{n-N}(1+t)^{n+N}\right].$$

From Lemma 3.7.7, we get that these are orthogonal polynomials of degree n with respect to the weight function given by

$$w_N(t) = \left(\frac{1-t}{1+t}\right)^N > 0$$

for $t \in (-1, 1)$ and

$${}_N\gamma_n = \frac{2}{2n+1}.$$

Here, we see the first problem due to which the Gauß quadrature with spin-weighted Legendre polynomials does not work. The weight function w_N is *not* integrable for $N \neq 0$.

If we ignore this and assume that the Gauß quadrature with spin-weighted Legendre polynomials is given for some F by

$$\int_{-1}^{1} F(t)w(t)\ \mathrm{d}t = \sum_{k=0}^{m} w_k F(t_k),$$

where the spin-weighted Legendre polynomials ${}_NP_m$ are zero for $t_1, \ldots, t_m$ and the weights w_k, $k = 1, \ldots, m \in \mathbb{N}$, we get the next problem by the calculation of the roots of the spin-weighted Legendre polynomials.

We get the roots of the spin-weighted Legendre polynomials ${}_NP_m$ with the recursion relations from Lemma 3.7.9

$$\begin{aligned} {}_NP_n(t) &= \frac{(2n-1)t\ {}_NP_{n-1}(t) - \sqrt{(n-1-N)(n-1+N)}\ {}_NP_{n-2}(t)}{\sqrt{(n-N)(n+N)}}, \\ {}_NP_n'(t) &= \frac{\sqrt{(n-N)(n+N)}\ {}_NP_{n-1}(t) - (nt-N)\ {}_NP_n(t)}{1-t^2} \end{aligned}$$

for $t \in (-1, 1)$, for $N \in \mathbb{N}$, and $n \in \mathbb{N}_0$, $n \geq |N|$, and with the starting points ${}_NP_{|N|}$ and ${}_NP_{|N|+1}$ and additionally, with the functions ${}_NP_{|N|}'$. Here, we only need spin weight $N = 0, \pm 1, \pm 2$. Therefore, we calculate the accordingly spin-weighted spherical harmonics. For CMB polarization, we only need spin weight $N = \pm 2$. So, we need mainly spin weight $N = \pm 2$. The starting points of the recursion as well as the function ${}_NP_{|N|}'$ are given from Corollary 3.7.12. There, we see for example that for spin weight $N = \pm 2$ we get that $t = \mp 1$ is a multiple root in cases like ${}_{\pm 2}P_2(t) = \sqrt{\frac{3}{8}}(1 \pm t)^2$ and ${}_{\pm 2}P_3(t) = \sqrt{\frac{15}{8}} t(1 \pm t)^2$ for $t \in [-1, 1]$. Hence, not all spin-weighted Legendre polynomials ${}_NP_m$ of degree $m \in \mathbb{N}$ have m pairwise distinct roots. On the other hand, there are also spin-weighted Legendre polynomials ${}_NP_m$ of degree $m \in \mathbb{N}$ that have m distinct roots such as ${}_1P_2(t) = -\sqrt{\frac{3}{8}}\,(t + t^2)$. So, in general, we do not get m hubs for the Gauß quadrature.

Moreover, note that here we start at $n = |N|$. Therefore, the number of orthogonal polynomials does not fit to the number of roots and we cannot describe the Lagrange polynomials in the spin-weighted Legendre polynomials. Alternatively, we can enlarge the degree of spin-weighted Legendre polynomials, but then, we cannot supply an analogous proof to Lemma A.1.5.

So, we can only calculate the Legendre weights by solving the system of linear equations

$$\sum_{k=1}^{m} w_k\ {}_NP_n(t_k) = \int_{-1}^{1} {}_NP_n(t) w_N(t)\ \mathrm{d}t$$

for $N \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, $n = |N|, \ldots, m-1$ (for the calculation of the left-hand side see Lemma 3.7.17). Here, we see the next problem. This system is underconstrained for $N \neq 0$, because we know that t_k, $k = 1, \ldots, m$, are the roots of ${}_NP_m$ (if they exist) and therefore, $n = |N|, \ldots, m-1$. So, we get a system of $m - 1 - |n|$ linear equations and $m - 1$ variables.

A.2 Integration over the Spherical Cap

We want to solve the problem of an integration over the spherical cap with radius b. So, we want to calculate

$$I(F) = \int_0^{2\pi} \int_b^1 F(t, \varphi) \, \mathrm{d}t \, \mathrm{d}\varphi$$

for $F \in \mathrm{L}^2(\Omega)$. This follows as the product of two Gauß quadratures [3].

The azimuthal integral can be calculated by Gauß-Tschebyscheff quadrature. So, for an arbitrary integral over $[0, 2\pi]$, we substitute $p = \cos\left(\frac{\varphi}{2}\right)$ and use Gauß-Tschebyscheff quadrature from Appendix A.1.1. Then, we get for polynomials G of degree less or equal $2m-1$, $m \in \mathbb{N}$,

$$\begin{aligned}\int_0^{2\pi} G(\varphi) \, \mathrm{d}\varphi &= \int_1^{-1} G(2\arccos p) \frac{-2}{\sqrt{1-p^2}} \, \mathrm{d}p \\ &= 2\int_{-1}^{1} G(2\arccos p) \frac{1}{\sqrt{1-p^2}} \, \mathrm{d}p.\end{aligned}$$

With Gauß-Tschebyscheff quadrature, this leads to

$$\int_0^{2\pi} G(\varphi) \, \mathrm{d}\varphi = 2 \, \frac{\pi}{m} \sum_{j=1}^{m} G(2\arccos p_j),$$

where

$$p_j = \cos\left(\frac{2j-1}{2m}\pi\right).$$

Lemma A.2.1. *Therefore, we get for the azimuthal integration for polynomials G of degree less or equal $2m-1$, $m \in \mathbb{N}$,*

$$\int_0^{2\pi} G(\varphi) \, \mathrm{d}\varphi = \frac{2\pi}{m} \sum_{j=1}^{m} G(\varphi_j),$$

where

$$\varphi_j = \frac{2j-1}{m}\pi$$

for $j = 1, \ldots, m$.

To determine the polar integration, we substitute $x = \frac{2}{1-b}t - \frac{1+b}{1-b}$ [75] such that for polynomials G of degree less or equal $2m-1$, $m \in \mathbb{N}$,

$$\begin{aligned}\int_b^1 G(t)\mathrm{d}t &= \int_{-1}^{1} G\left(\frac{(1-b)x+(1+b)}{2}\right) \frac{1-b}{2} \, \mathrm{d}x \\ &= \frac{1-b}{2} \int_{-1}^{1} G\left(\frac{(1-b)x+(1+b)}{2}\right) \, \mathrm{d}x.\end{aligned}$$

Lemma A.2.2. *With Gauß-Legendre quadrature from Appendix A.1.2, we obtain*

$$\int_b^1 G(t) \, \mathrm{d}t = \frac{1-b}{2} \sum_{k=1}^{m} w_k G(t_k),$$

where $t_k = \frac{(1-b)x_k+(1+b)}{2}$, $x_1, \ldots, x_m$ the roots of the Legendre polynomials P_m of degree m,

and

$$w_k = \frac{2}{(1-x_k^2)\,(P_m'(t_k))^2}$$

for $k = 1, \ldots, m \in \mathbb{N}$.

Lemma A.2.3. *So, we get that the integration over the spherical cap is given for functions* F *that are polynomial in* t *and* φ *up to degree* m*, where* $\xi = \xi(t,\varphi) \in \Omega$*, by*

$$I(F) = \int_0^{2\pi} \int_b^1 F(t,\varphi)\ \mathrm{d}t\ \mathrm{d}\varphi = \frac{(1-b)\pi}{m} \sum_{j=1}^{m} \sum_{k=1}^{m} w_k F\left(t_k, \varphi_j\right),$$

where $t_k = \frac{(1-b)x_k+(1+b)}{2}$, $x_1, \ldots, x_m$ *the roots of the Legendre polynomials* P_m *of degree* m*,* $\varphi_j = \frac{(2j-1)\pi}{m}$ *for* $j,k = 1, \ldots, m \in \mathbb{N}$*, and the Legendre weights*

$$w_k = \frac{2}{(1-x_k^2)\,(P_m'(x_k))^2}.$$

Note that there are fast algorithms to evaluate quadrature formulas and to compute quadrature weights for arbitrary point grids on the whole unit sphere exactly for spherical polynomials of degree L [36, 47]. This means that functions $F \in \mathrm{L}^2(\Omega)$ with the expansion

$$F(t,\varphi) = \sum_{n=0}^{L} \sum_{j=-n}^{n} F_{n,j} Y_{n,j}(\xi),$$

where $\xi = \xi(t,\varphi) \in \Omega$, can be integrated exactly. As an alternative to the quadrature method used in this thesis, one could discuss, in future research, how the approach from [36, 47] could be applicable to subdomains on the sphere such as spherical caps.

Appendix B

Point Grids on the Sphere

In this appendix, we want to introduce the point grids on the sphere that we use for the numerical experiments for the CMB polarization.

B.1 Integration Grid

With the integration grid, we denote the point grid on the sphere we obtained by the Gauß quadrature for the integration on the sphere over the spherical cap in Appendix A. The grid depends on a parameter $m \in \mathbb{N}$. So, we know that the polar coordinates of this point grid are given by

$$\varphi_j = \frac{2j-1}{m+1}\pi, \qquad j = 1, \ldots, m+1,$$
$$\vartheta_k = \arccos\left(\frac{(1-b)x_k + (1+b)}{2}\right), \qquad k = 1, \ldots, m,$$

where x_k are the roots of the Legendre polynomials P_m of degree m, which we calculate by Newton's method (see Appendix A.1.2). The grid consists of $m(m+1)$ points. In Figure B.1, we see the integration grid for $m = 20$.

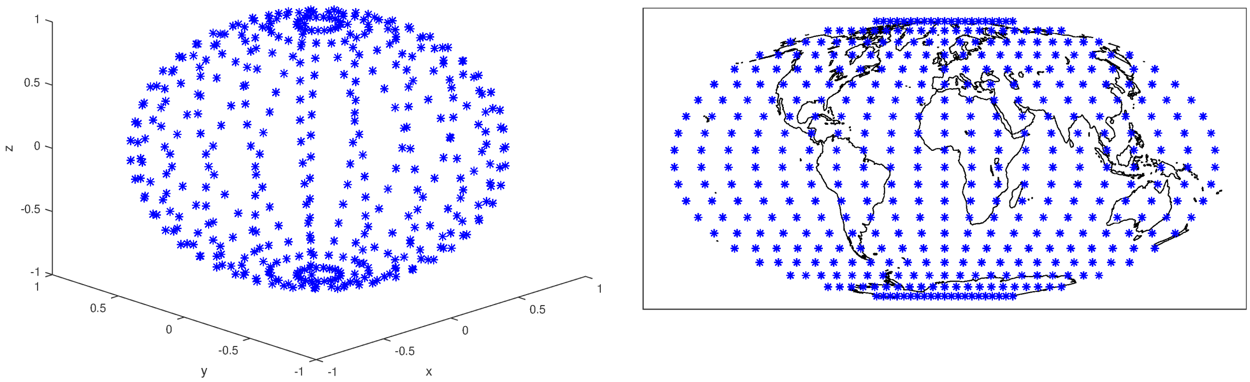
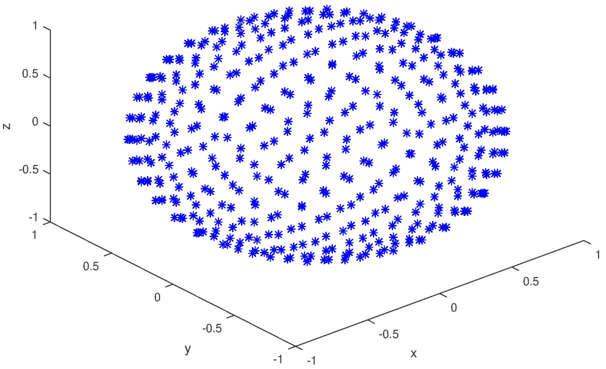


Figure B.1: Integration grid for $m = 20$.



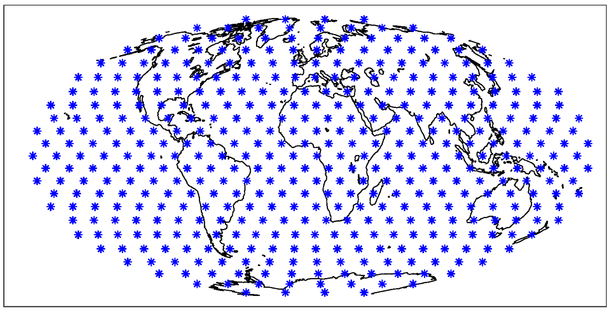


Figure B.2: HEALPix grid for $m_S = 6$.

B.2 HEALPix Grid

The HEALPix (Hierarchical Equal Area isoLatitude Pixelization) grid [29, 35] is related to CMB problems. Therefore, we use this grid for our numerical experiments for the CMB polarization. The HEALPix grid depends on a paramter $m_S \in \mathbb{N}$. For the northern cap, the polar coordinates are given by

$$\varphi_j = \frac{\pi}{2k}\left(j - 2k(k-1) - \frac{1}{2}\right),$$
$$\vartheta_j = \arccos\left(1 - \frac{k^2}{3m_S^2}\right),$$

where $j = 1, \ldots, m_{\mathrm{cap}} = 2m_S(m_S - 1)$ and

$$k = \left\lfloor \sqrt{\frac{j}{2} - \sqrt{\left\lfloor \frac{j}{2} \right\rfloor}} \right\rfloor + 1.$$

For the remaining northern hemisphere and the equator, the polar coordinates are given by

$$\varphi_{m_{\mathrm{cap}}+j} = \frac{\pi}{2m_S}\left((j-1) \bmod (4m_S) + 1 - \frac{r}{2}\right),$$
$$\vartheta_{m_{\mathrm{cap}}+j} = \arccos\left(\frac{4}{3} - \frac{2k}{3m_S}\right),$$

where $j = 1, \ldots, m_{\mathrm{belt}} = 4m_S(m_S + 1)$, $r = (k - m_s + 1) \bmod 2$, and

$$k = \left\lfloor \frac{j-1}{4m_S} \right\rfloor + m_S.$$

For the southern hemisphere, the polar coordinates are given by

$$\varphi_{N-j+1} = \varphi_j,$$
$$\vartheta_{N-j+1} = \pi - \vartheta_j,$$

where $j = 1, \ldots, m_{\mathrm{south}} = m_{\mathrm{cap}} + m_{\mathrm{belt}} - 4m_S$.

This grid consists of 12 m_S^2 points. In Figure B.2, we see the HEALPix grid for $m_S = 6$.

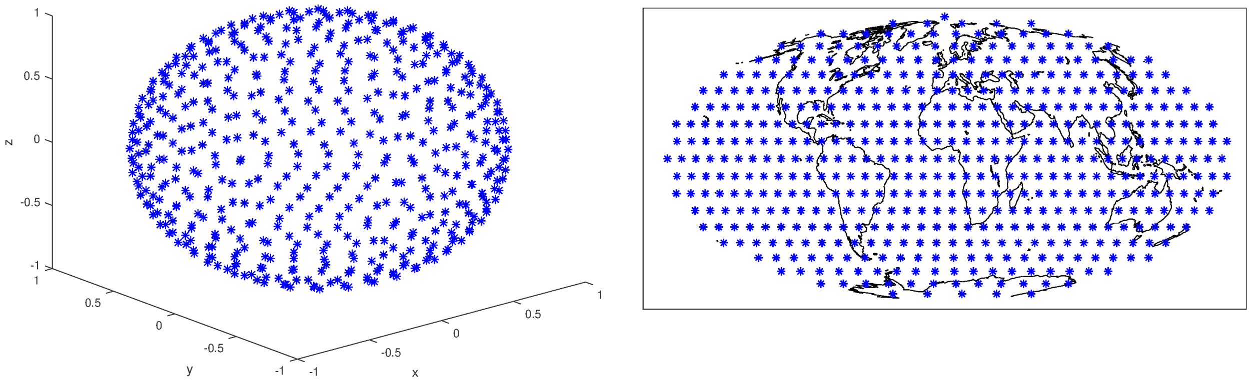


Figure B.3: Reuter grid for $m = 20$.

B.3 Reuter Grid

The Reuter grid [58, 28, 33] is an equidistributed grid on the sphere and depends on a parameter $m \in \mathbb{N}_0$. Its coordinates are given by

$$
\begin{aligned}
\vartheta_j &= \frac{\pi}{m} i, \qquad && i = 0, \dots, m, \\
\gamma_0 &= 1 \\
\gamma_i &= \left\lfloor \frac{2\pi}{\arccos\left(\frac{\cos\left(\frac{\pi}{m}\right) - \cos^2 \vartheta_i}{\sin^2 \vartheta_i} \right)} \right\rfloor, \qquad && i = 1, \dots, m-1, \\
\gamma_m &= 1, \\
\varphi_{0,1} &= 0, \\
\varphi_{i,j} &= \left(j - \frac{1}{2} \right) \frac{2\pi}{\gamma_i}, \qquad && i = 1, \dots, m-1, \ j = 1, \dots, \gamma_i, \\
\varphi_{m,1} &= 0.
\end{aligned}
$$

This grid consists of less or equal $2 + \frac{4m^2}{\pi}$ points. In Figure B.3, we see the Reuter grid for $m = 20$.

B.4 Driscoll-Healy Grid

The Driscoll-Healy grid [58] is an equiangular grid on the sphere and depends on a parameter $m \in \mathbb{N}_0$. Its polar coordinates are given by

$$
\begin{aligned}
\varphi_i &= \frac{2\pi}{m} i, \qquad && i = 0, \dots, m, \\
\vartheta_j &= \frac{\pi}{m} j, \qquad && j = 0, \dots, m.
\end{aligned}
$$

This grid consists of $(m+1)^2$ points. In Figure B.4, we see the Driscoll-Healy grid for $m = 20$.

We use the Driscoll-Healy grid as a plot grid not only for the CMB polarization but also for the tensor Slepian functions.

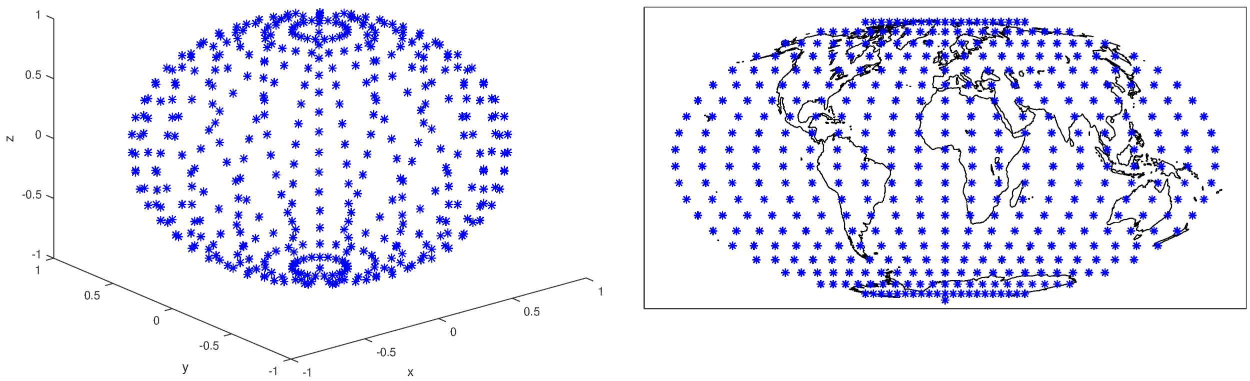


Figure B.4: Driscoll-Healy grid for $m = 20$.

Appendix C

List of Symbols

Basic Notations

Symbol	Explanation	Chapter
$\mathbb{N}_0$	set of the non-negative integers	2.1
$\mathbb{N}$	set of the positive integers	2.1
$\mathbb{Z}$	set of the integers	2.1
$\mathbb{R}$	set of the real numbers	2.1
$\mathbb{R}^+$	set of the positive real numbers	2.1
$\mathbb{C}$	set of the complex numbers	2.1
$\mathbb{R}^3$	three-dimensional Euclidean space	2.1
$\mathbb{C}^3$	three-dimensional complex space	2.1
F, G	scalar-valued functions	2.1
f, g	vector-valued functions	2.1
$\boldsymbol{f}$, $\boldsymbol{g}$	tensor-valued functions	2.1
ε^1, ε^2, ε^3	canonical orthonormal system on $\mathbb{R}^3$	2.1
$\mathrm{C}^{(k)}$, $\mathrm{c}^{(k)}$, $\mathbf{c}^{(k)}$	classes of continuous, differentiable functions	2.1
$\\|F\\|_{\mathrm{C}(D)}$, $\\|f\\|_{\mathrm{c}(D)}$, $\\|\boldsymbol{f}\\|_{\boldsymbol{c}(D)}$	C-, c-, $\boldsymbol{c}$-norm on D	2.1
$\langle x, y\rangle = x \cdot \overline{y}$	inner product of two vectors	2.1
$\delta_{n,n'}$	Kronecker symbol	2.1
$\lfloor \cdot \rfloor$	Gaussian rounding function	2.1
α, β, γ	angles	2.1
$\boldsymbol{s} \otimes \boldsymbol{t}$	tensor product of two tensors	2.2
$\boldsymbol{s} : \boldsymbol{t}$	double dot product of two tensors	2.2
ker(.)	delivers the kernel of the argument	3.6

Spherical Notations

Symbol	Explanation	Chapter
Ω	the unit sphere	2.3
ξ, η	elements of Ω	2.3
ε^r, ε^φ, ε^t	orthonormal triad on Ω	2.3
φ, ϑ	polar coordinates of a $\xi \in \Omega$	2.3
$\boldsymbol{i}_{\tan}$, $\boldsymbol{j}_{\tan}$	spherical tangential tensors	2.3
∇	gradient	2.3
Δ	Laplace operator	2.3
∇^*	surface gradient	2.3

L^*	surface curl gradient	2.3
Δ^*	Beltrami operator	2.3
$\mathrm{d}\omega$	surface element	2.3
L^p	classes of scalar-valued functions	2.3
l^p	classes of vector-valued functions	2.3
$\mathbf{l}^p$	classes of tensor-valued functions	2.3
$\langle F, G\rangle_{\mathrm{L}^2(\Omega)}$	L^2-inner product on Ω of two scalar-valued functions	2.3
$\langle f, g\rangle_{\mathrm{l}^2(\Omega)}$	l^2-inner product on Ω of two vector-valued functions	2.3
$\langle \boldsymbol{f}, \boldsymbol{g}\rangle_{\mathbf{l}^2(\Omega)}$	$\mathbf{l}^2$-inner product on Ω of two tensor-valued functions	2.3
f_{nor}	normal surface vector field	5.1
f_{tan}	tangential surface vector field	5.1
$\boldsymbol{f}_{\mathrm{nor,nor}}$	normal surface tensor field	6.1
$\boldsymbol{f}_{\mathrm{nor,tan}}$	left normal/right tangential surface tensor field	6.1
$\boldsymbol{f}_{\mathrm{tan,nor}}$	left tangential/right normal surface tensor field	6.1
$\boldsymbol{f}_{\mathrm{tan,tan}}$	tangential surface tensor field	6.1

Spherical Harmonics

Symbol	**Explanation**	**Chapter**
P_n	Legendre polynomials of degree n	2.4
$P_{n,j}$	associated Legendre functions of degree n and order j	2.4
$X_{n,j}$	fully normalized associated Legendre functions of degree n and order j	2.4
$Y_{n,j}$	(scalar) fully normalized spherical harmonics of degree n and order j	2.4
Hom_n	space of homogeneous polynomials of degree n	2.4
Harm_n	space of the scalar spherical harmonics of degree n	2.4
$c_{n,j}$	recursion coefficients for the spherical harmonics	2.4
$y_{n,j}^{(i)}$	vector spherical harmonics of type i, degree n, and order j	2.5
$\mu_n^{(i)}$	coefficients of the vector spherical harmonics of type i and degree n	2.5
$o^{(i)}$	vector operator of type i	2.5
0_i	minimal degree of the vector spherical harmonics of type i	2.5
$\mathrm{harm}_n^{(i)}$	space of the vector spherical harmonics of type i and degree n	2.5
harm_n	space of the vector spherical harmonics of degree n	2.5

$\boldsymbol{\Delta}^*$	vector Beltrami operator	2.5
$\boldsymbol{y}_{n,j}^{(i,k)}$	tensor spherical harmonics of type (i,k), degree n, and order j	2.6
$\boldsymbol{\mu}_n^{(i,k)}$	coefficients of the tensor spherical harmonics of type (i,k) and degree n	2.6
$\boldsymbol{q}^{(i,k)}$, $\boldsymbol{o}^{(i,k)}$	tensor operators of type (i,k)	2.6
0_{ik}	minimal degree of the tensor spherical harmonics of type (i,k)	2.6
$\mathbf{harm}_n^{(i,k)}$	space of the tensor spherical harmonics of type (i,k) and degree n	2.6
$\mathbf{harm}_n$	space of the tensor spherical harmonics of degree n	2.6
$\blacktriangle^*$	tensor Beltrami operator	2.6

Spin-Weighted Spherical Harmonics

Symbol	**Explanation**	**Chapter**
N	spin weight	3.1
${}_NF_n$	function of spin weight N and degree n	3.1
o^i, $\hat{o}^i$	function for spin weight description	3.1
sw(.)	delivers the spin weight of the argument	3.1
${}_NY_{n,j}$	spin-weighted spherical harmonics of spin weight N, degree n, and order j	3.2
$\eth_N$, $\overline{\eth_N}$	spin raising and lowering operator	3.2
$\eth_N^M$, $\overline{\eth_N}^M$	iterative use of the operators $\eth_N$, $\overline{\eth_N}$	3.2
$\alpha_{n,j}^N$	recursion coefficients for the spin-weighted spherical harmonics	3.3
$\Delta^{*,N}$	spin-weighted Beltrami operator	3.3
$D_{j,N}^n$	Wigner D-function	3.4
$d_{j,N}^n$	function, which defines the Wigner D-function	3.4
$Y_{n,j}^N$	generalized spherical harmonics of spin weight N, degree n, and order j	3.4
X^k	set of functions with special properties of the spin-weighted spherical harmonics	3.4
$P_n^{(\alpha,\beta)}$	Jacobi polynomials	3.4
Harm_n^N	set of the $(*,N)$-harmonic functions of spin weight N and degree n	3.6
${}_NP_n$	spin-weighted Legendre polynomials of spin weight N and degree n	3.7
${}_NP_{n,j}$	associated spin-weighted Legendre functions of spin weight N, degree n, and order j	3.7
${}_NX_{n,j}$	fully normalized associated spin-weighted Legendre functions of spin weight N, degree n, and order j	3.7
w_N	spin-weighted weight function of spin weight N	3.7

$\langle F, G\rangle_{w_N}$	weighted $\mathrm{L}^2([-1,1])$-inner product of two scalar functions with weight function w_N	3.7
${}_N\gamma_n$	weighted norm of the spin-weighted Legendre polynomials of spin weight N and degree n	3.7
$\beta^N_{n,j}$	recursion coefficients for the associated spin-weighted Legendre functions	3.7
${}_NA_n$	leading coefficient of the spin-weighted Legendre polynomials of spin weight N and degree n	3.7
$\tau_\pm$	orthonormal vectors on Ω	3.9

Slepian Functions

Symbol	**Explanation**	**Chapter**
K	kernel matrix of the scalar case	4.1
$\mathscr{K}$	kernel function for the scalar case	4.1
G	eigenvector of the scalar Slepian functions	4.1
$\mathscr{G}$	scalar Slepian function	4.1
S	Shannon number	4.3
k, k^{nor}, k^{tan}	kernel matrix of the vector case and its blocks parted into normal and tangential part	5.1
g	eigenvectors of k	5.1
g_{nor}	eigenvectors of k^{nor}	5.1.1
$\mathscr{k}^{\mathrm{nor}}$	kernel function to k^{nor}	5.1.1
$\mathscr{g}_{\mathrm{nor}}$	normal vector Slepian functions	5.1.1
$y^\pm$	combined vector spherical harmonics	5.1.2
$k^\pm$	transformed tangential kernel matrix blocks	5.1.2
$g^\pm_{\mathrm{tan}}$	eigenvectors of $k^\pm$	5.1.2
$\mathscr{k}^\pm$	kernel function to $k^\pm$	5.1.2
$\mathscr{g}^\pm_{\mathrm{tan}}$	tangential vector Slepian functions	5.1.2
y^i	transformed vector spherical harmonics	5.1.3
J	set of indices	5.1.3
N_i	spin-weight dependent on the type i of transformed vector spherical harmonics	5.1.3
k, $k^{ii} = k^i$	transformed kernel matrix of the vector case and its blocks	5.1.3
g, g^i	eigenvectors of the vector Slepian functions for the transformed vector spherical harmonics	5.1.3
$\mathscr{k}$	kernel functions for the transformed vector case	5.1.3
$\mathscr{g}$, $\mathscr{g}^i$	vector Slepian functions for the transformed vector spherical harmonics	5.1.3
$\boldsymbol{k}$, $\boldsymbol{k}^{\mathrm{nor,nor}}$, $\boldsymbol{k}^{\mathrm{nor,tan}}$, $\boldsymbol{k}^{\mathrm{tan,nor}}$, $\boldsymbol{k}^{\mathrm{tan,tan}}$	kernel matrix of the tensor case and its blocks	6.1
$\boldsymbol{g}$	eigenvectors of $\boldsymbol{k}$	6.1
$\boldsymbol{g}_{\mathrm{nor,nor}}$	eigenvectors of $\boldsymbol{k}^{\mathrm{nor,nor}}$	6.1.1
$\boldsymbol{\mathscr{k}}^{\mathrm{nor,nor}}$	kernel function to $\boldsymbol{k}^{\mathrm{nor,nor}}$	6.1.1

$\boldsymbol{\mathscr{g}}_{\mathrm{nor,nor}}$	normal tensor Slepian functions	6.1.1
$\boldsymbol{y}^{(1,\pm)}$	combined tensor spherical harmonics for the left normal/right tangential part	6.1.2
$\boldsymbol{k}^{(1,\pm)}$	transformed left normal/right tangential kernel matrix blocks	6.1.2
$\boldsymbol{g}^{(1,\pm)}_{\mathrm{nor,tan}}$	eigenvectors of $\boldsymbol{k}^{(1,\pm)}$	6.1.2
$\boldsymbol{\mathscr{k}}^{(1,\pm)}$	kernel function to $\boldsymbol{k}^{(1,\pm)}$	6.1.2
$\boldsymbol{\mathscr{g}}^{(1,\pm)}_{\mathrm{nor,tan}}$	left normal/right tangential tensor Slepian functions	6.1.2
$\boldsymbol{y}^{(\pm,1)}$	combined tensor spherical harmonics for the left tangential/right normal part	6.1.3
$\boldsymbol{k}^{(\pm,1)}$	transformed left tangential/right normal kernel matrix blocks	6.1.3
$\boldsymbol{g}^{(\pm,1)}_{\mathrm{tan,nor}}$	eigenvectors of $\boldsymbol{k}^{(\pm,1)}$	6.1.3
$\boldsymbol{\mathscr{k}}^{(\pm,1)}$	kernel function to $\boldsymbol{k}^{(\pm,1)}$	6.1.3
$\boldsymbol{\mathscr{g}}^{(\pm,1)}_{\mathrm{tan,nor}}$	left tangential/right normal tensor Slepian functions	6.1.3
$\boldsymbol{y}^{(\pm,\pm)}$	combined tensor spherical harmonics for parts of the tangential part	6.1.4
$\boldsymbol{k}^{(2,2)}$, $\boldsymbol{k}^{(3,3)}$, $\boldsymbol{k}^{(\pm,\pm)}$	transformed tangential kernel matrix blocks	6.1.4
$\boldsymbol{g}^{(2,2)}_{\mathrm{tan,tan}}$, $\boldsymbol{g}^{(3,3)}_{\mathrm{tan,tan}}$, $\boldsymbol{g}^{(\pm,\pm)}_{\mathrm{tan,tan}}$	eigenvectors of $\boldsymbol{k}^{(2,2)}$, $\boldsymbol{k}^{(3,3)}$, $\boldsymbol{k}^{(\pm,\pm)}$	6.1.4
$\boldsymbol{\mathscr{k}}^{(2,2)}$, $\boldsymbol{\mathscr{k}}^{(3,3)}$, $\boldsymbol{\mathscr{k}}^{(\pm,\pm)}$	kernel function to $\boldsymbol{k}^{(2,2)}$, $\boldsymbol{k}^{(3,3)}$, $\boldsymbol{k}^{(\pm,\pm)}$	6.1.4
$\boldsymbol{g}^{(2,2)}_{\mathrm{tan,tan}}$, $\boldsymbol{g}^{(3,3)}_{\mathrm{tan,tan}}$, $\boldsymbol{g}^{(\pm,\pm)}_{\mathrm{tan,tan}}$	tangential tensor Slepian functions	6.1.4
$\boldsymbol{y}^{i}$	transformed tensor spherical harmonics	6.1.5
$\boldsymbol{J}$	set of indices	6.1.5
$\mathbf{0}_i$	minimal degree of the transformed tensor spherical harmonics of type i	6.1.5
$\boldsymbol{N}_i$	spin-weight dependent on the type i of transformed tensor spherical harmonics	6.1.5
$\boldsymbol{k}$, $\boldsymbol{k}^{ii} = \boldsymbol{k}^{i}$	transformed kernel matrix of the tensor case and its blocks	6.1.5
$\boldsymbol{g}$, $\boldsymbol{g}^{i}$	eigenvectors of the tensor Slepian functions for the transformed tensor spherical harmonics	6.1.5
$\boldsymbol{\mathscr{k}}$	kernel functions for the transformed tensor case	6.1.5
$\boldsymbol{\mathscr{g}}$, $\boldsymbol{\mathscr{g}}^{i}$	tensor Slepian functions for the transformed tensor spherical harmonics	6.1.5
$\mathscr{H}$, H	spacelimited scalar Slepian functions and eigenvectors	7.1
$\mathscr{h}$, h^{i}	spacelimited vector Slepian functions and eigenvectors	7.2
$\boldsymbol{\mathscr{h}}$, $\boldsymbol{h}^{i}$	spacelimited tensor Slepian functions and eigenvectors	7.3
K^{N}	spin-weighted kernel matrix	8.1
$\mathscr{K}^{N}$	spin-weighted (scalar) kernel function	8.1

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$	Wigner $3j$-symbol	8.1
$\mathscr{I}^N$	spin-weighted commuting operator to $\mathscr{K}^N$	8.2
I^N	spin-weighted commuting matrix to K^N	8.3

CMB Polarization

Symbol	**Explanation**	**Chapter**
$\boldsymbol{p}$	polarization	9.1
$\boldsymbol{e}$, $\boldsymbol{b}$	electric and magnetic component of $\boldsymbol{p}$	9.1
$\boldsymbol{y}^{\mathrm{E}}_{n,j}$, $\boldsymbol{y}^{\mathrm{B}}_{n,j}$	electric and magnetic tensor spherical harmonics of degree n and order j	9.1
$\boldsymbol{k}$	kernel matrix for the transformed polarization eigenvalue problem	9.2
$\boldsymbol{g}$, $\boldsymbol{\mathscr{g}}$	eigenvector and eigenfunctions of $\boldsymbol{k}$	9.2
$\boldsymbol{\mathscr{g}}^{\mathrm{CMB}}$	tensor Slepian functions of the polarization	9.2
$\boldsymbol{\mathscr{g}}^{\mathrm{E}}$, $\boldsymbol{\mathscr{g}}^{\mathrm{B}}$	tensor Slepian functions of the electric and the magnetic part of the polarization	9.2
$\boldsymbol{p}^{\mathrm{CMB}}_{\alpha}$	coefficients of the tensor Slepian functions for the polarization	9.2
$\boldsymbol{g}^{\mathrm{E}}$, $\boldsymbol{g}^{\mathrm{B}}$	eigenvectors of the electric and magnetic Slepian functions	9.2

Integration by Quadrature

Symbol	**Explanation**	**Chapter**
w	weight function	A.1
$\langle F, G\rangle_w$	weighted L^2-inner product of two scalar functions with weight function w	A.1
w_k	quadrature weights	A.1
L_k	Lagrange polynomials	A.1
γ_n	weighted norm of the orthogonal quadrature polynomials of degree n	A.1
A_n	leading coefficient of the orthogonal quadrature polynomials of degree n	A.1.2

Bibliography

[1] M. Aich, T. Souradeep, and S. Basak. CMB temperature and polarization anisotropies on an incomplete sky. http://www.iucaa.ernet.in/~tarun/papers/papers/students/moumita_gradproject.pdf, 2007. Accessed 23. October 2017.

[2] A. Albertella, F. Sansò, and N. Sneeuw. Band-limited functions on a bounded spherical domain: The Slepian problem on the sphere. *Journal of Geodesy*, 73(9):436–447, 1999.

[3] K. Atkinson. Numerical integration on the sphere. *Journal of the Australian Mathematical Society*, 23(3):332–347, 1982.

[4] F. M. Becker, G. Boortz, V. Dietrich, L. Engelmann, C. Ernst, G. Fanghänel, H. Höhne, R. Lenertal, L. Meyer, C. Pews-Hocke, G. D. Schmidt, R. Stamm, and K. Weber. *Formeln und Tabellen für die Sekundarstufen I und II*. PAETEC Gesellschaft für Bildung und Technik mbH, Berlin, 11th edition, 2004.

[5] L. Bieberbach. *Theorie der Differentialgleichungen*. Springer, Heidelberg, 1979.

[6] A. O. Bouzas. Addition theorems for spin spherical harmonics: I. preliminaries. *Journal of Physics A: Mathematical and Theoretical*, 44(16):165301, 2011.

[7] A. O. Bouzas. Addition theorems for spin spherical harmonics: II. results. *Journal of Physics A: Mathematical and Theoretical*, 44(16):165302, 2011.

[8] R. Burridge. Spherically symmetric differential equations, the rotation group, and tensor spherical functions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 65(1):157–175, 1969.

[9] W. B. Campbell. Tensor and spinor spherical harmonics and the spin-s harmonics ${}_sY_{lm}(\theta, \phi)^*$. *Journal of Mathematical Physics*, 12(8):1763–1770, 1971.

[10] T. S. Chihara. *An Introduction to Orthogonal Polynomials*. Gordon and Breach, Science Publishers, New York, 1978.

[11] F. A. Dahlen and F. J. Simons. Spectral estimation on a sphere in geophysics and cosmology. *Geophysical Journal International*, 174(3):774–807, 2008.

[12] F. A. Dahlen and J. Tromp. *Theoretical Global Seismology*. Princeton University Press, Princeton, 1998.

[13] L. Dai, M. Kamionkowski, and D. Jeong. Total angular momentum waves for scalar, vector, and tensor fields. *Physical Review D*, 86(12):125013, 2012.

[14] S. V. Dhurandhar and M. Tinto. Astronomical observations with a network of detectors of gravitational waves - I. mathematical framework and solution of the five detector problem. *Monthly Notices of the Royal Astronomical Society*, 234(3):663–676, 1988.

[15] T. Dray. The relationship between monopole harmonics and spin-weighted spherical harmonics. *Journal of Mathematical Physics*, 26(5):1030–1033, 1985.

[16] J. R. Driscoll and D. M. Healy, Jr. Computing Fourier transforms and convolutions on the 2-sphere. *Advances in Applied Mathematics*, 15(2):202–250, 1994.

[17] M. Eastwood and P. Tod. Edth - a differential operator on the sphere. *Mathematical Proceedings of the Cambridge Philosophical Society*, 92(2):317–330, 1982.

[18] A. R. Edmonds. *Angular Momentum in Quantum Mechanics*. Princeton University Press, Princeton, 1957.

[19] W. Erb. An orthogonal polynomial analogue of the Landau-Pollak-Slepian time-frequency analysis. *Journal of Approximation Theory*, 166:56–77, 2013.

[20] W. Erb and S. Mathias. An alternative to Slepian functions on the unit sphere - A space-frequency analysis based on localized spherical polynomials. *Applied and Computational Harmonic Analysis*, 38(2):222–241, 2013.

[21] ESA. Goce. `http://www.esa.int/Our_Activities/Observing_the_Earth/GOCE`, 2018. Accessed 10. January 2018.

[22] M. Eshagh. Spatially restricted integrals in gradiometric boundary value problems. *Artificial Satellites*, 44(4):131–148, 2009.

[23] M. J. Fengler. *Vector Spherical Harmonic and Vector Wavelet Based Non-Linear Galerkin Schemes for Solving the Incompressible Navier-Stokes Equation on the Sphere*. PhD thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, 2005. Published by Shaker Verlag, Aachen.

[24] M. J. Fengler and W. Freeden. A nonlinear Galerkin scheme involving vector and tensor spherical harmonics for solving the incompressible Navier-Stokes equation on the sphere. *SIAM Journal on Scientific Computation*, 27(3):967–994, 2005.

[25] A. J. Fox and F. A. Johnson. On finding the eigenvalues of real symmetric tridiagonal matrices. *The Computer Journal*, 9(1):98–105, 1966.

[26] W. Freeden and T. Gervens. Vector spherical spline interpolation - basic theory and computational aspects. *Mathematical Methods in the Applied Sciences*, 16(3):151–183, 1993.

[27] W. Freeden, T. Gervens, and M. Schreiner. Tensor spherical harmonics and tensor spherical splines. *Manuscripta Geodaetica*, 19:70–100, 1994.

[28] W. Freeden, T. Gervens, and M. Schreiner. *Constructive Approximation on the Sphere With Applications to Geomathematics*. Oxford University Press, Oxford, 1998.

[29] W. Freeden and M. Gutting. *Special Functions on Mathematical (Geo-)Physics*. Springer, Basel, 2013.

[30] W. Freeden, V. Michel, and H. Nutz. Satellite-to-satellite tracking and satellite gravity gradiometry (advanced techniques for high-resolution geopotential field determination). *Journal of Engineering Mathematics*, 43(1):19–56, 2002.

[31] W. Freeden, V. Michel, and F. J. Simons. Spherical harmonics based special function systems and constructive approximation methods. In *Handbook of Mathematical Geodesy*. Springer, 2016. Submitted.

[32] W. Freeden and H. Nutz. Satellite gravity gradiometry as tensorial inverse problem. *GEM - International Journal on Geomathematics*, 2:177–218, 2011.

[33] W. Freeden and M. Schreiner. *Spherical Functions of Mathematical Geosciences*. Springer, Heidelberg, 2009.

[34] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan. Spin-s spherical harmonics and $\eth$. *Journal of Mathematical Physics*, 8(11):2155–2161, 1967.

[35] K. M. Górski, E. Hivon, A. J. Banday, B. D. Wandelt, and F. K. Hansen. HEALPix: A framework for high-resolution discretization and fast analysis of data distributed on the sphere. *The Astrophysical Journal*, 622(2):759–771, 2005.

[36] M. Gräf, S. Kunis, and D. Potts. On the computation of nonnegative quadrature weights on the sphere. *Applied and Computational Harmonic Analysis*, 27(1):124–132, 2009.

[37] F. A. Grünbaum, L. Longhi, and M. Perlstadt. Differential operators commuting with finite convolution integral operators: Some non-Abelian examples. *SIAM Journal on Applied Mathematics*, 42(5):941–955, 1982.

[38] M. Hanke-Bourgeois. *Grundlagen der Numerischen Mathematik und des Wissenschaftlichen Rechnens*. Teubner Verlag, Wiesbaden, second edition, 2006.

[39] D. M. Healy, Jr., D. N. Rockmore, P. J. Kostelec, and S. Moore. FFTs for the 2-sphere-improvements and variations. *Journal of Fourier Analysis and Applications*, 9(4):341–385, 2003.

[40] E. L. Hill. The theory of vector spherical harmonics. *American Journal of Physics*, 22(4):211–214, 1954.

[41] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, second edition, 2013.

[42] W. Hu and M. White. CMB anisotropies: Total angular momentum method. *Physical Review D*, 56(2):596–615, 1997.

[43] K. Jahn and N. Bokor. Revisiting the concentration problem of vector fields within a spherical cap: A commuting differential operator solution. *Journal of Fourier Analysis and Applications*, 20(2):421–451, 2014.

[44] M. Kamionkowski, L. Dai, and D. Jeong. Tensor-induced B modes with no temperature fluctuations. *Physical Review D*, 89(10):107302, 2014.

[45] M. Kamionkowski, A. Kosowsky, and A. Stebbins. A probe of primordial gravity waves and vorticity. *Physical Review Letters*, 78(11):2058–2061, 1997.

[46] M. Kamionkowski, A. Kosowsky, and A. Stebbins. Statistics of cosmic microwave background polarization. *Physical Review D*, 55(12):7368–7388, 1997.

[47] J. Keiner and D. Potts. Fast evaluation of quadrature formulae on the sphere. *Mathematics of Computation*, 77(261):397–419, 2008.

[48] Z. Khalid, R. A. Kennedy, and J. D. McEwen. Slepian spatial-spectral concentration on the ball. *Applied and Computational Harmonic Analysis*, 40(3):470–504, 2016.

[49] P. J. Kostelec, D. K. Maslen, D. M. Healy, and D. N. Rockmore. Computational harmonic analysis for tensor fields on the two-sphere. *Journal of Computational Physics*, 162(2):514–535, 2000.

[50] S. Kunis and D. Potts. Fast spherical Fourier algorithms. *Journal of Computational and Applied Mathematics*, 161(1):75–98, 2003.

[51] P. K. Kythe and M. R. Schäferkotter. *Handbook of Computational Methods for Integration.* Chapman & Hall/CRC Press, Boca Raton, 2005.

[52] H. J. Landau and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty - II. *Bell System Technical Journal*, 40(1):65–84, 1961.

[53] A. Lewis, A. Challinor, and N. Turok. Analysis of CMB polarization on an incomplete sky. *Physical Review D*, 65(2):023505, 2002.

[54] H. Lütkepohl. *Handbook of Matrices.* John Wiley & Sons Ltd, Chichester, 1996.

[55] W. Magnus, F. Oberhettinger, and R. P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics.* Springer, Berlin, 1966.

[56] Z. Martinec. Green's function solution to spherical gradiometric boundary-value problems. *Journal of Geodesy*, 77(1-2):41–49, 2003.

[57] J. Mathews. Gravitational multipole radiation. *SIAM Journal of the Society for Industrial and Applied Mathematics*, 10(4):768–780, 1962.

[58] V. Michel. *Lectures on Constructive Approximation. Fourier, Spline, and Wavelet Methods on the Real Line, the Sphere, and the Ball.* Birkhäuser Verlag, New York, 2013.

[59] V. Michel, S. Orzlowski, and N. Schneider. Vectorial slepian functions on the ball. arXiv 1707.00245, 2017.

[60] V. Michel and F. J. Simons. A general approach to regularizing inverse problems with regional data using Slepian wavelets. *Inverse Problems*, 33(125016), 2017.

[61] M. J. Mohlenkamp. A fast transform for spherical harmonics. *Journal of Fourier Analysis and Applications*, 5(2-3):159–184, 1999.

[62] V. Mukhanov. *Physical Foundations of Cosmology.* Cambridge University Press, Cambridge, 2005.

[63] E. T. Newman and R. Penrose. Note on the Bondi-Metzner-Sachs group. *Journal of Mathematical Physics*, 7(5):863–870, 1966.

[64] R. Penrose and W. Rindler. *Spinors and Space-Time Volume 1: Two-Spinor Calculus and Relativistic Fields.* Cambridge University Press, Cambridge, 1984.

[65] R. Penrose and W. Rindler. *Spinors and Space-Time Volume 2: Spinor and Twistor Methods in Space-Time Geometry.* Cambridge University Press, Cambridge, 1986.

[66] R. A. Phinney and R. Burridge. Representation of the elastic-gravitational excitation of a spherical Earth model by generalized spherical harmonics. *Geophysical Journal International*, 34(4):451–487, 1973.

[67] A. Plattner and F. J. Simons. Spatiospectral concentration of vector fields on a sphere. *Applied and Computational Harmonic Analysis*, 36(1):1–22, 2014.

[68] D. Potts, G. Steidl, and M. Tasche. Fast and stable algorithms for discrete spherical Fourier transforms. *Linear Algebra and its Applications*, 275-276:433–450, 1998.

[69] D. Potts, G. Steidl, and M. Tasche. Fast Fourier transforms for nonequispaced data: A tutorial. In *J. J. Benedetto and P. J. S. G. Ferreira (eds.): Modern Sampling Theory. Applied and Numerical Analysis*, pages 247–270. Birkhäuser, Boston, 2001.

[70] J. Rasch and A. C. H. Yu. Efficient storage scheme for precalculated Wigner 3j, 6j and Gaunt coefficients. *SIAM Journal on Scientific Computing*, 25(4):1416–1428, 2003.

[71] T. Regge and J. A. Wheeler. Stability of a Schwarzschild singularity. *Physical Review*, 108(4):1063–1069, 1957.

[72] R. Rummel and M. van Gelderen. Spectral analysis of the full gravity tensor. *Geophysical Journal International*, 111(1):159–169, 1992.

[73] V. D. Sandberg. Tensor spherical harmonics on S^2 and S^3 as eigenvalue problems. *Journal of Mathematical Physics*, 19(12):2441–2446, 1978.

[74] M. Schreiner. *Tensor Spherical Harmonics and Their Application in Satellite Gradiometry*. PhD thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, 1994.

[75] H. R. Schwarz and N. Köckler. *Numerische Mathematik*. Teubner Verlag, Wiesbaden, 6th edition, 2006.

[76] U. Seljak and M. Zaldarriaga. Signature of gravity waves in the polarization of the microwave background. *Physical Review Letters*, 78(11):2054–2057, 1997.

[77] S. K. Sharma and U. Khanal. Perturbation of FRW spacetime in NP formalism. *International Journal of Modern Physics D*, 23(2):1450017, 2014.

[78] F. J. Simons. Slepian functions and their use in signal estimation and spectral analysis. In *W. Freeden, M. Zuhair Nashed and T. Sonar (eds.): Handbook of Geomathematics*, pages 893–923. Springer, Heidelberg, 2010.

[79] F. J. Simons, 2016. personal communication.

[80] F. J. Simons and F. A. Dahlen. Spherical Slepian functions and the polar gap in geodesy. *Geophysical Journal International*, 166(3):1039–1061, 2006.

[81] F. J. Simons and F. A. Dahlen. A spatiospectral localization approach to estimating potential fields on the surface of a sphere from noisy, incomplete data taken at satellite altitudes. *SPIE Proceedings*, 6701(6):60117, 2007.

[82] F. J. Simons, F. A. Dahlen, and M. A. Wieczorek. Spatiospectral concentration on a sphere. *SIAM Review*, 48(3):504–536, 2006.

[83] F. J. Simons and A. Plattner. Scalar and vector Slepian functions, spherical signal estimation and spectral analysis. arXiv 1306.3184, 2013.

[84] D. Slepian. Prolate spheroidal wave functions, Fourier analysis and uncertainty - IV: Extensions to many dimensions; generalized prolate spheroidal functions. *Bell System Technical Journal*, 43(6):3009–3057, 1964.

[85] D. Slepian and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty - I. *Bell System Technical Journal*, 40(1):43–63, 1961.

[86] R. Suda and M. Takami. A fast spherical harmonics transform algorithm. *Mathematics of Computation*, 71(238):703–715, 2001.

[87] G. Szegö. *Orthogonal Polynomials*. American Mathematical Society, Providence, Rhode Island, 1939.

[88] I. Tamm. Die verallgemeinerten Kugelfunktionen und die Wellenfunktionen eines Elektrons im Felde eines Magnetpoles. *Zeitschrift für Physik*, 71(3-4):141–150, 1931.

[89] R. Telschow. *An Orthogonal Matching Pursuit for the Regularization of Spherical Inverse Problems*. PhD thesis, Geomathematics Group, Department of Mathematics, University of Siegen, 2014. Published by Verlag Dr. Hut, München 2015.

[90] K. S. Thorne. Multipole expansions of gravitational radiation. *Reviews of Modern Physics*, 52(2):299–339, 1980.

[91] G. F. Torres del Castillo. Spin-weighted spherical harmonics and their applications. *Revista Mexicana de Fisica*, 53(2):125–134, 2007.

[92] M. Turner and C. M. Will. Post-Newtonian gravitational Bremsstrahlung. *The Astrophysical Journal*, 220:1107–1124, 1978.

[93] D. A. Varshalovich, A. Moskalev, and V. Khersonskii. *Quantum Theory of Angular Momentum*. World Scientific Publishing Co Pte Ltd, Singapore, 1988.

[94] A. Voigt and J. Wloka. *Hilberträume und elliptische Differentialoperatoren*. Bibliographisches Institut, Mannheim, 1975.

[95] G. von Winckel. Legendre-Gauss quadrature weights and nodes. `https://de.mathworks.com/matlabcentral/fileexchange/4540-legendre-gauss-quadrature-weights-and-nodes`, 2004. Accessed 23. October 2017.

[96] Y. Wiaux, L. Jacques, and P. Vandergheynst. Fast spin ±2 spherical harmonics transforms and application in cosmology. *Journal of Computational Physics*, 226(2):2359–2371, 2007.

[97] Y. Wiaux, L. Jacques, P. Vielva, and P. Vandergheynst. Fast directional correlation on the sphere with steerable filters. *The Astrophysical Journal*, 652(1):820–832, 2006.

[98] T. T. Wu and C. N. Yang. Some properties of monopole harmonics. *Physical Review D*, 16(4):1018–1021, 1977.

[99] M. Zaldarriaga and U. Seljak. All-sky analysis of polarization in the microwave background. *Physical Review D*, 55(4):1830–1840, 1997.

[100] F. J. Zerilli. Gravitational field of a particle falling in a Schwarzschild geometry analyzed in tensor harmonics. *Physical Review D*, 2(10):2141–2160, 1970.

[101] F. J. Zerilli. Tensor harmonics in canonical form for gravitational radiation and other applications. *Journal of Mathematical Physics*, 11(7):2203–2208, 1970.

[102] W. Zhao and D. Baskaran. Separating E and B types of polarization on an incomplete sky. *Physical Review D*, 82(2):023001, 2010.

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The spin-weighted spherical harmonics of Newman and Penrose (1966) form an orthonormal basis of $L^2(\Omega)$ on the unit sphere Ω and have a huge field of applications.

We present a unified mathematical theory. Here, we not only collect already known properties in a mathematical way, but also show new ones as well. All of this is connected to the notation of the spherical harmonics. In addition, we use spin-weighted spherical harmonics to construct tensor Slepian functions on the sphere.

Slepian functions are spatially concentrated and spectrally limited. Their concentration within a chosen region of the sphere allows for local inversions when only regional data are available, or enable the extraction of regional information.

By using spin-weighted spherical harmonics, our theory offers several numerical advantages. Furthermore, we present a method for an efficient construction of tensor Slepian functions for spherical caps. In this context, we are able to construct a localized basis on the spherical cap for the cosmic microwave background (CMB) polarization.

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