Borders of the Probabilistic Symbol, Time-Inhomogeneity, and Generalized Semimartingales

DISSERTATION

zur Erlangung des Grades eines Doktors

der Naturwissenschaft

vorgelegt von

M.Sc. Sebastian Rickelhoff

eingereicht bei der Naturwissenschaftlich-Technischen Fakultät

der Universität Siegen

Siegen 2023

Betreuer und erster Gutachter
Univ-Prof. Dr. rer. nat. Alexander Schnurr
Universität Siegen

Zweiter Gutachter
Univ-Prof. Dr. rer. nat. habil. Anita Behme
Technische Universität Dresden

Tag der mündlichen Prüfung
15. Dezember 2023

Abstract

When investigating semimartingales, both the characteristics and the probabilistic symbol play important roles. They allow for the analysis of various significant properties and, in some cases, the characterization of the underlying process. For instance, the symbol, which is related to the right derivative of the characteristic function of the one-dimensional marginals, of a Lévy process, coincides with the characteristic exponent, and for Feller processes with the symbol of the operator. The most general class for which the symbol exists is Itô processes.

In this thesis, we show that within the class of Hunt semimartingales, Itô processes are precisely those for which the probabilistic symbol exists. Furthermore, we point out that the applicability of the symbol can be lost for processes that are not Hunt semimartingales, even if the symbol exists.

Investigating beyond time-homogeneity, we add a time component to the symbol to analyze non-homogeneous processes. We show the existence of such a time-dependent symbol for non-homogeneous Itô processes. Additionally, for this class of processes, we derive maximal inequalities, which we apply to extend the Blumenthal-Getoor indices to the non-homogeneous case. These allow for the derivation of various properties concerning the paths of the process.

Lastly, we generalize semimartingales by introducing a 'point of no return' or 'killing point', as known in the Markovian context, within this framework. To this end, the development of a new characteristic to describe this phenomenon is required. We present a theory concerning these generalized semimartingales by extending some of the most important classical results with the help of the new characteristic, and integrate the probabilistic symbol into this context. Additionally, we present a natural way in which a killing can occur in the semimartingale framework.

Zusammenfassung

Bei der Untersuchung von Semimartingalen spielen sowohl deren Charakteristiken als auch das probabilistische Symbol eine wichtige Rolle, denn sie ermöglichen nicht nur die Analyse verschiedener wichtiger Eigenschaften, sondern in einigen Fällen auch die Charakterisierung des zugrunde liegenden Prozesses. Beispielsweise stimmt das Symbol bei Lévyprozessen mit dem jeweiligen charakteristischen Exponenten überein und bei Feller-Prozessen mit dem Symbol des Operators. Die allgemeinsten Prozesse, für die das Symbol existiert, sind Itô-Prozesse.
In dieser Arbeit zeigen wir, dass das probabilistische Symbol eines Hunt-Semimartingals genau dann existiert, wenn der betrachtete Prozess ein Itô-Prozess ist. Darüber hinaus führen wir aus, warum das Symbol für Prozesse, die keine Hunt-Semimartingale sind, seine Anwendbarkeit verliert.
Lässt man die zeitliche Homogenität der bis zu diesem Zeitpunkt betrachteten Prozesse hinter sich, müssen wir dem Symbol, um auch zeitlich inhomogene Prozesse analysieren zu können, eine Zeitkomponente hinzufügen. Wir zeigen die Existenz eines solchen zeitabhängigen Symbols für inhomogene Itô-Prozesse und leiten für diese Klasse von Prozessen Maximal-Ungleichungen her, die wir anwenden, um die so genannten Blumenthal-Getoor-Indizes auf den inhomogenen Fall zu erweitern. Diese wiederum ermöglichen die Ableitung verschiedener Pfadeigenschaften des Prozesses.
Schließlich verallgemeinern wir Semimartingale durch die Einführung eines sogenannten „Killing Point“ , wie er im Markov-Kontext bereits bekannt ist. Dies erfordert die Einführung einer neuen Semimartingal-Charakteristik zur Beschreibung des genannten Phänomens. Wir stellen eine Theorie dieser verallgemeinerten Semimartingale vor, indem wir einige der wichtigsten klassischen Ergebnisse mit Hilfe der neuen Charakteristik verallgemeinern. Außerdem stellen wir eine natürliche Art und Weise vor, wie so ein „Killing Point“ im Kontext von Semimartingalen auftreten kann.

Contents

Introduction

Doléans-Dade and Meyer introduced semimartingales, as they are known today, in 1970 (cf. [19]) to detach stochastic integration from the theory of Markov processes. Nowadays, semimartingales are one of the most important classes of stochastic processes. Not only do semimartingales contain processes like Lévy processes, Feller processes or martingales but they are widely used in various fields of research such as stochastic analysis, mathematical finance, or probability theory.

Since, historically, semimartingales emerged from the theory of Markov processes, it is of no surprise that numerous researchers transferred Markovian concepts to the theory of semimartingales. This includes the 'infinitesimal' behavior of Markov processes: In [74], Kolmogorov showed that continuous Markov processes depend essentially on the speed of the drift and on the size of the purely random part. Similarly, Lévy processes can be characterized by their characteristic exponent (cf. [61] Thm. 8.1 and Cor. 11.6), and, therefore, by the Lévy triplet (ℓ, Q, ν), representing a linear drift, a Brownian motion and a jump measure. Additionally, the infinitesimal generator $A : D(A) \to B_b(E)$ of the associated semigroup $(T_t)_{t\geq 0}$ defined by

$$Au := \lim_{t\downarrow 0} \frac{T_t u - u}{t},$$

for $u \in D(A)$, where

$$D(A) := \left\{ u \in B_b(\mathbb{R}^d) : \lim_{t\downarrow 0} \frac{T_t u - u}{t} \text{ exists w.r.t } \|\cdot\|_\infty \right\}$$

plays an important role to describe and even characterize the Markov process.

To establish similar tools in the framework of semimartigales, the characteristics of a semimartingale were introduced by Jacod and Mémin [35] in 1976. Generalizing the Lévy triplet, these characteristics describe the predictable 'drift' of the process through time, the 'volatility' of the continuous martingale part, and the 'rate' of jumps.

In 1998 Jacob (cf. [33]) inserted an exponential function into the generator of a Markov process in the sense of Blumenthal and Getoor and obtained

$$p(x, \xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x e^{i(X_t - x)'\xi} - 1}{t}, \quad x, \xi \in \mathbb{R}^d,$$

This expression is called the probabilistic symbol and allows for an analysis of the infinitesimal generator as well as various properties concerning the process. In [69], Schnurr was able to leave Markovianity behind and prove the existence of a generalized version of the probabilistic symbol for a subclass of semimartingales.

Keeping these considerations in mind, we begin this work by introducing Markov processes and discussing some closely related concepts like transition semigroups and the infinitesimal generator. Thereafter, two of the most fundamental classes of Markov processes, Lévy and Feller processes, are introduced. We conclude the first chapter by discussing non-homogeneous Markov processes and the associated space-time process.

In the second chapter, we take a short detour via the theory of local martingales, processes of finite variation and random measures to establish semimartingales, their characteristics and stochastic integration in Section 2.3 to 2.5. In addition to classical definitions and results, in these sections, we state some historical facts about semimartingales and their characteristics. Section 2.6 concludes this chapter, by introducing Markov semimartingales together with some important results on this topic, including auxiliary results preparing the theory developed in the following chapter.
In the third chapter, we formally introduce the probabilistic symbol of a Markov semimartingale as proposed by Schnurr in [67]. In contrast to the definition above, Schnurr included an exit time T to the definition of the symbol, i.e. considering

$$p(x,\xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x e^{i(X_t^T - x)'\xi} - 1}{t}$$

in order to deal with Markov semimartingales that are not bounded. For Itô processes, being Markov semimartingales with characteristics of the form

$$\begin{aligned} B_t(\omega) &= \int_0^t \ell(X_s(\omega))\, ds, \\ C_t(\omega) &= \int_0^t Q(X_s(\omega))\, ds, \\ \nu(\omega; ds, dw) &= N(X_s(\omega), dw)\, ds \end{aligned}$$

it is known that the probabilistic symbol is of the form

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y) \right) N(x, dy)$$

under some mild conditions to the characteristics.
Taking these considerations as starting point, the main objective of this chapter is the following: When considering Hunt semimartingales, we prove that the symbol exists if and only if the process under consideration is an Itô processes. In addition, we observe that when leaving Hunt semimartingales, in particular the quasi-left continuity, behind the symbol loses its applicability. In the second part of this chapter, the symbol is utilized to define the so-called Blumenthal-Getoor indices for Itô processes as was done by Schilling in [66]. These indices enable us to derive maximal-inequalities for the mentioned class of processes, which in turn provide a wide range of properties of the underlying stochastic process.
The fourth chapter, consisting of two sections, generalizes the probabilistic symbol to the non-homogeneous case by adding a time-component. One obtains

$$p(\tau, x, \xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^{\tau,x} e^{i\left(X_{\tau+t}^T - x\right)'\xi} - 1}{t}.$$

To this end, in the first section we develop a theory of non-homogeneous Markov semimartingales: We consider the time-dependent symbol of additive processes and rich Feller evolution processes, before we show the existence of the time-dependent symbol for non-homogeneous Itô processes.

In the second section, we use the theory developed so far to prove maximal-inequalities, i.e. inequalities for the probabilities

$$\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq\tau+t}\|X_s - x\| \geq R\right) \text{ and } \mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq\tau+t}\|X_s - x\| < R\right)$$

of a non-homogeneous Itô process X, $x \in \mathbb{R}^d, \tau \geq 0$ and $R > 0$. With the help of these inequalities, we generalize the Blumenthal-Getoor indices introduced in the previous chapter. These indices, as in the homogeneous case, enable us to prove a selection of properties of such processes like the asymptotic behavior of the sample paths.
Finally, inspired again by the theory of Markov processes we introduce a 'killing' to the semimartingale framework. For this class of processes this is technically more demanding than in the Markovian context and needs a separation into an 'explosion' and a 'sudden' killing. Subsequently, we generalize the theory of semimartingales by adding a new characteristic to the theory, which describes the sudden killing, and accordingly generalize some, in our point of view, of the most import results concerning semimartingales and their characteristics.
The second section treats the generalization of the probabilistic symbol in this context, and aims only to give a short introduction into this topic. The last section of this chapter provides a natural way in which the killing of semimartingales can occur. To this end, path-dependent killing by multiplicative functionals, known from the theory of Markov processes, is introduced into the semimartingale framework.
In the appendix, we state some basic notations and definitions from calculus and probability theory. In addition, we give some auxiliary results from advanced analysis on the real line.

Markov Processes

1

1.1. Probability Theory and Stochastic Processes

This section serves as an introduction of the notations and definitions from probability theory and the theory of stochastic processes essential for this thesis. We refer to [4] and [40] for a fundamental overview of measure and probability theory in combination with an introduction into the theory of general stochastic processes. For notations we refer to [36]. One can find basic notations from calculus and basic probability theory in Appendix A.1.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We refer to the measurable space $(E, \mathcal{E})$ as the *state space*, and unless otherwise specified, it is assumed to be $\mathbb{R}^d$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Let us note that we write E instead of $\mathbb{R}^d$ in contexts where the classical literature does not require $E = \mathbb{R}^d$. A mapping $X : \Omega \times \mathbb{R}_+ \to (E, \mathcal{E})$ is called a *stochastic process* if $X(\cdot, t)$ is a random variable for every $t \geq 0$. We denote it by $(X_t)_{t\geq 0}$ or simply X. For a stochastic process X, we define the mapping $t \mapsto X_t(\omega)$ as the *path* or *trajectory* of the process for fixed $\omega \in \Omega$. We call a process X *càdlàg* if its paths are right-continuous and have left-limits. A process X is said to be *left-continuous*, *increasing*, etc. if all paths of X possess the respective properties. For a càdlàg process X, we define the process $X_- = (X_{t-})_{t\geq 0}$ as follows: $X_{0-} := X_0$ and $X_{t-} := \lim_{s<t} X_s$ for $t > 0$. This process is called the *left-continuous version* of X. Additionally, we define $\Delta X := (\Delta X_t)_{t\geq 0}$, where $\Delta X_t := X_t - X_{t-}$ for $t \geq 0$, to represent the so-called *jump process* of X. Furthermore, for $t \geq 0$, we denote

$$X_t^* := \sup_{0\leq s\leq t} \|X_s\|$$

and refer to X^* as the *maximum process* of X.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a family of sub-σ-fields of $\mathcal{F}$ that is increasing, i.e., for $0 \leq s \leq t$, we have $\mathcal{F}_s \subset \mathcal{F}_t$. Such a family $(\mathcal{F}_t)_{t\geq 0}$ is called a *filtration*, and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration is referred to as a *stochastic basis*. A filtration is said to be *right-continuous* if

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s,$$

and *complete with respect to* $\mathbb{P}$ if $\mathcal{F}_0$ contains all subsets of $\mathbb{P}$-null sets. A stochastic process X is called *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if the random variable X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$. Moreover, the *natural filtration* of the process X is denoted by $(\mathcal{F}_t^X)_{t\geq 0}$, and the *natural σ-algebra* of the process X is denoted by $\mathcal{F}^X$, and defined as

$$\mathcal{F}_t^X := \sigma(X_s : s \leq t) \quad \text{and} \quad \mathcal{F}^X := \sigma(X_t : t \geq 0).$$

A family of σ-fields $(\mathcal{G}_t^\tau)_{0\leq\tau\leq t}$ is called a *two-parameter* or *double filtration* if $\mathcal{G}_s^\tau \subset \mathcal{G}_t^\tau$ for all $0 \leq \tau \leq s \leq t$ and $\mathcal{G}_t^{\tau_1} \subset \mathcal{G}_t^{\tau_2}$ for $0 \leq \tau_1 \leq \tau_2 \leq t$. The *natural double filtration* of X is denoted by $\big((\mathcal{F}^X)_t^\tau\big)_{0\leq\tau\leq t}$ and is defined as

$$\big(\mathcal{F}^X\big)_t^\tau := \sigma(X_s : \tau \leq s \leq t).$$

The σ-field referred to as the *optional σ-algebra*, and denoted by $\mathcal{O}$, is the σ-field on $\Omega \times \mathbb{R}_+$ which is generated by all càdlàg adapted processes. A stochastic process or a random set, i.e., a subset of $\Omega \times \mathbb{R}_+$, is *optional* if it is $\mathcal{O}$-measurable. The *predictable σ-field* $\mathcal{P}$ denotes the σ-field on $\Omega \times \mathbb{R}_+$ generated by all left-continuous adapted processes. A stochastic process or a random set is called *predictable* if it is $\mathcal{P}$-measurable.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic basis. A mapping $T : \Omega \to \overline{\mathbb{R}}_+$ is called a *stopping time* if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. For a stopping time T, the σ-algebra $\mathcal{F}_T$ is defined by

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Let X be an adapted, right-continuous stochastic process. For an open set $B \subset \mathbb{R}^d$, the stopping times

$$T := \inf\{t \geq 0 : X_t \in B\} \text{ and } S := \inf\{t \geq 0 : X_t \in B^c\}$$

are called the *hitting time of the set B* and the *first exit-time of X from B*, respectively. For two stopping times S, T, the random sets

- $[\![S, T]\!] := \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) \leq t \leq T(\omega)\}$,
- $[\![S, T[\![:= \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) \leq t < T(\omega)\}$,
- $]\!]S, T]\!] := \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) < t \leq T(\omega)\}$,
- $]\!]S, T[\![:= \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) < t < T(\omega)\}$

are called *stochastic intervals*, and instead of $[\![T, T]\!]$, we write $[\![T]\!]$.
We call the process X^T defined by

$$X_t^T := X_{t \wedge T} = X_t 1_{\{t \leq T\}} + X_T 1_{\{T > t\}}$$

for all $t \geq 0$ the *process stopped at time T* or simply the *stopped process*. An adapted càdlàg process X is called *quasi-left-continuous* if for any increasing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ with limit T, we have

$$\lim_{n \to \infty} X_{T_n} = X_T \text{ a.s. on } \{T < \infty\}.$$

A random set A is called *evanescent* if

$$\{\omega \in \Omega : \exists t \in \mathbb{R}_+ \text{ with } (\omega, t) \in A\}$$

is a $\mathbb{P}$-null set. Two processes X and Y are called *indistinguishable* if the random set

$$\{X \neq Y\} := \{(\omega, t) : X_t(\omega) \neq Y_t(\omega)\}$$

is evanescent, i.e. if almost all paths of X and Y are the same. We call X and Y *modifications* of each other, if for every $t \geq 0$

$$X_t = Y_t \quad \mathbb{P}\text{-a.s.}$$

holds. Indeed, for càdlàg processes if X and Y are modifications of each other the processes are indistinguishable and vice versa. From now on, unless mentioned otherwise, we write

$X = Y$ for two process X and Y if X and Y are indistinguishable.

A stochastic process $(X_t)_{t\geq 0}$ is called a *martingale* with respect to a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ if X_t is integrable for all $t \geq 0$, and

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s \quad \text{a.s.}$$

for all $0 \leq s \leq t$. With $\geq$ resp. $\leq$ in the previous equality we call the process *submartingale* resp. *supermartingale*. We denote by $\mathcal{M}$ the set of all *uniformly integrable martingales*. Moreoever, if $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, every martingale admits a càdlàg modification (see Corollary 5.1.9 [14]). For an introduction to the vast theory of martingales we refer to Chapters 9-11 of [40].

Let $\mathcal{C}$ be a class of processes. We define the *localized class $\mathcal{C}_{loc}$* as the set of all processes X for which there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times such that $\lim_{n\to\infty} T_n = \infty$ a.s. and $X^{T_n} \in \mathcal{C}$ for all $n \in \mathbb{N}$. We call such a sequence $(T_n)_{n\in\mathbb{N}}$ *localizing sequence* for X.

Example 1.1.1. We call a continuous adapted process B on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *Brownian motion* if the following properties hold:

(i.) $B_0 = 0$.

(ii.) $\mathbb{E}(B_t^2) < \infty$ and $\mathbb{E}(B_t) = 0$ for each $t \geq 0$.

(iii.) $B_t - B_s$ is independent of the σ-field $\mathcal{F}_s$ for all $0 \leq s \leq t$.

If B additionally fulfills $\mathbb{E}(X_t^2) = t$, we call the process B a *standard Brownian motion*.

Remark 1.1.2. In contrary to the classical literature, a Brownian motion, as defined in the previous example, is not time-homogeneous.

Example 1.1.3. (a.) Let $(\sigma_k)_{k\in\mathbb{N}}$ be a family of independent exponential distributed random variables with parameter $\lambda > 0$. We call the process $(N_t)_{t\geq 0}$ defined by

$$N_t = \sum_{k=1}^{\infty} 1_{[0,t]}(\tau_k),$$

where $\tau_k = \sigma_1 + ... + \sigma_k$ a *Poisson process in $\mathbb{R}$ with jump height 1 and intensity λ*.

(b.) Let $(N_t)_{t\geq 0}$ be a Poisson process in $\mathbb{R}$ with jump height 1 and intensity λ, and $(H_k)_{k\in\mathbb{N}}$ be a family of independent random variables with distribution μ and independent from $(N_t)_{t\geq 0}$. We call the process $(C_t)_{t\geq 0}$ defined by

$$C_t := \sum_{k=1}^{N_t} H_k$$

a *compound Poisson process with jump distribution μ and intensity λ*.

1.2. Markov Processes and Transition Functions

In this section, we introduce Markov processes and discuss related concepts. Although the following results and definitions are crucial for this thesis, they only provide a brief introduction to the extensive topic of Markov processes. Therefore, we recommend [8],[22] and [73] for a more throughout overview.
The literature covers various different concepts of Markov processes. In this thesis we mostly follow the approach presented in [8]:
Let us consider $(E, \mathcal{E})$ with an additional element ∂ not belonging to E. Let $E_\partial := E \cup \{\partial\}$ and $\mathcal{E}_\partial$ be the σ-field generated by $\mathcal{E}$ and $\{\partial\}$. We define a stochastic process $X = (X_t)_{t\geq 0}$ on the family of stochastic bases $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E_\partial}$ with state space $(E_\partial, \mathcal{E}_\partial)$, which satisfies the following conditions: if $X_s(\omega) = \partial$, then $X_t(\omega) = \partial$ for all $t \geq s$, and $X_\infty(\omega) = \partial$. Additionally, let Ω have a distinguished point ω_∂ such that $X_t(\omega_\partial) = \partial$ for all $t \geq 0$. For each $t \in \mathbb{R}_+$, we have the mapping $\theta_t \colon \Omega \to \Omega$.

Definition 1.2.1. We call a family $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E_\partial}$ a *(universal time-homogeneous) Markov process* if the following conditions hold:

(M_1) The mapping $x \mapsto \mathbb{P}^x(X_t \in B)$ is $\mathcal{E}$-measurable for all $t \geq 0$ and $B \in \mathcal{E}$.

(M_2) For all $t, h \in \mathbb{R}_+$ we have $X_t \circ \theta_h = X_{t+h}$.

(M_3)
$$\mathbb{P}^x[X_{t+s} \in B \mid M_t] = \mathbb{P}^{X_t}(X_s \in B) \tag{MP}$$
for all $x \in E_\partial, B \in \mathcal{E}_\partial$, and $s, t \geq 0$.

We call (MP) the *Markov property* and the filtration $(\mathcal{M}_t)_{t\geq 0}$ the *Markov filtration*. In addition, $(\theta_t)_{t\geq 0}$ is referred to as *(time-)shift-operator*.

We assume each Markov process to be *normal*, i.e. $\mathbb{P}^x(X_0 = x) = 1$ for all $x \in E_\partial$. Unless otherwise mentioned, we assume the σ-algebra $\mathcal{E}$ to be complete with respect to the family of measures $\{\mu; \mu$ is a finite measure on $(E, \mathcal{E})\}$. Furthermore, we can assume without loss of generality (cf. Chapter I.5 of [8]) that $\mathcal{M}$ is complete with respect to the family of measures $\{\mathbb{P}^x,\ x \in E_\partial\}$, and $\mathcal{M}_t$ is complete within $\mathcal{M}$ with respect to $\{\mathbb{P}^x,\ x \in E_\partial\}$.
When we consider the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$ and the σ-algebra $\mathcal{F}^X$, we assume them to be complete with respect to the family of measures $\{\mathbb{P}^\mu; \mu$ is a finite measure on $\mathcal{E}_\partial\}$, where $\mathbb{P}^\mu(\cdot) := \int \mathbb{P}^x(\cdot)\, d\mu(x)$ for a finite measure μ on $\mathcal{E}_\partial$.

Note that the Markov property (MP) holds true if and only if
$$\mathbb{E}^x\left[f(X_{t+s}) \mid \mathcal{M}_t\right] = \mathbb{E}^{X_t}(f(X_s)) \tag{MP1}$$
for all $x \in E_\partial, s, t \geq 0$ and all bounded $\mathcal{E}_\partial$-measurable functions f. In addition, we can state the Markov property using the time-shift operator as follows:
$$\mathbb{P}^x(X_s \circ \theta_t \in B \mid M_t) = \mathbb{P}^{X_t}(X_s \in B).$$
Indeed, (MP) holds if and only if
$$\mathbb{E}^x\left[Y \circ \theta_t \mid \mathcal{M}_t\right] = \mathbb{E}^{X_t}(Y) \tag{MP2}$$

for all $x \in E_\partial, t \geq 0$ and a bounded $\mathcal{F}$-measurable random variable Y (cf. Theorem I.3.6 of [8] for both equivalences stated above).

Definition 1.2.2. We call a family $(P_{s,t})_{t\geq s\geq 0}$ of functions mapping $E_\partial \times \mathcal{E}_\partial$ to $[0,1]$ *Markov transition function* if the following conditions hold:

(a.) $A \mapsto P_{s,t}(x, A)$ is a probability measure on $\mathcal{E}_\partial$ for all $s, t \geq 0$ and $x \in E_\partial$.

(b.) $x \mapsto P_{s,t}(x, A)$ is $\mathcal{E}_\partial$-measurable for all $s, t \geq 0$ and $A \in \mathcal{E}_\partial$.

(c.) The *Chapman-Kolmogorov equation* holds true, i.e.,

$$P_{t,u}(x, A) = \int P_{s,u}(y, A)\ P_{t,s}(x, dy)$$

for all $0 \leq t < s < u$, $x \in E$ and $A \in \mathcal{E}_\partial$.

If for a Markov transition function the equation

$$P_{s,t}(x, A) = P_{s+h,t+h}(x, A)$$

is fulfilled for all x, A and $0 \leq t < s$ and $h \geq 0$, we call it *homogeneous,* and write

$$P_t(x, A) := P_{t,0}(x, A).$$

Additionally, we call a homogeneous Markov transition function P *conservative,* if $P_t(x, E) = 1$ for all $t \geq 0$.

A Markov process X defines a homogeneous transition function N_t on $(E_\partial, \mathcal{E}_\partial)$ by

$$N_t(x, A) := \mathbb{P}^x(X_t \in A).$$

This leads to an interpretation of a transition function: It represents the probabilities of a particle starting at x at any given moment in time. Conversely, when starting with a homogeneous transition function $(P_t)_{t\geq 0}$, one can utilize Kolmogorov's extension theorem to construct a Markov process X on the path-space $E^{\mathbb{R}+}$ that satisfies the property (cf. Theorem 4.3 of [8])

$$\mathbb{P}^x(X_t \in A) = P_t(x, A).$$

Remark 1.2.3. By the definition of a Markov process, it is evident that $N_t(\partial, \partial) = 1$ for all $t \geq 0$. Therefore, the Markov transition function is completely determined by its restriction to $(E, \mathcal{E})$. We refer to a transition function on $(E, \mathcal{E})$ as *sub-Markovian* if $N_t(x, E) < 1$ for some $t \geq 0$ and $x \in E$. On the other hand, adding a point $\partial \notin E$ to E provides a simple way to extend a sub-Markovian transition function P to a Markovian one, denoted as $\tilde{P}$, on E_∂. To this end, one defines $\tilde{P}_t$ on $(E_\partial, \mathcal{E}_\partial)$ as follows:

$$\tilde{P}_t(x, A) := \begin{cases} P_t(x, A) & , x \in E, A \in \mathcal{E}_\partial \text{ and } A \subset E \\ 1 - P_t(x, E) & , x \in E, A = \{\partial\} \\ \delta_\partial(A) & , x = \partial. \end{cases}$$

Remark 1.2.4. For most Markov processes its shift-operator is given implicit. However, when considering a Markov process on the path-space $E^{\mathbb{R}_+}$, for example when obtained by a transition function, the meaning of the shift operator $(\theta_t)_{t\geq 0}$ is much more transparent: For the process X given by $X_t(\omega) = \omega(t)$ the time-shift operator is given by

$$\theta_s(\omega(\cdot)) := \omega(s + \cdot).$$

Aside from transition functions there are several other ways in which Markov processes occur naturally:

- Using stochastic differential equations: Under certain assumptions the solution of a stochastic differential equation is Markovian. Indeed, the original motivation of Itô for the development of stochastic integration was to examine Markov processes. We refer to Chapter V.6 of [54] for more insights.
- Via the martingale problem: Any solution $(X_t)_{t\geq 0}$ to the martingale problem is a Markov process (cf. [22] Lemma IV.4.2(i)).
- Using Dirichlet forms (cf. Chapter 3.3 of [11] or Chapter IV.2 of [47]).

Occasionally, it is necessary to impose a slightly stronger assumption on a Markov process. In many settings, we consider the Markov property not only for fixed times t but for stopping times T. Let us mention, that $\mathcal{E}_\partial^*$ denotes the σ-algebra of universally measurable sets over $(E_\partial, \mathcal{E}_\partial)$.

Definition 1.2.5. Let $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E_\partial}$ be a Markov process with state space $(E, \mathcal{E})$. We call X a *strong Markov process* if X_T is $\mathcal{M}_T$-$\mathcal{E}_\partial^*$-measurable and

$$\mathbb{E}^x \left[f(X_{t+T}) \mid M_T \right] = \mathbb{E}^{X_T}(f(X_t)) \tag{SMP}$$

for all $t \geq 0$, $x \in E$, each stopping time T and all bounded, measurable functions f.

For a strong Markov process the equation

$$\mathbb{E}^x \left[Y \circ \theta_T \mid M_T \right] = \mathbb{E}^{X_T}(Y)$$

holds true for all bounded $\mathcal{F}^X$-measurable random variables Y (cf. Theorem I.8.6 of [8]).

Definition 1.2.6. We call a strong Markov process X *Hunt process* if X is càdlàg and quasi-left-continuous.

1.3. Lévy Processes and Feller Processes

The present section introduces two of the most fundamental classes of Markov processes: Lévy and Feller processes. These classes play a crucial role for the understanding of many concepts introduced in the following, and are often utilized as standard examples. Of course the theorys of Lévy and Feller processes are vast fields of research in their own right. This section provides only a short insight into the respective fields.

1.3.1. Lévy Processes

So-called 'Lévy processes' are stochastic processes which increments are stationary and independent and are càdlàg. Lévy processes are one of the most fundamental class of processes. Not only do they contain famous examples like the Brownian motion, Poisson processes, and stable processes, but they represent a class of processes which exhibits many of the interesting phenomena that appear in the theory of stochastic processes like properties of their distributions or the behavior of the sample paths. Indeed, many important classes of processes were developed as generalization of Lévy processes. These include Markov processes and semimartingales.

The term 'Lévy process' honors the work of the french mathematician Lévy who played an important role in characterizing this particular class of processes. Some extraordinary works of the early time are Lévy [46], Kolmogorov [41], and Khintchine [39]. We note that the early literature used various different names when addressing this class of processes. Lévy himself referred to them as a sub-class of 'processus additifs' (cf. Definition 1.3.2). In the 1960s and 1970s most researchers used the term 'processes with stationary and independent increments' (cf. for example [36]). The term Lévy process became standard in the 1980s (cf. page 2 of [44]).

The main aim of this section is to briefly introduce the class of Lévy process together with some of the most import results regarding this topic. Since Lévy processes are well understood there are many textbook dealing with them. This section mostly follows [61], [5] and [42].

Definition 1.3.1. A stochastic process $(L_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}^L, (\mathcal{F}^L_t)_{t\geq 0}, \mathbb{P})$ with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called *Lévy processes* if the following conditions hold true

(1.) $L_0 = 0$ a.s..

(2.) For all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < ... < t_n$ the random variables

$$L_{t_0}, L_{t_1} - L_{t_0}, ..., L_{t_n} - L_{t_{n-1}}$$

are independent. We say that L has *independet increments*.

(3.) L has *stationary increments*, i.e., the distribution of $L_{s+t} - L_s$ does not depend on s, for $s, t \geq 0$.

(4.) L is càdlàg.

Definition 1.3.2. We call a process $(Z_t)_{t\geq 0}$ *additive* or *PII* if the conditions (1.),(2.) and (4.) of the previous definition hold.

Remark 1.3.3. Let us mention that (2.) is equivalent to the independence of $L_t - L_s$ and $\mathcal{F}^L_s$ for $s, t \geq 0$. In addition, the literature sometimes states the property

$$\lim_{t\to 0} \mathbb{P}(|L_t - L_0| > \varepsilon) = 0 \quad \forall \varepsilon > 0 \tag{4'.}$$

instead of (4.) in Definition 1.3.1. We call (4'.) *continuity in probability* and observe that (4.) $\Rightarrow$(4'.). On the other hand, one is able to prove that for every process satisfying (1.),(2.),(3.) and (4'.) there exists a version which is a Lévy process (cf. Theorem 11.1 of [61]).

In order to explore the markovian nature of a Lévy process L, we extend the underlying measurable space by setting

$$\tilde{\Omega} := \Omega \times \mathbb{R}^d \text{ and } \tilde{\mathcal{A}} := \mathcal{F}^L \otimes \mathcal{B}(\mathbb{R}^d).$$

On this space we consider the family of probability measures $(\mathbb{P}^x)_{x\in\mathbb{R}^d}$ defined by $\mathbb{P}^x := \delta_x \otimes \mathbb{P}$. The stochastic process $\tilde{L}_t(\omega, x) := x + L_t(\omega)$ possesses independent and stationary increments and is càdlàg. We refer to it as *Lévy process starting in x*. Moreover, one can observe that

$$\tilde{L} := \left(\tilde{\Omega}, \tilde{\mathcal{A}}, (\mathcal{F}^{\tilde{L}_t})_{t\geq 0}, (\tilde{L}_t)_{t\geq 0}, \mathbb{P}^x\right)_{x\in\mathbb{R}^d} \tag{1.1}$$

is a Markov process with transition function $P_t(x, B) := \mathbb{P}(x + L_t \in B)$ that is identical in law to L. For more details see Lemma 4.4 and Theorem 4.6 of [42] or Chapter 2.10 of [61].

The following theorem is one of the most important results concerning Lévy processes. It is due to Lévy and Khinchine and, therefore, commonly known as *Lévy-Khintchine formula*. The proof can be found in Chapter 2.8 of [61].

Theorem 1.3.4. *Let $(L_t)_{t\geq 0}$ be a Lévy processes. There exists a unique triplet (ℓ, Q, ν), where $\ell \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ a positive semi-definite and symmetric matrix, and ν a measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ with*

$$\int_{\mathbb{R}^d\setminus\{0\}} (|y|^2 \wedge 1)\ \nu(dy) < \infty$$

such that

$$\mathbb{E}\left(e^{iL_t'\xi}\right) = e^{-t\psi(\xi)} \tag{1.2}$$

for all $\xi \in \mathbb{R}^d, t \geq 0$ and

$$\psi(\xi) := -i\ell'\xi + \frac{1}{2}\xi' Q\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy'\xi} + iy'\xi\chi(y))\ \nu(dy). \tag{1.3}$$

In (1.3) χ is a cut-off function, i.e., $\chi : \mathbb{R}^d \to \mathbb{R}$ measurable and $1_{B_r(0)} \leq \chi \leq 1_{B_{2r}(0)}$ for some $r > 0$.
Conversely, we are able to define a Lévy process $(L_t)_{t\geq 0}$ for any function ψ defined in (1.3) via (1.2).

Definition 1.3.5. For a Lévy process $(L_t)_{t\geq 0}$ we call the function ψ as defined in (1.3) the *characteristic exponent* of L and the triplet (ℓ, Q, ν) the *Lévy triplet*.

Example 1.3.6. (a.) The *deterministic drift* $X_t \equiv t\ell$, for $\ell \in \mathbb{R}^d$ is a Lévy process with characteristic exponent $\psi(\xi) = -i\ell'\xi$.

(b.) The Brownian motion with covariance matrix $Q \in \mathbb{R}^{d\times d}$ is a Lévy process with characteristic exponent $\psi(\xi) = \frac{1}{2}\xi' Q\xi$.

(c.) The compound Poisson process with jump distribution μ and intensity λ possess the characteristic exponent

$$\psi(\xi) = \lambda \int_{\{y\neq 0\}} 1 - e^{iy'\xi}\ \mu(dy).$$

Remark 1.3.7. A Lévy process L as defined above is conservative. In terms of the characteristic exponent that is equivalent to $\psi(0) = 0$ because

$$\mathbb{P}(X_t \in E) = \mathbb{E}\left(e^{iL_t'0}\right) = e^{-t\psi(0)}.$$

However, if a Lévy process L is not required to be conservative, as done for Markov processes, the characteristic exponent in (1.3) is of the form

$$\Psi(\xi) = a - i\ell'\xi + \frac{1}{2}\xi' Q\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy'\xi} + iy'\xi\chi(y))\ \nu(dy)$$

where (ℓ, Q, ν) is a Lévy triplet as above and $a \geq 0$. On the other hand, every function Ψ defines a Lévy process L which is not conservative if $a > 0$. We see that

$$\mathbb{P}(X_t \notin E) = \mathbb{E}\left(e^{iL_t'0}\right) = e^{-t\psi(0)} = e^{-ta}.$$

We derive that the time when L leaves the state space is exponentially distributed with parameter a. Hence, we call L *Lévy process with exponential killing.*

The last result we want to mention concerns the paths of a Lévy process, and is known as the *Lévy-Itô decomposition*.

Theorem 1.3.8. *Let $(L_t)_{t\geq 0}$ be a Lévy process with Lévy triplet (ℓ, Q, ν). Then there exists a standard Brownian motion $(B_t)_{t\geq 0}$ such that*

$$L_t = \ell t + \sqrt{Q}B_t + \int_0^t \int_{\{0<|y|<1\}} y\ (N(dy, ds) - \nu(dy)ds) + \int_0^t \int_{\{|y|\geq 1\}} y\ N(dy, ds) \quad (1.4)$$

for all $t \geq 0$, where there measure N on $\mathbb{R}_+ \times \mathbb{R}^d$ is defined by

$$N((0,t] \times B) := N_\omega((0,t] \times B) = \#\{s \in (0,t] : L_s(\omega) - L_{s-}(\omega) \in B\}.$$

Proof. See Theorem 9.9 of [42]. □

In decomposition (1.4), the term

$$\sqrt{Q}B_t + \int_0^t \int_{\{0<|y|<1\}} y\ (N(dy, ds) - \nu(dy)ds)$$

is an L^2-martingale, ℓt is the drift part and $\int_0^t \int_{\{|y|\geq 1\}} y\ N(dy, ds)$ is the pure-jump part.

1.3.2. Operator Semigroups and Feller Processes

Let us consider a conservative Markov process X with transition function P. We are able to define a family of operators $(T_t)_{t\geq 0}$ on the Banach space of bounded, $\mathcal{E}$-measurable functions $(B_b(E), \|\cdot\|_\infty)$ by

$$T_t u(x) := \int u(y)\ P_t(x, dy) = \mathbb{E}^x(u(X_t)) \tag{1.5}$$

for $u \in B_b(E)$. The Markov property provides that $(T_t)_{t\geq 0}$ induces a semigroup. In literature, $(T_t)_{t\geq 0}$ is often used to derive properties of the corresponding process.

Definition 1.3.9. (a.) A family $(P_t)_{t\geq 0}$ of linear operators on the bounded Borel-measurabe functions is called *(one-parameter operator) semigroup* if

$$P_0 = id \text{ and } P_t \circ P_s = P_{t+s}$$

for all $s, t \geq 0$.

(b.) We call a semigroup *sub-Markov semigroup* if it is *positivity preserving*, i.e., $P_t u \geq 0$ for all $0 \leq u \in B_b(E)$, and *sub-markovian*, i.e., $P_t u \leq 1$ for all $u \in B_b(E)$ with $u \leq 1$.

(c.) We call a sub-Markov semigroup *Markov semigroup* if it is *conservative*, i.e., $P_t 1 = 1$.

(d.) We call a semigroup *strongly continuous* if

$$\lim_{t\to 0} \|P_t f - f\|_\infty = 0$$

for all $f \in C_\infty(E)$, i.e. for all bounded, continuous functions vanishing at infinity.

(e.) We call a strongly continuous Markov semigroup *Feller semigroup* if $P_t : C_\infty(E) \to C_\infty(E)$, and *strong Feller* semigroup if $P_t : B_b(E) \to C_\infty(E)$.

For the general theory of one parameter semigroups we refer to [16] or [34].
The semigroup $(T_t)_{t\geq 0}$ as defined in (1.5) is a Markov semigroup for which it holds true that

$$\|T_t u\|_\infty \leq \|u\|_\infty.$$

We call such a semigroup a *contraction Markov semigroup.*
If the process X is a Lévy-process, $(T_t)_{t\geq 0}$ is a Feller semigroup, and (1.5) equals

$$T_t u(x) := \int u(x+y)\ \mathbb{P}_{X_t}(dy)$$

Moreover, if the transition probabilities $\mathbb{P}(X_t \in dy)$ are absolutely continuous with respect to the Lebesgue measure, $(T_t)_{t\geq 0}$ is a strong Feller semigroup (cf. Example 1.3 of [11]).
On the other hand, for each Feller semigroup $(P_t)_{t\geq 0}$ the Riesz representation theorem provides the existence of a Markov transition function $N_t(x, dy)$, such that

$$P_t f(x) = \int f(y)\ N_t(x, dy).$$

For proof see Lemma 5.2 of [42]. Now, it is natural to consider stochastic processes whose semigroups are a Feller semigroups:

Definition 1.3.10. We call a càdlàg Markov process $(X_t)_{t\geq 0}$ *Feller process* if the corresponding semigroup $(T_t)_{t\geq 0}$ is a Feller semigroup. If additionally the semigroup is strong Feller, we call the process *strong Feller process*.

The classical literature does not demand a Feller process to be càdlàg. However, this is no restriction because due to a fundamental theorem (cf. Theorem 1.19 of [11] or Chapter II.2 of [56]) every Feller process has a càdlàg modification which is also a Feller process.

A central notion in the theory of one parameter semigroups of operators is that of the so-called generator. That is because the generator poses a way to construct a strongly continuous semigroup and vice verse. Hence, the analysis of the generators provides insights into the behavior of semigroups, and, therefore, their associated Markov processes. For a short introduction see Chapter I.1 of [47].

Definition 1.3.11. Let X be a Markov process with semigroup $(T_t)_{t\geq 0}$. The *(infinitisimal) generator* $A : D(A) \to B_b(E)$ of the semigroup is the linear operator defined by

$$Au := \lim_{t\downarrow 0} \frac{T_t u - u}{t}$$

for $u \in D(A)$, where

$$D(A) := \left\{ u \in B_b(E) : \lim_{t\downarrow 0} \frac{T_t u - u}{t} \text{ exists w.r.t } \|\cdot\|_\infty \right\}.$$

We call $(A, D(A))$ a *Feller generator* if X is a Feller process.

Definition 1.3.12. We call a Feller process X *rich* if the domain of its generator contains the test functions $C_c^\infty(E)$, i.e., the infinitely-often differentiable functions with compact support on E.

Example 1.3.13. (a.) The generator of the *deterministic drift* $X_t \equiv t\ell$ is given by $Au(x) = \ell'\nabla u(x)$ where $\ell \in \mathbb{R}^d$ and $C_\infty^k(\mathbb{R}^d) \subset D(A)$ for $k = 1$.

(b.) The Brownian motion with covariance matrix $Q \in \mathbb{R}^{d\times d}$ has the generator

$$Au(x) = \frac{1}{2}\mathrm{tr}(Q(x)\nabla^2 u(x))$$

and $C_\infty^2(\mathbb{R}^d) \subset D(A)$.

(c.) The generator of a Lévy process with characteristic exponent ψ and triplet (ℓ, Q, ν) is given by

$$\begin{aligned} Au(x) = &-\ell'\nabla u(x) + \frac{1}{2}\sum_{j,k=1}^d q^{(jk)}\partial_{jk}u(x) \\ &+ \int_{\mathbb{R}^d\setminus\{0\}} u(x+y) - u(x) - \nabla u(x)'y\chi(|y|)\,\nu(dy) \end{aligned} \tag{1.6}$$

If the domain of the generator is sufficiently rich, i.e. $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$, a classical result of Courrège and van Waldenfels (cf. Theorem 2.21 of [11]) shows that the generator is a so-called *pseudo-differential operator*, when restricted to $C_c^\infty(\mathbb{R}^d)$. That is, we can represent A as follows:

$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} q(x,y)\hat{u}(\xi)\, d\xi \tag{1.7}$$

for all $u \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. The function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is given by

$$q(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy'\xi} + i\xi' y\chi(|y|)\, \nu(x,dy) \tag{1.8}$$

where $(\ell(x), Q(x), v(x,dy))$ is a Lévy-triplet for fixed $x \in \mathbb{R}^d$ and χ is a cut-off function such that $0 \le 1 - \chi(s) \le \kappa(s \wedge 1)$ for some $\kappa > 0$ and $s\chi(s)$ stays bounded. We call the function $q(x,\xi)$ the *symbol of the operator*. Moreover, q is locally bounded and for fixed x a continuous negative definite function in the sense of Schoenberg. This is equivalent to saying that q admits a Lévy-Khintchine representation. For the theory of negative definite functions see Chapter 2.2 of [11].
Furthermore, calculating the expression in (1.7) with the representation of q as in (1.8) we conclude that

$$\begin{aligned} Au(x) &= \ell(x)'\nabla u(x) + \frac{1}{2}\sum_{j,k=1}^{d} q^{(jk)}(x)\partial_{jk}u(x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} u(x+y) - u(x) - \nabla u(x)' y\chi(|y|)\, \nu(x,dy). \end{aligned} \tag{1.9}$$

In the framework of rich Feller processes, the growth and sector conditions (G) and (S) play an important role: The growth condition (G) is fulfilled if there exists a $c > 0$ such that

$$\sup_{x\in\mathbb{R}^d} |q(x,\xi)| \le c(1 + \|\xi\|^2) \tag{G}$$

for every $\xi \in \mathbb{R}^d$. The sector condition (S) is fulfilled if there exists a $c_0 > 0$ such that for every $x, \xi \in \mathbb{R}^d$

$$|\mathrm{Im}(q(x,\xi))| \le c_0 \,\mathrm{Re}(p(x,\xi)), \tag{S}$$

where Re resp. Im denote the real resp. the imaginary part.

1.4. Non-homogeneous Markov Processes and the Space-time Process

This section is devoted to non-homogeneous Markov processes. For a Markov process as defined above we have seen a one to one correspondence between time-homogeneous transition function and Markov process. However, since the time-homogeneous case is only a special case of a transition function, we want to introduce non-homogeneous Markov processes. For an overview over the theory of non-homogeneous Markov processes consider

[76] and [26]. Let us mention that whenever non-homogeneous processes are involved $\tau \geq 0$ never denotes a stopping time but rather the starting time of the process.

Definition 1.4.1. Let $(\Omega, \mathcal{M})$ be a measurable space equipped with the two-parameter filtration $(\mathcal{M}_t^\tau)_{0 \leq \tau \leq t}$. We call a stochastic process X *adapted to the two-parameter filtration* if for all $0 \leq \tau \leq t$

$$\left(\mathcal{F}^X\right)_t^\tau \subset \mathcal{M}_t^\tau.$$

For the following definition we assume $\mathbb{P}^{\tau,x}$ to be a probability measure on $(\Omega, \vee_{t \geq \tau} \mathcal{M}_t^\tau)$, where $\vee_{t \geq \tau} \mathcal{M}_t^\tau := \sigma\left(\cup_{t \geq \tau} \mathcal{M}_t^\tau\right)$.

Definition 1.4.2. A stochastic process $(\Omega, \mathcal{M}, (\mathcal{M}_t^\tau)_{0 \leq \tau \leq t}, (X_t)_{t \geq 0}, \mathbb{P}^{\tau,x})_{\tau \geq 0, x \in E}$ is called a *non-homogeneous Markov process* if

$$\mathbb{E}^{\tau,x}\left[f(X_t) \mid \mathcal{M}_s^\tau\right] = \mathbb{E}^{s,X_s}\left(f(X_t)\right) \quad \mathbb{P}^{\tau,x}\text{-a.s.} \tag{1.10}$$

for all $\tau \leq s \leq t, x \in E$ and all bounded Borel-measurable functions f. As in the homogeneous case, we call the measurable function $\theta_s : \Omega \to \Omega$ for $s \geq 0$ fulfilling

$$X_t \circ \theta_s = X_{t+s}, \quad t \geq 0$$

the *time-shift operator*.

Remark 1.4.3. When considering a non-homogeneous Markov process on the stochastic basis

$$(\Omega, \vee_{\tau \geq t} \mathcal{M}_t^\tau, (\mathcal{M}_t^\tau)_{\tau \leq t}, \mathbb{P}^{\tau,x})$$

for some $\tau \geq 0$ and $x \in E$, we see that X_t only defines a measurable function for $t \geq \tau$. Hence, when considering the process with respect to the measure $\mathbb{P}^{\tau,x}$ the process is properly defined by $X = (X_t)_{t \geq \tau}$.

As in the homogeneous case $P(\tau, x, t, A) := \mathbb{P}^{\tau,x}(X_t \in A)$ defines a transition function. Vice versa, Kolmogorov's extension theorem provides for every (non-homogeneous) transition function a non-homogeneous Markov process on the path space by

$$\mathbb{P}^{\tau,x}(X_t \in A) := P(\tau, x, t, A)$$

for $0 \leq \tau \leq t \leq T$ and $A \in \mathcal{E}$. For a throughout proof see Section 1.4 of [26].

Although, the previous definition of a non-homogeneous Markov process is clearly more general than that of a homogeneous Markov process, the classical literature (cf. [8] and [22]) deals with the homogeneous case. This is partly due to the fact that it is possible to transform any non-homogeneous Markov process into a homogeneous one, and reducing, in most scenarios, the non-homogeneous case to a homogeneous one. For this purpose, we add a time component to the non-homogeneous Markov process in an additional dimension. Thus, artificially homogenizing it. Subsequently, we want to carry out the mentioned construction following [10]:
Let $\hat{\Omega} := \mathbb{R}_+ \times \Omega$ and the σ-field $\hat{\mathcal{M}} := \{B \subset \hat{\Omega} : B_s \in \mathcal{M} \ \forall s \in \mathbb{R}_+\}$ where B_s denotes

the s-slice of B for $s \geq 0$, i.e. $B_s := \{\omega \in \Omega : (s, \omega) \in B\}$. We define a process $\hat{X}$ on this measurable space with values in $\mathbb{R}_+ \times E$ with

$$\hat{X}_t(\hat{\omega}) = \hat{X}_t((c, \omega)) := (t + c, X_{t+c}(\omega)).$$

Moreover, we set

$$\hat{\theta}_\tau : \hat{\Omega} \to \hat{\Omega}; (s, \omega) \mapsto (s + \tau, \omega)$$

and

$$\mathbb{P}^{(\tau,x)}(B) := \mathbb{P}^{\tau,x}\left(\pi_0^{-1}\left(\hat{\theta}_\tau^{-1}(\{\tau\} \times B_\tau)\right)\right) \tag{1.11}$$

where $\pi_0 : \Omega \to \hat{\Omega}; \omega \mapsto (0, \omega)$ and $B \in \hat{\mathcal{M}}$. We call the homogeneous Markov process

$$\hat{X} := \left(\hat{\Omega}, \hat{\mathcal{M}}, (\mathcal{F}^{\hat{X}})_{t\geq 0}, (\hat{X}_t)_{t\geq 0}, (\hat{\theta}_t)_{t\geq 0}, \hat{\mathbb{P}}^{(\tau,x)}\right)_{(\tau,x)\in\mathbb{R}_+\times E}$$

the *space-time process associated with* X. The transition probability function of X is given by

$$\hat{P}(t, (\tau, x), A) := P(\tau, x, t + \tau, A_{t+\tau}), \tag{1.12}$$

and for any $\hat{\mathcal{M}}$-measurable random variable Y it holds true that

$$Y = Y \circ \hat{\theta}_\tau \circ \pi_0, \quad \hat{\mathbb{P}}^{(\tau,x)}\text{-a.s..}$$

That is, for $\hat{\mathbb{P}}^{(\tau,x)}$-almost all $(c, \omega) \in \hat{\Omega}$:

$$Y(c, \omega) = Y(\tau, \omega). \tag{1.13}$$

This will be used frequently when calculating with the space-time process (cf. Chapter 4).

The construction of a space-time process can be found in various classical textbooks (see Section II.16 of [73] or Chapter 4.6 of [21]). If we look at the previous construction more closely, we see that it is sufficient to specify the transition function as in (1.12) in order to define the space-time process. From a constructional point of view, this would lead to defining the space-time process on the path-space, that is $(\mathbb{R}_+ \times E)^{\mathbb{R}_+}$, with the help of Komogorov's extension theorem. In this case the space-time process would be defined by $\hat{X}_t(\phi, \omega) = (\phi(t), \omega(t))$ for $(\phi, \omega) \in (\mathbb{R}_+ \times E)^{\mathbb{R}_+}$. This includes the possibility of the first component $t \mapsto \phi(t)$ to be non-measurable, which makes the defined space-time process practically useless.
Regarding this topic, we refer the reader to Section 1.5 of [26], where one can find detailed comments concerning the space-time process. In contrast to the construction above, in [26] the space-time process is defined by

$$\tilde{X}_t(\hat{\omega}) = \tilde{X}_t(s, \omega) := (s + t, X_s(\omega))$$

possessing the shift-operator $\tilde{\theta}_t : \hat{\Omega} \to \hat{\Omega}; (s, \omega) \mapsto (s, \theta_t(\omega))$.
Although this definition of a space-time process seems more intuitive than the one we presented above, it leads to various problems when taking the semimartingale property into account as we do in Chapter 4.

2 Semimartingales

The aim of this section is threefold. First, we aim to introduce the theory of semimartingales and their characteristics, along with the closely related fields of local martingales, processes of finite variation, random measures, and stochastic integration. We will present some of the most important definitions and results of the respective areas. Semimartingales and their characteristics are essential for the understanding of the remainder of this thesis. Of course, it is not possible to state the entire theory of semimartingales. Instead, we refer to the textbooks [36], [14], and [54]. Throughout, we mainly follow [36] and use the notations therein, which are standard in most works on semimartingales. When considering so-called 'Markov semimartingales', we refer to the seminal paper [12]. The second aim of this section is to make some historical comments concerning semimartingales and their characteristics. Finally, we want to provide some technical results concerning 'Markov semimartingales' that will be utilized later in this thesis.
In this chapter, the historical comments concerning semimartingales and their characteristics are taken from [72] where one finds a more throughout treatment.

As previously, let $(X_t)_{t\geq 0}$ be a stochastic process on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

2.1. Local Martingales and Processes of Total Variation

For the understanding of semimartingales, it is essential to closely examine the classes of processes considered in this subsection: local martingales and processes of finite variation. Both classes possess a well-developed theory associated with them. Local martingales are strongly intertwined with martingales, and we recommend Section II.4b of [36] and Section 5.6 of [14] for a detailed overview. Let us mention that Itô and Watanabe (cf. [29]) introduced local martingales in 1965 while researching additive functionals of Markov processes. On the other hand, processes of finite variation, i.e. processes where almost all paths are of finite variation on finite intervals, allow a relatively straightforward definition of a pathwise stochastic integral. That is, because the path do not 'vary' to much, allowing a usage of the classical Lebesgue-Stieltjes integral (cf. page 131-136 of [38]).

Definition 2.1.1. (a.) We denote by $\mathcal{M}_{loc}$ the localized class of $\mathcal{M}$, i.e. of the uniformly integrable martingales. We call a stochastic process $M \in \mathcal{M}_{loc}$ a *local martingale*. Additionally, we denote by $\mathcal{L}$ the set of local martingales starting in zero.

(b.) Let $\mathcal{H}^2$ be the set of all *square-integrable martingales* M, i.e. M is a martingale with

$$\sup_{s\geq 0} \mathbb{E}(M_s^2) < \infty.$$

We denote by $\mathcal{H}^2_{loc}$ the corresponding localized class.

Remark 2.1.2. The optional stopping theorem provides that the spaces $\mathcal{M}$, $\mathcal{M}_{loc}$ and $\mathcal{H}^2$, $\mathcal{H}^2_{loc}$ are stable under stopping, and, in addition, $\mathcal{M} \subsetneq \mathcal{M}_{loc}$ (cf. Section 6 of [54]). One is able to drop the uniform integrability in the previous definition because one can always replace the original localization sequence (T_n) by $(T_n \wedge n)$ to obtain a bounded martingale which, therefore, is uniformly integrable.

The following theorem provides a decomposition of a local martingale. For proof, see Theorem I.4.18 of [36].

Theorem 2.1.3. *Let M be a local martingale. Then M possesses a unique decomposition of the form*

$$M = M_0 + M^c + M^d,$$

where $M_0^d = M_0^c = 0$, M^c is a continuous local martingale, and M^d is a purely discontinuous local martingale, i.e. $M_0^d = 0$ and for all continuous local martingales N the product $M^d N$ is a local martingale.

Definition 2.1.4. A stochastic process $(A_t)_{t\geq 0}$ with values in $\mathbb{R}$ is called a process of *finite variation* on the interval $[0,t]$ for $t \geq 0$, if

$$\sup \left(\sum_{i\in\mathbb{N}} |A_{t_{i+1}} - A_{t_i}| \right) < \infty \quad \text{a.s.},$$

where the supremum is taken over all increasing sequences $(t_i)_{i\in\mathbb{N}}$ in $[0,t]$.

We want to mention that the previous definition of the finite variation of a process is pathwise, and, therefore, not affected by a change of measure.

Definition 2.1.5. (a.) Let $\mathcal{V}$ denote the set of all real, adapted, càdlàg processes A with $A_0 = 0$ and finite variation on the intervals $[0,t]$ with $t \in \mathbb{R}_+$.

(b.) Let $\mathcal{V}^+$ denote the set of all real, adapted, càdlàg and increasing processes A with $A_0 = 0$.

After introducing the classes $\mathcal{V}$ and $\mathcal{V}^+$, we want to consider them more closely, and in particular examine their relation. Therefore, we define a new kind of process in the next definition.

Definition 2.1.6. For a process $A \in \mathcal{V}$ we define the *variation process* $\mathrm{Var}(A)$ of A by

$$\mathrm{Var}(A)_t(\omega) := \lim_{n\to\infty} \sum_{k=1}^{n} \left| A_{t\frac{k}{n}}(\omega) - A_{t\frac{k-1}{n}}(\omega) \right|$$

for every $t \geq 0$ and $\omega \in \Omega$.

For proof of the following proposition, see Proposition 3.3 of [36].

Proposition 2.1.7. *Let $A \in \mathcal{V}$. Then there exists a unique pair (B, C) of adapted, increasing processes such that $A = B - C$ and $\text{Var}(A) = B + C$. In particular, if A is predictable then $B,\ C$ and $\text{Var}(A)$ are predictable.*

Definition 2.1.8. (a.) Let $\mathcal{A}^+$ be the set of all integrable processes $A \in \mathcal{V}^+$, i.e. $\mathbb{E}(A_t) < \infty$ for all $t \geq 0$.

(b.) Let $\mathcal{A}$ be the set of all $A \in \mathcal{V}$ with integrable variation, i.e. $\mathbb{E}(\text{Var}(A)_\infty) < \infty$.

(c.) Let $\mathcal{A}^+_{loc}$ and $\mathcal{A}_{loc}$ be the localized classes of $\mathcal{A}^+$ and $\mathcal{A}$. We call a process in $\mathcal{A}^+_{loc}$ *locally integrable adapted increasing process* and a process in $\mathcal{A}_{loc}$ *adapted process with locally integrable variation*.

The following inclusions hold:

$$\mathcal{A}^+ \subset \mathcal{A}^+_{loc} \subset \mathcal{V}^+ \text{ and } \mathcal{A} \subset \mathcal{A}_{loc} \subset \mathcal{V}.$$

The localizing procedure does not extend $\mathcal{V}^+$ and $\mathcal{V}$. That is, $\mathcal{V}_{loc} = \mathcal{V}$ and $\mathcal{V}^+_{loc} = \mathcal{V}^+$. Moreover, $\mathcal{V},\ \mathcal{V}^+,\ \mathcal{A}$ and $\mathcal{A}^+$ are stable under stopping.

Remark 2.1.9. As we mentioned earlier, processes of finite variation are particularly interesting because they provide an intuitive way to define a stochastic integral. We only want to point out the rough idea of this integration since we are going to introduce a more general approach in Section 2.4. Consider an increasing stochastic process $A \in \mathcal{V}^+$. The idea is to fix ω and to consider the paths of A, namely $t \mapsto A_t(\omega)$. This function induces a family of measure $dA_s(\omega)$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ by defining

$$dA_\cdot(\omega)((s,t]) := A_t(\omega) - A_s(\omega)$$

for $0 \leq s < t$ and almost all $\omega \in \Omega$. Thus, for a jointly measurable function $H : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ we are able to define a stochastic process by the following pathwise definition

$$(H \bullet A)_t(\omega) := \int_0^t H(\omega, s)\ dA_s(\omega) := \int_0^t H(\omega, s)\ dA_\cdot(\omega).$$

In order to extent the class of possible integrators, we recall that every finite-variation process $A \in \mathcal{V}$ can be represented as the difference of two increasing processes $B, C \in \mathcal{V}^+$. Thus, we proceed analogously for $A \in \mathcal{V}$ and define

$$\int_0^t H(\omega, s)\ dA_s(\omega) := \int_0^t H(\omega, s)\ dB_s(\omega) - \int_0^t H(\omega, s)\ dC_s(\omega).$$

For further reference, see Section 8.1 of [14] or Section 7 of [54].

We conclude this subsection by introducing the notion of the compensator of a process with locally integrable variation and the quadratic covariation of two local martingales.

Theorem 2.1.10. *Let $A \in \mathcal{A}_{loc}$. Then there exists a unique predictable process $A^p \in \mathcal{A}_{loc}$ such that $A - A^p$ is a local martingale.*

Proof. See Theorem I.3.18 of [36]. □

Definition 2.1.11. For a process $A \in \mathcal{A}_{loc}$ we call the unique predictable process $A^p \in \mathcal{A}_{loc}$ in the previous theorem the *(predictable) compensator* of A.

We now introduce the quadratic covariation of two local martingales.

Theorem 2.1.12. *Let $M, N \in \mathcal{H}^2_{loc}$. Then there exists a unique predictable process $\langle M, N\rangle \in \mathcal{V}$ such that $MN - \langle M, N\rangle \in \mathcal{M}_{loc}$.*
Moreover,

$$\langle M, N\rangle = \frac{1}{4}(\langle M+N, M+N\rangle - \langle M-N, M-N\rangle) \tag{2.1}$$

holds true.
If $M, N \in \mathcal{H}^2$, then $\langle M, N\rangle \in \mathcal{A}$ and $MN - \langle M, N\rangle \in \mathcal{M}$. The process $\langle M, M\rangle$ is increasing.

Proof. See Theorem I.4.2 of [36] □

Definition 2.1.13. We call the process $\langle M, N\rangle$ defined in the previous theorem the *predictable quadratic covariation or angle bracket* of the pair (M, N).

It is easy to see that $\mathcal{H}^2$ is a Hilbert space with the scalar product

$$(M, N)_{\mathcal{H}^2} := \mathbb{E}(M_\infty N_\infty),$$

where M_∞ and N_∞ are the unique terminal variables of $M \in \mathcal{H}^2$ and $N \in \mathcal{H}^2$. In particular, the norm of a process $M \in \mathcal{H}^2$ is given by $\|M\|_{\mathcal{H}^2} := \|M_\infty\|_{\mathbf{L}^2}$.
Intuitively, the quadratic covariation $\langle M, N\rangle$ locally behaves similar to an inner product on $\mathcal{H}^2$.

Lemma 2.1.14. *Let $M, N \in \mathcal{H}^2$. Then the following properties hold:*

(i.) *The mapping $\langle \cdot, \cdot\rangle$ is bilinear and symmetric.*

(ii.) *$\langle M, N\rangle = 0$ if M or N is of finite variation and one of them is continuous.*

(iii.) *Let M^c be the continuous part of M as defined in Theorem 2.1.3, then*

$$\langle M^c, M^c\rangle = \langle M, M\rangle^c.$$

The proofs of the previous properties can be found in Section 11.2 of [14].

2.2. Random Measures

Let us mention again that $E = \mathbb{R}^d$ and $\mathcal{E} = \mathcal{B}(\mathbb{R}^d)$ in this work. Indeed, for the theory of random measures as presented in the following it is sufficient for $(E, \mathcal{E})$ to be a Blackwell space (cf. Definition III.24 of [18]).

Definition 2.2.1. A *random measure* on $\mathbb{R}_+ \times E$ is a family $\mu = (\mu(\omega; dt, dx))_{\omega\in\Omega}$ of measures on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E})$ such that $\mu(\omega; \{0\} \times E) = 0$ for all $\omega \in \Omega$.

Now, since we are dealing with $\Omega \times \mathbb{R}_+ \times E$ in the context of random measures, let us introduce the following notation:

- We consider the sample space $\Omega \times \mathbb{R}_+ \times E$ together with the σ-fields $\tilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{E}$ and $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{E}$.
- A function W on $\Omega \times \mathbb{R}_+ \times E$ that is $\tilde{\mathcal{O}}$-measurable (resp. $\tilde{\mathcal{P}}$-measurable) is called an *optional* (resp. *predictable*) function.
- Let μ be a random measure and W an optional function on $\Omega \times \mathbb{R}_+ \times E$. Then the *integral process* $W * \mu$ defined by

  $$W * \mu_t := \begin{cases} \int_{[0,t]\times E} W(\cdot, s, x)\mu(\cdot; ds, dx), & \text{if } \int_{[0,t]\times E} |W(\cdot, s, x)|\mu(\cdot; ds, dx) < \infty \\ \infty, & \text{else} \end{cases}$$

  for $t \geq 0$.
- A random measure μ is called *optional* (resp. *predictable*) if the process $W * \mu$ is optional (resp. predictable) for every optional (resp. predictable) function W.
- An optional random measure μ is called *$\tilde{\mathcal{P}}$-σ-finite* if there exists a strictly positive, predictable function V on $\tilde{\Omega}$ such that the random variable $V * \mu_\infty$ is integrable.

Using the foregoing definitions we generalize the theory of the compensator in the sense of random measures. This generalization will play an important role for the definition of the characteristics of a semimartingale.

Theorem 2.2.2. *Let μ be an optional $\tilde{\mathcal{P}}$-σ-finite random measure. Then there exists a $\mathbb{P}$-unique, predictable random measure μ^p, called the compensator of μ, for which the following equivalent properties hold:*

(i.) *We have*

$$\mathbb{E}(W * \mu^p_\infty) = \mathbb{E}(W * \mu_\infty)$$

for every non-negative predictable function W on $\Omega \times \mathbb{R}_+ \times E$.

(ii.) *For every predictable function W on $\Omega \times \mathbb{R}_+ \times E$ such that $|W| * \mu \in \mathcal{A}^+_{loc}$, the process $|W| * \mu^p$ belongs to $\mathcal{A}^+_{loc}$, and $W * \mu^p$ is the compensator of the process $W * \mu$ in the sense of Theorem 2.1.10.*

In particular, there exists a predictable process $A \in \mathcal{A}^+$ and a transition kernel $K(\omega, t; dx)$ from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ to $(E, \mathcal{E})$ such that

$$\mu^p(\omega; dt, dx) = dA_t(\omega)K(\omega, t; dx) \tag{2.2}$$

for every $\omega \in \Omega$.

Proof. See Theorem II.18 of [36]. □

Although we stated the general definition of a random measure above, for this work it is sufficient to consider the following for an adapted, càdlàg stochastic process X

$$\mu^X(\omega; dt, dx) := \sum_{s\geq 0} 1_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s(\omega))}(dt, dx). \tag{2.3}$$

Indeed, μ^X defines a random measure on $\mathbb{R}_+ \times E$ which is an optional $\tilde{\mathcal{P}}$-σ-finite random measure such that $\mu(\cdot\,, A)$ takes values in $\overline{\mathbb{N}}$ for all $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}$ (cf. Proposition II.1.16 of [36]). One easily sees that the random measure μ^X is induced by the jumps of the process. In addition, Theorem 2.2.2 provides the existence of the compensator ν of μ^X.

Example 2.2.3. For a Lévy process $(L_t)_{t\geq 0}$ with Lévy triplet (ℓ, Q, μ) the compensator ν of μ^L is deterministic and of the form

$$\nu(\cdot\,; ds, dy) = \mu(dy)ds$$

For further information on random measure we refer to Chapter 13 of [14].

2.3. Semimartingales

While the term 'semimartingale' or 's-martingale' was first used by Doob in [20] to denote what we call nowadays sub- and supermartingale, the notion of semimartingales was introduced by Fisk in [23] under the name 'quasimartingale' to investigate necessary and sufficient conditions for a stochastic process to possess a decomposition into the sum of a martingale and a process with paths of finite variation. However, semimartingales and their application in stochastic integration were first introduced by Meyer in [52] in 1967, where he defined a stochastic process X to be a semimartingale if it is right-continuous and possesses a decomposition

$$X = M + A$$

into a martingale M and a stochastic process A with paths of finite variation. Indeed, Meyer was inspired by the famous paper [43] by Kunita and Watanabe who investigated a more general version of Itô's formula, namely replacing the Brownian motion by martingales which are not necessarily continuous. The non-continuity led to a much more complicated form.

In his paper, which was one of four on the topic of stochastic integration, Meyer was able to leave the Markovian framework behind. Meyer's definition as stated above, is not the most general definition as it is known today, and stated below. This was introduced by Doléans-Dade and Meyer [19] in 1970.

For a much more detailed look into the development of stochastic integration, semimartingales and mathematical finance until 1970 we refer to the interesting article [37].

Definition 2.3.1. (a.) We call a stochastic process $(X_t)_{t\geq 0}$ *semimartingale* if it possesses a decomposition of the form

$$X = X_0 + M + A \tag{2.4}$$

where X_0 is finite-valued and $\mathcal{F}_0$-measurable, $M \in \mathcal{L}$ and $A \in \mathcal{V}$. We denote by $\mathcal{S}$ the space of semimartingales. Moreover, $\mathcal{S}^d$ denotes the set of all random vectors $X = (X^{(1)}, ..., X^{(d)})$ where $X^{(i)} \in \mathcal{S}$ for all $i = 1, ..., d$, and we call $X \in \mathcal{S}^d$ a *d-dimensional semimartingale*.

(b.) We call a semimartingale $(X_t)_{t\geq 0}$ *special semimartingale* if it possesses a decomposition (2.4) where the process A is predictable. We call this decomposition the *canonical decomposition* of X. We denote by $\mathcal{S}_p$ the set of special semimartingales.

The decomposition in (2.4) is not unique, but there is at most one such decomposition with A being predictable. On the other hand, the canonical decomposition is unique (cf. Proposition I.3.17 of [36]). The inclusions

$$\mathcal{M}_{loc} \subset \mathcal{S} \text{ and } \mathcal{V} \subset \mathcal{S}$$

are trivial. In addition, $\mathcal{S} = \mathcal{S}_{loc}$, and the following characterization holds true:

Theorem 2.3.2. *The process X is a semimartingale if and only if there exists a localizing sequence $(T_n)_{n\in\mathbb{N}}$ and a sequence of semimartingales $(Y(n))_{n\in\mathbb{N}}$ such that $X = Y(n)$ on each interval $[\![0, T_n[\![$.*

Moreover, all semimartingales are càdlàg and adapted by definition, and the space of semimartingales forms a vector space (cf. Theorem 2.1 of [54]).

Remark 2.3.3. In most textbooks semimartingales are introduced as above. As pointed out, this resembles the historical development of semimartingales. A different route was taken by Protter in [54]. In contrast to Definition 2.3.1, Protter defined semimartingales to be those processes for which the stochastic integral is continuous (for a more precise definition see Chapter II.2 of [54]). Indeed, this provides some advantages over the classical procedure since the proofs of some important results are much more intuitive. In the preface to the first edition of his book, Protter emphasizes that this approach is originally due to Dellacherie [17]. By the famous theorem of Bichteler and Dellacherie (see Theorem 43 of [54]), both definitions of semimartingales are equivalent.

Example 2.3.4. (a.) Every Lévy process L with Lévy triplet (ℓ, Q, ν) is a semimartingale. This is due to the Lévy-Itô decomposition 1.3.8, where

$$\sqrt{Q}B_t + \int_0^t \int_{\{0<|y|<1\}} y \, (N(dy, ds) - \nu(dy)ds)$$

is a L^2-martingale and

$$\ell t + \int_0^t \int_{\{|y|\geq 1\}} y \, N(dy, ds)$$

belongs to $\mathcal{V}$.

(b.) Let X be a conservative Feller process with generator $(A, D(A))$ such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$. Then X is a semimartingale. For proof see Theorem 3.5 of [66] or Theorem 3.1 of [67].

The following theorem connects the decomposition of a local martingale examined in Theorem 2.1.3 and the Decomposition (2.4) of a semimartingale. The statement is taken from Proposition I.4.27 of [36].

Theorem 2.3.5. *Let X be a semimartingale. Then there exists a unique continuous local martingale X^c with $X_0^c = 0$ so that for any decomposition*

$$X = X_0 + M + A$$

as described in decomposition (2.4), the equation $M^c = X^c$ holds. Here M^c denotes the continuous local martingale part of M as mentioned in Theorem 2.1.3. We call X^c the continuous martingale part of X.

2.4. Stochastic Integration with Respect to a Semimartingale

As previously stated, Meyer utilized semimartingales in order to treat stochastic integration. Indeed, the class of semimartingales is the largest class with respect to which stochastic integration is possible such that useful results like the dominated convergence theorem hold. In this subsection, we want to state the definition and some major properties of the stochastic integral.

Let $\mathfrak{S}$ be the set of all processes H of the form

$$H_t := H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{\rrbracket T_i, T_{i+1} \rrbracket}(t), \tag{2.5}$$

where $0 = T_1 \leq ... \leq T_{n+1} < \infty$ is a finite sequence of stopping times and H_i is $\mathcal{F}_{T_i}$-measurable and $|H_i| < \infty$ a.s. for $0 \leq i \leq n$. A process $H \in \mathfrak{S}$ is called *simple predictable*. Let H be simple predictable and $X \in \mathcal{S}$. We define the *integral process* $H \bullet X_t$ or $\int_0^t H_s \, dX_s$ as

$$H \bullet X_t := H_0 X_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1} \wedge t} - X_{T_i \wedge t})$$

where H has the representation

$$H = H_0 1_{\{0\}} + \sum_{i=1}^{n} H_i 1_{\rrbracket T_i, T_{i+1} \rrbracket}$$

as in (2.5) and $t \geq 0$.

We extend the class of integrands with the next theorem (cf. Theorem I.4.31 of [36]).

Theorem 2.4.1. *Let $X \in \mathcal{S}$. The map $H \mapsto H \bullet X$ as defined on $\mathfrak{S}$ has an extension to the space of all locally bounded predictable processes H which is still denoted by $H \mapsto H \bullet X$ and as before called the stochastic integral of H with respect to X. The following properties hold:*

(a.) *$H \bullet X$ is a càdlàg adapted process.*

(b.) $H \mapsto H \bullet X$ *is linear.*

(c.) *For a sequence* $(H(n))_n$ *of predictable processes which converges pointwise to a limit* H *and for which* $|H(n)| \leq K$ *for a locally bounded predictable process* K*, the integral process* $H(n) \bullet X_t$ *converges to* $H \bullet X_t$ *in measure for all* $t \in \mathbb{R}_+$.

Moreover, this extension is unique and in $(c.)$ *the integral process* $H(n) \bullet X$ *converges to* $H \bullet X$ *in measure, uniformly on finite intervals, i.e.,*

$$(H(n) \bullet X - H \bullet X)_t^* \overset{\mathbb{P}}{\to} 0$$

for all $t \geq 0$.

We now state some properties of the stochastic integral which are used extensively when calculating with the stochastic integral in the following chapters. The proofs can be found in Chapter I.4d of [36].

Theorem 2.4.2. *Let* $X \in \mathcal{S}$ *and* H, K *be locally bounded predictable processes. Then the following properties hold:*

(i.) $H \bullet K$ *is a semimartingale.*

(ii.) *If* X *is a local martingale, so is* $H \bullet X$.

(iii.) *If* $X \in \mathcal{V}$*, then* $H \bullet X$ *also belongs to* $\mathcal{V}$ *and equals the Stieltjes-Integral.*

(iv.) $(H \bullet X)_0 = 0$.

(v.) $\Delta(H \bullet X) = H \Delta X$.

(vi.) $(H \bullet X)^T = (H 1_{[\![0,T]\!]}) \bullet X$ *for all stopping times* T.

(vii.) $K \bullet (H \bullet X) = (KH) \bullet X$ *(Associativity).*

For a throughout introduction of stochastic integration we refer to Section I.4d of [36] or Sections II.4 – II.7 of [54].

The next theorem is the most general version of the famous Itô formula. This change of variable formula for stochastic integrals was introduced by Itô in [30] in 1951 for Brownian motions and extended to semimartingales by Kunita and Watanabe [43] and Meyer [51, 52]. Indeed, Itô's formula is frequently used in the following chapters.

Theorem 2.4.3. *Let* $X := (X^{(1)}, ..., X^{(d)})$*, where* $X^{(1)}, ..., X^{(d)} \in \mathcal{S}$ *and* $f \in C^2(\mathbb{R}^d)$*. Then* $f(X)$ *is a semimartingale and the following formula holds for* $t \geq 0$*:*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \left(\frac{\partial}{\partial x^{(i)}} f(X_{t-}) \right) \bullet X^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(X_{t-}) \right) \bullet \langle X^{(i),c}, X^{(j),c} \rangle$$

$$+ \sum_{s \leq t} \left[f(X_s) - f(X_{s-}) - \sum_{i=1}^{d} \left(\frac{\partial}{\partial x^{(i)}} f(X_{s-}) \right) \Delta X_s^{(i)} \right].$$

Proof. See Theorem I.4.57 in [36]. □

Similar to the quadratic covariation for locally square-integrable martingales, we conclude this section by defining the following object. Proofs of the properties mentioned below can be found in Section I.4.e of [36].

Definition 2.4.4. Let $X, Y \in \mathcal{S}$. The *quadratic co-variation* of X and Y is defined by

$$[X, Y] := XY - X_0 Y_0 - X_- \bullet Y - Y_- \bullet X.$$

We call $[X, X]$ the *quadratic variation* of X and the following polarization property holds

$$[X, Y] = \frac{1}{4}([X + Y, X + Y] - [X - Y, X - Y]).$$

Moreover, the following properties hold true:

(a.) $[X, Y] \in \mathcal{V}$ and $[X, X] \in \mathcal{V}^+$.

(b.) $\Delta[X, Y] = \Delta X \Delta Y$.

(c.) If Y is predictable and $X \in \mathcal{M}_{loc}$, then $[X, Y]$ is a local martingale.

(d.) If X or Y is continuous, then $[X, Y] = 0$.

(e.) X belongs to $\mathcal{H}^2$ (resp. $\mathcal{H}^2_{loc}$) if and only if $[X, X]$ belongs to $\mathcal{A}$ (resp. $\mathcal{A}_{loc}$).

2.5. Characteristics of Semimartingales

It is well-known that Markov processes can be analyzed by their generator. Lévy processes are characterized by their characteristic exponent (cf. [61] Thm. 8.1 and Cor. 11.6). To possess similar 'infinitesimal' tools to treat semimartingales, this led to the concept of semimartingale characteristics, which, in some sense, also describe the predictable 'drift', the 'volatility' and the 'rate of jumps' of the process. All objects mentioned above describe the infinitesimal behavior of the stochastic process.
A truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded, measurable function which coincides with the identity in a neighborhood of zero. Most of the time we write $\chi \cdot id$ for the truncation function h, where χ is a cut-off function. A possible way to choose the *cut-off functions* χ in different dimensions $m \in \mathbb{N}$ is as follows: Take a one-dimensional cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ and define for $x \in \mathbb{R}^m$: $\tilde{\chi}(x) := \chi(x^{(1)}) \cdots \chi(x^{(m)})$ as the product of the one-dimensional cut-off function. Although, in this work, we are mostly using cut-off functions, sometimes the usage of truncation functions seems more natural or more standard like in the following definition of the characteristics of a semimartingale.

Let X be a d-dimensional semimartingale and h a truncation function. We define two processes as follows:

$$\begin{aligned} \check{X}(h)_t &:= \sum_{s \le t} (\Delta X_s - h(\Delta X_s)), \quad t \ge 0, \\ X(h) &:= X - \check{X}(h). \end{aligned}$$

A closer look on these processes and the truncation function h provides that $(\Delta X - h(\Delta X)) \neq 0$ only if there exists an $\varepsilon > 0$ such that $|\Delta X| > \varepsilon$. It follows that $\check{X}$ is well-defined, since it is the sum of the big jumps of X of which only countably many exist pathwise. Moreover, the process belongs to $\mathcal{V}^d := \mathcal{V} \times ... \times \mathcal{V}$, and we deduce that $X(h)$ is a d-dimensional semimartingale. By observing the jumps of $X(h)$, we easily see that

$$\Delta X(h) = \Delta X - \Delta \check{X}(h) = h(\Delta X).$$

So it follows that $\Delta X(h)$ is bounded since h is. Therefore, $X(h)$ is a special semimartingale by Proposition I.4.24 of [36] and possesses the canonical decomposition

$$X(h) = X_0 + M(h) + B(h), \tag{2.6}$$

where $M(h) \in \mathcal{L}^d := \mathcal{L} \times .... \times \mathcal{L}$ and $B(h) \in \mathcal{V}^d$ is predictable. With this, we are now able to define the characteristics of a semimartingale.

Definition 2.5.1. Let h be a truncation function and X be a d-dimensional semimartingale.

(i.) We define $B := (B^{(1)}, ..., B^{(d)})'$ to be the predictable process $B(h)$ defined in (2.6).

(ii.) We define $C := (C^{(ij)})_{i,j \le d}$ to be the continuous process belonging to $\mathcal{V}^{d \times d}$ defined by

$$C^{(ij)} := \langle X^{(i),c}, X^{(j),c} \rangle$$

for $i, j \in \{1, ..., d\}$.

(iii.) We define the predictable random measure ν on $\mathbb{R}_+ \times \mathbb{R}^d$ to be the *predictable compensator* of the integer-valued random measure μ^X.

We call the triplet (B, C, ν) the *characteristics* of X.

In the previous sections we have seen, that the processes B, X^c, and, therefore, $\langle X^{i,c}, X^{j,c} \rangle$, and the random measure ν are unique up to a $\mathbb{P}$-null set.

Remark 2.5.2. To our knowledge, the first time semimartinale characteristics have been defined in the modern way was by Jacod and Mémin [35] in 1976. In their work, the authors defined the characteristics almost as above but only considering the truncation function $h(x) = x1_{[0,1]}(|x|)$. They investigated how a change of measure effects the characteristics of a semimartingale.
The first idea for the characteristics of a semimartingale dates back to Grigelionis [24] in 1971 or in English language in [25] in 1972. In order to investigate problems like nonlinear filtering of stochastic processes or absolute continuity of measures corresponding to stochastic processes, Grigelionis wanted to consider a wide class of stochastic processes for which one could naturally define local coefficients of drift, diffusion and Lévy measure. He called these processes locally infinitely divisible. In modern times, the characteristics of such a process would be of the following form:

$$\begin{aligned} B_t &= \int_0^t b_s \, ds, \\ C_t &= \int_0^t c_s \, ds, \\ \nu(\omega; dt, dx) &= dt \; \Pi(\omega, t; dx). \end{aligned}$$

Let us mention that estimating the characteristics of such processes by observing the process in a high frequency regime has been a fruitful question in the theory of statistics of stochastic processes (cf. [1], [3] and the references given therein).

Example 2.5.3. (a.) For a Lévy process $(L_t)_{t\geq 0}$ the Lévy-Itô decomposition provides the following characteristics:

$$\begin{aligned} B_t &= \ell t, \\ C_t &= Qt, \\ \nu(\omega; ds, dy) &= N(dy)\, ds, \end{aligned}$$

where (ℓ, Q, N) is the Lévy triplet of L.

(b.) In Theorem 3.10 of [67] it is shown that every rich Feller process possesses characteristics of the form

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\, ds, \\ C_t^{(jk)}(\omega) &= \int_0^t Q^{(jk)}(X_s(\omega))\, ds, \\ \nu(\omega; ds, dy) &= N(X_s(\omega), dy)\, ds, \end{aligned}$$

where $(\ell(x), Q(x), N(x, dy))$ is a Lévy triplet for fixed $x \in \mathbb{R}^d$ and with respect to a fixed cut-off function χ.

Definition 2.5.4. Let X be a semimartingale. We call X a *homogeneous diffusion with jumps* (cf. [36] Definition III.2.18) if it possesses characteristics of the form

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}(X_s(\omega))\, ds, \\ C_t^{(jk)}(\omega) &= \int_0^t Q^{(jk)}(X_s(\omega))\, ds, \\ \nu(\omega; ds, dw) &= N(X_s(\omega), dw)\, ds, \end{aligned} \tag{2.7}$$

with respect to a fixed cut-off function χ. Here, $\ell(x) = (\ell^{(j)}(x))_{1\leq j\leq d} \in \mathbb{R}^d$, $Q(x) = (Q^{(jk)}(x))_{1\leq j,k\leq d}$ is a symmetric positive semidefinite matrix, $N(x, dw)$ is a measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{w\neq 0}(1 \wedge \|w\|^2)\, N(x, dw) < \infty$. We call ℓ, Q and $n := \int_{w\neq 0}(1 \wedge \|w\|^2)\, N(\cdot, dw)$ the *differential characteristics* of the process.

Let us mention that we do not want to end this section with the most important results concerning semimartingales and their characteristics as we have done in the sections before. That is, because we will state generalizations of these statements in Chapter 5. From these, the original statements could be derived easily.

2.6. Markov Semimartingales

In this section, we want to consider a combination of semimartingales and Markov processes. Indeed, the intersection of semimartingales and Markov processes gives rise to a rich theory, in particular, when considering the characteristics of a 'Markov semimartingale'. To this end, let $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$ be a Markov process, and let $(Y_t)_{t\geq 0}$ be a semimartingale over $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)$ for every $x \in E$. Similarly, we define other classes of processes on the family $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)$. Let us mention that $E = \mathbb{R}^d$.

Definition 2.6.1. Let $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$ be a Markov process and let $\mathcal{C}$ be a class of stochastic processes. When considering a stochastic process X on $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$, we say $X \in \mathcal{C}^x$ if $X \in \mathcal{C}$ for all $\mathbb{P}^x$, $x \in E$.

Definition 2.6.2. Let X be a Markov process.

(a.) We call X *Markov semimartingale* if it is a semimartingale for all $\mathbb{P}^x$, $x \in E$.

(b.) If X is a Hunt process and a semimartingale for all $\mathbb{P}^x$, $x \in E$, we call it *Hunt semimartingale.*

(c.) If X is a homogeneous diffusion with jumps we call it *Itô process*.

A priori, for a semimartingale Y on $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)$, one would expect all properties discussed in the previous sections, like the decomposition in (2.4) or the characteristics of Y, to be dependent on the measure $\mathbb{P}^x$. Fortunately, [12] shows that such properties can be defined to be the same for all $\mathbb{P}^x$.

Theorem 2.6.3. *Let Y be a semimartingale on $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$.*

(i.) There exists a $M \in \mathcal{L}^x$ and $A \in \mathcal{V}^x$ such that $Y = M + A$.

(ii.) If Y is a special semimartingale on $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$, then there exists $M \in \mathcal{L}^x$ and a predictable $A \in \mathcal{V}^x$ such that $Y = M + A$.

(iii.) If $Y \in \mathcal{A}_{loc}$ on $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$, then there exists a predictable process $Y^p \in \mathcal{V}^x$ which is a version of the $\mathbb{P}^x$-compensator of Y.

(iv.) There exists an continuous $Y^c \in \mathcal{L}^x$ which is a version of the $\mathbb{P}^x$-continuous local martingale part of Y for every $x \in \mathbb{R}^d$.

(v.) Let μ be an optional $\tilde{\mathcal{P}}$-σ-finite random measure. There exists a predictable random measure $\tilde{\mu}$ which is a version of the compensator μ^p of μ for every $\mathbb{P}^x$.

Proof. See Theorems 3.12 and 6.6 of [12]. □

Hence, from now on, we assume the decomposition and the characteristics of a Markov semimartingale to be the same for all $\mathbb{P}^x$, $x \in \mathbb{R}^d$. For the theory provided in this section the notion of additive functionals is essential.

Definition 2.6.4. Let X be a Markov process.

(a.) We call a process $Y = (Y_t)_{t\geq 0}$ *additive functional (AF)* if (i.) $Y_0 = 0$ $\mathbb{P}^x$-a.s. for all $x \in \mathbb{R}^d$;

(ii.) for every $s, t \geq 0$, we have

$$Y_{s+t} = Y_s + Y_t \circ \theta_s, \quad \mathbb{P}^x\text{-a.s.}$$

for all $x \in \mathbb{R}^d$.

(b.) We call an additive functional Y *strong* if $(ii.)$ holds true for every $t \geq 0$ and all $(\mathcal{M}_t)$-stopping times ρ instead of s.

(c.) We call an AF Y *perfect* if $\bigcup_{s,t\geq 0} \{Y_{s+t} \neq Y_t + Y_s \circ \theta_t\}$ is null for all $\mathbb{P}^x, x \in \mathbb{R}^d$.

Let $\mathcal{C}$ be a class of processes. We denote by $\mathcal{C}_{ad}$ the class of processes $X \in \mathcal{C}$ which are additive functionals.

Remark 2.6.5. At this point, let us mention that X is a Markov semimartingale if and only if $X - X_0$ is an additive functional and a semimartingale for all $\mathbb{P}^x$, $x \in E$.

Remark 2.6.6. The previous definition also holds true if we allow X and Y to take values in any measurable space $(E, \mathcal{E})$.

The classical literature (see for example [8] and [73]) mostly demands an additive functional to be right-continuous and increasing. We do not need these properties in the following. The exceptional set in (ii.) of (a.) in the previous definition is, in general, dependent on both s and t. However, any right-continuous additive functional Y of a strong Markov process with its natural filtration is strongly additive. Moreover, in this case, Y is indistinguishable to a perfect additive functional (see Proposition 3.21 of [12]).

Definition 2.6.7. Let $(Y_t)_{t\geq 0}$ be a stochastic process. We define the *big shifts* $(\Theta_s)_{s\geq 0}$ by

$$(\Theta_s Y)_t := (Y_{t-s} \circ \theta_s) 1_{[s,\infty)}(t)$$

for $s, t \geq 0$.

Note that a process Y is an additive functional if and only if $Y_0 = 0$ and $(\Theta_s Y)_t = Y_t - Y_{t\wedge s}$ for all $s, t \geq 0$, where all equalities are meant $\mathbb{P}^x$-a.s. for all $x \in \mathbb{R}^d$.
When considering Markov semimartingales, a natural question which arises is how objects defined in the semimartingale framework react when concatenated with the time-shift operator $(\theta_t)_{t\geq 0}$. We aim to clarify this in the next proposition.

Proposition 2.6.8. *Let $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$ be a Markov semimartingale.*

(a.) If X is a special semimartingale, it holds for $s, t \geq 0$ that $M_{s+t} = M_t \circ \theta_s$ and $V_{s+t} = V_t \circ \theta_s$ where $X = X_0 + M + A$ is the canonical decomposition of X.

(b.) It holds true that $X^c_t \circ \theta_s = X^c_{t+s}$ for $s, t \geq 0$.

Proof. (a.) By definition, the special semimartingale X possesses a unique decomposition

$$X = X_0 + M + A,$$

where M is a local martingale and $A \in \mathcal{V}$. Let $s \geq 0$, then $(Y_t)_{t\geq 0} := (X_{t+s})_{t\geq 0}$ is a special semimartingale which possesses the canonical decomposition

$$Y_t = X_0 + M_{t+s} + A_{t+s} \quad \forall t \geq 0.$$

Moreover, $X_{t+s} = X_t \circ \theta_s = (\Theta_s Y)_t$.
Theorem 3.15 of [12] provides that $\Theta_s Y$ is a special semimartingale possessing the canonical decomposition

$$\begin{aligned} & (\Theta_s Y)_t = \Theta_s M_{t+s} + \Theta_s A_{t+s} \\ \Leftrightarrow \quad & X_t \circ \theta_s = M_t \circ \theta_s + A_t \circ \theta_s \end{aligned}$$

for all $t \geq 0$. Hence, by the uniqueness of the decomposition, we derive $M_{t+s} = M_t \circ \theta_s$ and $A_{t+s} = A_t \circ \theta_s$.
(b.) Since X^c is the continuous martingale part of X, one derives that $(X^c_{s+t})_{t\geq 0}$ is the continuous martingale part of $(Y_t := X_{s+t})_{t\geq 0}$. Theorem 3.15 (iv) of [12] provides that

$$X^c_{s+t} = (X_{s+t})^c = (X_t \circ \theta_s)^c = (\Theta_s Y)^c_t = \Theta_s(Y^c_t) = \Theta_s(X^c_{s+t}) = X^c_t \circ \theta_s$$

for $t \geq s \geq 0$. □

The following theorem is Theorem 6.27 of [12].

Theorem 2.6.9. *Let $X = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$ be a strong Markov process, and let Y be an additive d-dimensional semimartingale which is $\mathbb{P}^x$-quasi-left-continuous for all $x \in E$. Then the characteristics (B, C, ν) of Y possess a version of the form*

$$B = \ell(X) \bullet F, \quad C = Q(X) \bullet F, \quad \nu(\omega; dt, dy) = dF_t(\omega) N(X_t(\omega), dy), \tag{2.8}$$

where

(i.) $F \in \mathcal{V}^+(\mathcal{F}^X_t)$ is an additive functional and continuous,

(ii.) $\ell = (\ell^{(i)})_{1\leq i\leq d}$ is $\mathcal{E}$-measurable,

(iii.) $Q = (Q^{(ij)})_{1\leq i,j\leq d}$ is $\mathcal{E}$-measurable with values in the set of all symmetric non-negative matrices,

(iv.) $N(x, dy)$ is a positive kernel from $(E, \mathcal{E})$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying $N(x, \{0\}) = 0$ and $\int (|y|^2 \wedge 1)\, N(x, dy) < \infty$.

Since it is easily deduced that the semimartingales X and $X - X_0$ possess the same characteristics, the previous theorem provides for every Hunt semimartingale the existence of a continuous, strictly increasing, strongly additive* functional $F \in \mathcal{V}$, measurable

*Although in 2.6.9 F is stated to be additive only, by the remark below Theorem 6.14 of [12], one can assume F to be strongly additive.

functions $\ell : \mathbb{R}^d \to \mathbb{R}^d$, $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and a positive kernel N from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that the characteristics of X are of the form

$$B_t^{(j)} = \int_0^t \ell^{(j)}(X_s)\, dF_s, \tag{2.9}$$

$$C_t^{(jk)} = \int_0^t Q^{(jk)}(X_s)\, dF_s, \tag{2.10}$$

$$\nu(\omega; dt, dx) = dF_t(\omega) N(X_t(\omega), dx). \tag{2.11}$$

The following two theorems, although of interest in their own right, serve as auxiliary results to prove Theorem 3.1.7 of the following chapter.

Lemma 2.6.10. *Let $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}}$ be a Hunt semimartingale with characteristics stated in (2.9)-(2.11), and let σ be defined by*

$$\sigma := \sigma_R^x := \inf\{t \geq 0 : \|X_t - x\| > R\}$$

for $x \in \mathbb{R}^d$ and $R > 0$. Then the following equality holds true for $\xi \in \mathbb{R}$:

$$\begin{aligned}\mathbb{E}^x\left(e^{i(X_t^\sigma - x)\xi} - 1\right) = \mathbb{E}^x \Bigg[\int_0^t 1_{[\![0,\sigma[\![} \Bigg(& i\xi e^{i(X_{s-}^\sigma - x)\xi} \ell(X_s) - \frac{1}{2}\xi^2 e^{i(X_{s-}^\sigma - x)\xi} Q(X_s) \\ &+ \int_{\{y\neq 0\}} \left(e^{i(X_{s-}-x)\xi}(e^{i\xi y} - 1 - i\xi y\chi(y))\right) N(X_s(\omega), dy)\Bigg)\, dF_s(\omega)\Bigg].\end{aligned}$$

Proof. The left-continuous process $(X_{t-}^\sigma)_{t\geq 0}$ is bounded, the stopped jumps $(\Delta X)^\sigma$ are the jumps of the stopped process (ΔX^σ) and X^σ admits the stopped characteristics:

$$\begin{aligned} B_t^\sigma(\omega) &= \int_0^t \ell(X_s(\omega)) 1_{[\![0,\sigma]\!]}(\omega, s)\, dF_s(\omega), \\ C_t^\sigma(\omega) &= \int_0^t Q(X_s(\omega)) 1_{[\![0,\sigma]\!]}(\omega, s)\, dF_s(\omega), \\ \nu^\sigma(\omega; dF_s, dy) &:= 1_{[\![0,\sigma]\!]}(\omega, s)\, N(X_s(\omega), dy)\, dF_s(\omega). \end{aligned}$$

We can now set $1_{[\![0,\sigma]\!]}$ to $1_{[\![0,\sigma[\![}$ in the previous equalities, as we are integrating with respect to a continuous measure. Moreover, using Itô s formula we obtain

$$\begin{aligned} \mathbb{E}^x\left(e^{i(X_t^\sigma - x)\xi} - 1\right) &= \mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma - x)\xi}\, dX_s^\sigma\right) \\ &+ \mathbb{E}^x\left(\frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}^\sigma - x)\xi}\, d\langle X^\sigma, X^\sigma\rangle_s^c\right) \\ &+ \mathbb{E}^x\left(e^{-ix\xi}\sum_{0<s\leq t}\left(e^{i\xi X_s^\sigma} - e^{i\xi X_{s-}^\sigma} - i\xi e^{i\xi X_{s-}^\sigma}\Delta X_s^\sigma\right)\right) \\ &= \mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma - x)\xi}\, dX_s^\sigma\right) \qquad (2.12) \\ &+ \mathbb{E}^x\left(\frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}^\sigma - x)\xi}\, dC_s^\sigma\right) \end{aligned}$$

$$+\,\mathbb{E}^x\left(\int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)\xi}(e^{i\xi y}-1-i\xi y\chi(y))\right)\mu^{X^\sigma}(\cdot;ds,dy)\right)$$
$$+\,\mathbb{E}^x\left(\int_{]0,t]\times\{y\neq 0\}}\left(e^{i(X_{s-}-x)\xi}(-i\xi y\cdot(1-\chi(y)))\right)\mu^{X^\sigma}(\cdot;ds,dy)\right).$$

Now, we want to consider the first summand from above. To this end, we use the canonical decomposition of the semimartingale (see [36] Theorem II.2.34):

$$\begin{aligned}X_t =&X_0+(X_t^\sigma)^c+\int_0^t\chi(y)y\ \left(\mu^{X^\sigma}(\cdot\,;ds,dy)-\nu^\sigma(\cdot\,;ds,dy)\right)\\ &+\int_0^t y(1-\chi(y))\,\mu^{X^\sigma}(\cdot\,;ds,dy)+B_t^\sigma.\end{aligned}$$

Thus, the linearity of the integral provides

$$\begin{aligned}&\mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma-x)\xi}\,dX_s^\sigma\right)\\ =&\,\mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma-x)\xi}\,dX_s^{\sigma,c}\right)\\ &+\mathbb{E}^x\left(\int_0^t\int_{\{y\neq 0\}} i\xi\left(e^{i(X_{s-}-x)\xi}\chi(y)y\right)\ \left(\mu^{X^\sigma}(\cdot;ds,dy)-\nu^\sigma(\cdot;ds,dy)\right)\right)\\ &+\mathbb{E}^x\left(\int_{]0,t]\times\{y\neq 0\}}\left(i\xi e^{i(X_{s-}-x)\xi}y(1-\chi(y))\right)\ \mu^{X^\sigma}(\cdot;ds,dy)\right)\\ &+\mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma-x)\xi}\,dB_s^\sigma\right).\end{aligned}$$

First, we show that

$$\mathbb{E}^x\left[\int_{0+}^t i\xi e^{i(X_{s-}-x)\xi}\,d\left(X_s^c\right)^\sigma\right]=0.$$

The integral $e^{i(X_{t-}-x)\xi}\bullet X_t^c$ is a local martingale, since X_t^c is a local martingale. To see that it is indeed a martingale, we calculate the following: In the first two lines the integrand is now bounded because ℓ and Q are locally bounded and $\|X_s^\sigma(\omega)\|<R$ on $[0,\sigma(\omega)[$ for every $\omega\in\Omega$. For the martingale preservation in the first term we obtain

$$\begin{aligned}\left[e^{i(X^\sigma-x)\xi}\bullet X^{\sigma,c},e^{i(X^\sigma-x)\xi}\bullet X^{\sigma,c}\right]_t &=\left[e^{i(X^\sigma-x)\xi}\bullet X^c,e^{i(X^\sigma-x)\xi}\bullet X^c\right]_t^\sigma\\ &=\left(\int_0^t(e^{i(X_s^\sigma-x)\xi})^2\,d[X^c,X^c]_s\right)^\sigma\\ &=\int_0^t\left(e^{i(X_s^\sigma-x)\xi}\right)^2 1_{[\![0,\sigma[\![}(s)\,d[X^c,X^c]_s\\ &=\int_0^t\left(e^{i(X_s^\sigma-x)\xi}\right)^2 1_{[\![0,\sigma[\![}(s)\,d\left(\int_0^s Q(X_r)\,dF_r\right)\\ &=\int_0^t\left(e^{i(X_s^\sigma-x)\xi}\right)^2 1_{[\![0,\sigma[\![}(s)Q(X_s)\,dF_s,\end{aligned}$$

where we used several properties of the quadratic variation and the fact that $X^{\sigma,c} := (X^\sigma)^c = (X^c)^\sigma =: X^{c,\sigma}$. Hence,

$$\mathbb{E}^x\left(\left[e^{i(X^\sigma - x)\xi} \bullet X^{\sigma,c}, e^{i(X_t^\sigma - x)\xi} \bullet X_t^{\sigma,c}\right]_t\right) < \infty, \quad t \geq 0,$$

and Property (e.) of Definition 2.4.4 provides that $e^{i(X_t^\sigma - x)\xi} \bullet X_t^{\sigma,c}$ is an L^2-martingale which is zero at zero and, therefore, its expected value is constantly zero.
Moreover, one easily sees that $e^{i(X_{s-}-x)\xi} y\chi(y)$ is in the class F_p^2 of Ikeda and Watanabe ([28], Section 2.3), and we conclude that

$$\int_0^t \int_{y\neq 0} \left(e^{i(X_{s-}-x)\xi}\chi(y)y\right)\mu^{X^\sigma}(\cdot; ds, dy) - \nu^\sigma(\cdot; ds, dy))$$

is also a martingale. Thus, in summation with equality (2.12) we obtain

$$\begin{aligned}
\mathbb{E}^x\left(e^{i(X_t^\sigma - x)\xi} - 1\right) &= \mathbb{E}^x\left(\int_{]0,t]\times\{y\neq 0\}} i\xi e^{i(X_{s-}-x)\xi} y(1-\chi(y))\, \mu^{X^\sigma}(\cdot; ds, dy)\right)\\
&+ \mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma - x)\xi}\, dB_s^\sigma\right) + \mathbb{E}^x\left(\frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}^\sigma - x)\xi}\, dC_s^\sigma\right)\\
&+ \mathbb{E}^x\left(\int_{]0,t]\times\{y\neq 0\}} \left(e^{i(X_{s-}-x)\xi}(e^{i\xi y} - 1 - i\xi y\chi(y))\right) \mu^{X^\sigma}(\cdot; ds, dy)\right)\\
&+ \mathbb{E}^x\left(\int_{]0,t]\times\{y\neq 0\}} \left(e^{i(X_{s-}-x)\xi}(-i\xi y\cdot(1-\chi(y)))\right) \mu^{X^\sigma}(\cdot; ds, dy)\right)\\
&= \mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma - x)\xi}\, dB_s^\sigma\right) + \mathbb{E}^x\left(\frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}^\sigma - x)\xi}\, dC_s^\sigma\right)\\
&+ \mathbb{E}^x\left(\int_{]0,t]\times\{y\neq 0\}} \left(e^{i(X_{s-}-x)\xi}(e^{i\xi y} - 1 - i\xi y\chi(y))\right) \mu^{X^\sigma}(\cdot; ds, dy)\right).
\end{aligned}$$

By the associativity of the stochastic integral and the properties of the compensator of a random measure the following equality holds true

$$\begin{aligned}
&\mathbb{E}^x\left(e^{i(X_t^\sigma - x)\xi} - 1\right)\\
&= \mathbb{E}^x\Bigg[\int_0^t \Bigg(i\xi e^{i(X_{s-}^\sigma - x)\xi}\ell(X_s)1_{[\![0,\sigma[\![} - \frac{1}{2}\xi^2 e^{i(X_{s-}^\sigma - x)\xi} Q(X_s)1_{[\![0,\sigma[\![}\\
&+ \int_{\{y\neq 0\}} \left(e^{i(X_{s-}-x)\xi}(e^{i\xi y} - 1 - i\xi y\chi(y))\right)1_{[\![0,\sigma[\![}\, N(X_s(\omega), dy)\Bigg)\, dF_s(\omega)\Bigg].
\end{aligned}$$

$\square$

From advanced analysis (cf. Section A.2 of the appendix) we know that every continuous, increasing function $f : [a, b] \to \mathbb{R}$ can be expressed in the form $f = g + s$, where f is increasing and absolutely continuous and s is increasing and singular. Moreover, g and s are unique up to additive constants. The following lemma provides a similar statement for an AF of a strong Markov process.

Lemma 2.6.11. *Let $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}}$ be a Hunt process. Let $(F_t)_{t\geq 0}$ be an increasing, continuous and strongly additive functional adapted to $(\mathcal{F}_t)_{t\geq 0}$.*
Then the processes

$$(F_t)_{t\geq 0} \text{ and } \left(\int_0^t g(X_s)ds + S_t\right)_{t\geq 0}$$

are indistinguishable, where g is a positive $\mathcal{M}$-measurable function and $(S_t)_{t\geq 0}$ is an increasing, continuous, singular, perfect strongly additive functional adapted to $(\mathcal{F}_t)_{t\geq 0}$.

Proof. Let $B_t := t$ for all $t \geq 0$. Combination of Exercise 66.17 and Theorem 8.6 of [73] provides the existence of a set $G \in \mathcal{B}(\mathbb{R}^d)$ and a measurable function g such that $1_G(X) \bullet F = g(X) \bullet B$. Moreover, the strongly additive functional $S := 1_{G^c}(X) \bullet F$ is singular. We obtain

$$F = (1_G(X) + 1_{G^c}(X)) \bullet F = g(X) \bullet B + S.$$

Since F is increasing, continuous and adapted to $(\mathcal{F}_t^X)_{t\geq 0}$, so is S, and Proposition 3.21 of [12] provides that S is indistinguishable from a perfect strongly additive functional. $\square$

In order to clarify the relationships of the most important classes of processes introduced up to this point, we conculde this chapter by stating the following diagram (cf. Figure 1 of [72]). Therein, the abbreviation h.d.w.j stands for homogeneous diffusion with jumps. Moreover, every inclusion is strict.

$$\begin{array}{ccccccccc}
 & & & & \text{h.d.w.j} & & \subset & & \text{semimartingale} \\
 & & & & \cup & & & & \cup \\
\text{Lévy} & \subset & \begin{matrix}\text{rich}\\ \text{Feller}\end{matrix} & \subset & \text{Itô} & \subset & \begin{matrix}\text{Hunt}\\ \text{semimartingale}\end{matrix} & \subset & \begin{matrix}\text{Markov}\\ \text{semimartingale}\end{matrix} \\
 & & \cap & & & & \cap & & \cap \\
 & & \text{Feller} & & & \subset & \text{Hunt} & \subset & \text{Markov}
\end{array}$$

Fig. 2.1.: Relations between the classes of processes under consideration

3 The Symbol of a Stochastic Process and Generalized Blumenthal-Getoor Indices

The aim of this chapter is twofold: In the first part, we introduce the notion of the 'symbol' of a stochastic process, which proves to be one of the fundamental concepts of this work. Roughly speaking, the probabilistic symbol is the right-hand side derivative of the characteristic functions corresponding to the one-dimensional marginals of a stochastic process. This object, as long as the derivative exists, provides crucial information concerning the stochastic process: For example, as we will see, for Lévy processes, the symbol equals the characteristic exponent, whereas for a (rich) Feller process it coincides with the classical symbol of the operator as defined in (1.7). The most general class of processes for which the symbol still exists are Itô processes. We prove that further generalizations within the Hunt framework are not possible and see that when leaving Hunt semimartingales, in particular the quasi-left continuity, behind the symbol loses its applicability.
In the second part, the symbol is utilized to generalize the concept of Blumenthal-Getoor indices, well-known from the theory of Lévy processes, to the class of Itô processes. These indices enable us to derive maximal-inequalities for the mentioned class of processes, which in turn provide a wide range of properties for the underlying stochastic process.
Most of the historical insights regarding the probabilistic symbol and the Blumenthal-Getoor indices are drawn from [72]. The mathematical results presented in this chapter can be found in [58]. Both articles are collaborative work with A. Schnurr. The proof of the main theorem of this chapter is due to the author of this work.

3.1. The Symbol of a Stochastic Process

Let $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$ be a normal, conservative Markov process, where we assume $\mathcal{M}$ and $(\mathcal{M}_t)_{t\geq 0}$ to be complete with respect to $\{\mathbb{P}^x : x \in \mathbb{R}^d\}$. Moreover, we assume all filtrations encountered in the following to be right-continuous. Additionally, let X take values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ is the σ-field of Lebesgue sets. Let us consider a stochastic process $X = (X_t)_{t\geq 0}$ starting in $x \in \mathbb{R}^d$. Historically, the *probabilistic symbol* is defined as the negative of the right-hand side derivative of the characteristic function of $(X_t - x)_{t\geq 0}$, that is,

$$q(x,\xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x e^{i(X_t-x)'\xi} - 1}{t}. \tag{3.1}$$

The characteristic function carries all information of the marginal distribution at time t. It is a natural idea to analyze its infinitesimal behavior to derive properties of the process. Hence, it is not surprising that (3.1) offers a unique way to describe the distribution at a certain point in time. From the point of view of Markov processes, Formula (3.1) can be

interpreted as follows: one plugs a (complex) exponential function into the generator of the process.
The idea of using a probabilistic formula in order to calculate the (functional analytic) symbol defined above is due to Jacob (see [33]) and was generalized by Schillling in [64] to rich Feller processes satisfying the properties (G) and (S) (cf. Example 3.1.2 (ii.)). The focus therein still was to present a new way to calculate the functional analytic symbol $q(x,\xi)$ in a context where it already existed. Neglecting the Feller property and (G), the symbol for quite general Markov processes, precisely Itô processes, was calculated in [67] by Schnurr. Unlike in earlier papers, the proof relied on the semimartingale structure only, and in particular on the semimartingale characteristics (2.7). On the other hand, the earlier results where included, since every rich Feller process is an Itô process. To overcome that processes not fulfilling (G) are possibly unbounded a stopping time σ was added to the expression in (3.1) in order to have the process bounded at least on $[\![0,\sigma[\![$. The new idea of proof suggested to define the symbol for general Markov semimartingales (cf. [67] Definition 4.3):

Definition 3.1.1. Let X be a conservative Markov semimartingale. For a fixed starting point $x \in \mathbb{R}^d$ we define $\sigma := \sigma_R^x$ to be the first exit time from the compact $B_R(x)$:

$$\sigma := \inf\{t \geq 0 : \|X_t - x\| > R\}.$$

We call the function $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by

$$p(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t^\sigma - x)'\xi} - 1}{t} \tag{3.2}$$

the *(probabilistic) symbol* of the process, if the limit exists and coincides for every R.

Instead of the compact balls $B_R(x)$, one can use general compact neighborhoods of x in the previous definition as described in the Remark after Theorem 4.4 of [67].

Example 3.1.2. (i.) Let $(L_t)_{t\geq 0}$ be a Lévy process on $(\Omega, \mathcal{F}^L, (\mathcal{F}_t^L)_{t\geq 0}, \mathbb{P})$ with characteristic exponent ψ, and let

$$\tilde{L} := \left(\tilde{\Omega}, \tilde{\mathcal{A}}, (\mathcal{F}^{\tilde{L}_t})_{t\geq 0}, (\tilde{L}_t)_{t\geq 0}, \mathbb{P}^x\right)_{x\in\mathbb{R}^d}$$

be the corresponding Lévy process starting in x, which is a Markov process as defined in (1.1). The symbol calculates as follows:

$$-\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(\tilde{L}_t - x)'\xi} - 1}{t} = -\lim_{t\downarrow 0} \mathbb{E} \frac{e^{iL_t'\xi} - 1}{t} = -\lim_{t\downarrow 0} \frac{e^{-t\phi(\xi)} - 1}{t} = \psi(\xi).$$

(ii.) Let $(Y_t)_{t\geq 0}$ be a rich Feller process with generator $(A, \mathcal{D}(A))$. The generator A is (restricted to the test functions $C_c^\infty(\mathbb{R}^d)$) a pseudo differential operator, i.e., A can be written as

$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} q(x,\xi)\widehat{u}(\xi)\, d\xi, \qquad u \in C_c^\infty(\mathbb{R}^d), \tag{3.3}$$

where $\widehat{u}(\xi) = (2\pi)^{-d} \int e^{-iy'\xi} u(y)dy$ denotes the Fourier transform and $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is the symbol of the operator. If the differential characteristics ℓ, Q and n of Y are continuous, the probabilistic symbol p and the symbol of the operator q coincide (cf. Corollary 4.5 of [67]).

(iii.) Let $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ be locally Lipschitz continuous and bounded, and let $(Z_t)_{t\geq 0}$ be a n-dimensional Lévy process with characteristic exponent ϕ. Then the solution of the SDE

$$dX_t = \Phi(X_t)dL_t$$

exists for every $x \in \mathbb{R}^d$ with $X_0 = 0$ and possesses the symbol $q(x,\xi) = \phi(\Phi(x)'\xi)$ (cf. Theorem 3.8 in [11]).

Remark 3.1.3. Historically, it is important to note that the concept of the probabilistic symbol has not been invented in the theory of stochastic processes: The term 'symbol' was initially introduced in the framework of singular integral operators by S.G. Michlin in the 1940s. Later on, in the 1960s pseudo-differential operators emerged in analysis (c.f. e.g. Hörmander [27]). These pseudo-differential operators are defined by their symbols. Courrège [15] showed that the generator A of a rich Feller process (restricted to the test functions $C_c^\infty(\mathbb{R}^d)$) is a *pseudo differential operator*. In fact, Courrège did not deal with stochastic processes: rather, he proved that operators satisfying the so called positive maximum principle are operators of this kind (cf. in this context also von Waldenfels [78] and [77]). It was Jacob who recognized that the mentioned generators always fulfill this principle. Hence, Jacob ([31], [32]) introduced the notion of pseudo differential operators into the theory of Markov processes.

As we have pointed out before, the most general result concerning the existence of the symbol in the Markovian context was given by Schnurr in [67] Theorem 4.4. We state the result in the following. Let us mention, that a definition for the notion of fine continuity which appears in the following theorem can be found in Section II.4 of [8].

Theorem 3.1.4. *Let X be an Itô process such that the differential characteristics ℓ, Q and n are finely continuous and locally bounded. In this case the limit*

$$-\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t^\sigma - x)'\xi} - 1}{t}$$

exists and the symbol of X is given by

$$p(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{\{y\neq 0\}} \left(e^{iy'\xi} - 1 - iy'\xi\cdot\chi(y)\right) N(x,dy) \tag{3.4}$$

for $x, \xi \in \mathbb{R}^d$.

Remark 3.1.5. One is able to state the previous theorem for homogeneous diffusions with jumps with exactly the same proof (cf. [69]). However, since for this work Markovianity is crucial we do not consider this case any further.

Taking a closer look at (3.4), we observe that $\xi \mapsto p(x,\xi)$ is negative definite function for all $x \in \mathbb{R}^d$ (cf. Section 1.3.2). Moreover, it holds true that if the symbol exists for every

$\xi \in \mathbb{R}^d$ pointwise, the function $\xi \mapsto p(x, \xi)$ is a negative definite function for every $x \in \mathbb{R}^d$ (cf. Theorem 4.6 a) of [67]). Hence, for over a decade, it has been an educated guess that Itô processes are the most general Markov processes for which the symbol exists and can be used. This guess is stated as point (d) in the open problems section of the monograph [11]. The following theorem provides a conclusive answers to this question in the framework of Hunt processes. The proof relies to a great extend on the auxiliary Lemmas 2.6.10 and 2.6.11 which we stated in the previous chapter. Moreover, several concepts and results from advanced analysis on the real line, like Dini derivatives or singular functions, are needed. We denote Dini derivatives by D^+, D_+, D^-, D_-. For further information see Appendix A.2.

Remark 3.1.6. In order to calculate the symbol for Itô processes, fine continuity (and local boundedness) are the most general requirements on the differential characteristics of the process. However, since classical continuity is much more natural, we demand this property from now on. Continuity implies both: fine continuity and local boundedness. Hence, Theorem 3.6 of [69] yields that the symbol exists for every Itô process with continuous differential characteristics.

Theorem 3.1.7. *Let* $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$ *be a Hunt semimartingale with characteristics*

$$\begin{aligned} B_t^{(j)} &= \int_0^t \ell^{(j)}(X_s)\, dF_s, \\ C_t^{(jk)} &= \int_0^t Q^{(jk)}(X_s)\, dF_s, \\ \nu(\omega; dt, dx) &= dF_t(\omega) N(X_t(\omega), dx), \end{aligned}$$

where $F \in \mathcal{V}$ *is a continuous, strictly increasing, and a strongly additive functional,* $\ell : \mathbb{R}^d \to \mathbb{R}^d$ *is continuous and measurable,* $Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ *is continuous and measurable, and* N *is a positive kernel* N *from* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *into* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *such that*

$$x \mapsto \int_{\{y\neq 0\}} (1 \wedge y^2)\, N(x, dy)$$

is continuous. If the symbol of X*,*

$$p(x,\xi) = -\lim_{t\downarrow 0} \mathbb{E}^x \left[\frac{e^{i(X_t^\sigma - x)'\xi} - 1}{t} \right] \tag{3.5}$$

exists for all $x, \xi \in \mathbb{R}^d$*, then* X *is an Itô process.*

Proof. In this proof we only consider the one-dimensional case. The multivariate one works alike, but it is notationally more involved. The stopping time σ is defined by

$$\sigma := \inf\{t \geq 0 : \|X_t - x\| > k\}$$

for some starting point x and $k \in \mathbb{R}_+$. The symbol does not depend on the particular choice of k. Hence, we will be able to choose a particular k later in this proof.
At first, let us consider the expression

$$\mathbb{E}^x \left[\frac{e^{i(X_t^\sigma - x)\xi} - 1}{t} \right].$$

Lemma 2.6.10 provides the following:

$$\begin{aligned} \mathbb{E}^x \left[e^{i(X_t^\sigma - x)\xi} - 1 \right] =& \mathbb{E}^x \Bigg[\int_0^t i\xi e^{i(X_{s-} - x)\xi} \ell(X_s) 1_{[\![0,\sigma[\![} - \frac{1}{2}\xi^2 e^{i(X_{s-}-x)\xi} Q(X_s) 1_{[\![0,\sigma[\![} \\ &+ \int_{\{y\neq 0\}} e^{i(X_{s-}-x)\xi} 1_{[\![0,\sigma[\![} \left(e^{i\xi y} - 1 - i\xi y \chi(y) \right) \, N(X_s, dy) \, dF_s \Bigg]. \end{aligned}$$

So, when considering the symbol we obtain

$$\begin{aligned} p(x,\xi) &= -\lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}^x \left[e^{i(X_t^\sigma - x)\xi} - 1 \right] \\ &= \lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}^x \Bigg[\int_0^t -i\xi e^{i(X_{s-}-x)\xi} \ell(X_s) 1_{[\![0,\sigma[\![} + \frac{1}{2}\xi^2 e^{i(X_{s-}-x)\xi} Q(X_s) 1_{[\![0,\sigma[\![} \\ &\quad - \int_{\{y\neq 0\}} e^{i(X_{s-}-x)\xi} 1_{[\![0,\sigma[\![} \left(e^{i\xi y} - 1 - i\xi y\chi(y) \right) \, N(X_s, dy) \, dF_s \Bigg]. \end{aligned}$$

To avoid lengthy notations, we define

$$\begin{aligned} Y_s^{\xi,x} :=& -i\xi e^{i(X_s - x)\xi} \ell(X_s) + \frac{1}{2}\xi^2 e^{i(X_s - x)\xi} Q(X_s) \\ &- \int_{\{y\neq 0\}} e^{i(X_s - x)\xi} \left(e^{i\xi y} - 1 - i\xi y\chi(y) \right) \, N(X_s, dy). \end{aligned}$$

Hence, the existing limit in (3.5) equals

$$p(x,\xi) = \lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}^x \left[\int_0^t Y_s^{\xi,x} 1_{[\![0,\sigma[\![} \, dF_s \right]. \tag{3.6}$$

Let us consider the process $(F_t)_{t\geq 0}$. By Lemma 2.6.11 this process is indistinguishable to

$$\left(\int_0^t g(X_u) du + S_t \right)_{t\geq 0}$$

where g is a positive, measurable function and $(S_t)_{t\geq 0}$ is an increasing, continuous, singular, perfect strongly AF. Thus, with the associativity of the stochastic integral we decompose (3.6) as follows:

$$p(x,\xi) = \lim_{t\downarrow 0} \left(\frac{1}{t} \mathbb{E}^x \left[\int_0^t Y_u^{\xi,x} g(X_u) 1_{[\![0,\sigma[\![} \, du \right] + \frac{1}{t} \mathbb{E}^x \left[\int_0^t Y_u^{\xi,x} 1_{[\![0,\sigma[\![} \, dS_u \right] \right) \tag{3.7}$$

For the first summand of (3.7), Theorem 4.4 of [67] provides the existence of the limit, and since (3.7) exists, the limit

$$\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^x\left[\int_0^t Y_u^{\xi,x}1_{[\![0,\sigma[\![}\,dS_u\right]$$

exists for all $\xi, x \in \mathbb{R}$, and, therefore, the same is true for

$$\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^x\left[\int_0^t \mathrm{Re}\left(Y_u^{\xi,x}\right)1_{[\![0,\sigma[\![}\,dS_u\right], \tag{3.8}$$

and

$$\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^x\left[\int_0^t \mathrm{Im}\left(Y_u^{\xi,x}\right)1_{[\![0,\sigma[\![}\,dS_u\right]. \tag{3.9}$$

First, we consider the case where there exists a $x_0 \in \mathbb{R}$ such that $Q(x_0) \neq 0$ or $N(x_0, dy) \neq 0$: The function $q^{\xi,x} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$z \mapsto \mathrm{Re}\left(-i\xi e^{i(z-x)\xi}\ell(z) + \frac{1}{2}\xi^2 e^{i(z-x)\xi}Q(z) - \int_{\mathbb{R}} e^{i(z-x)\xi}\left(e^{i\xi y} - 1 - i\xi y\chi(y)\right)\,N(z,dy)\right)$$

is continuous, and satisfies for a $\xi_0 \in \mathbb{R}$

$$q^{\xi,x_0}(x_0) = \frac{1}{2}\xi_0^2 Q(x_0) + \int_{\mathbb{R}} 1 - \cos(y\xi_0)\;N(x_0,dy) > 0.$$

The continuity of the differential characteristics provides the existence of an $\varepsilon > 0$ and a $\delta > 0$ such that $q^{\xi_0,x_0}(z) > \varepsilon$ for all $\|z - x_0\| \leq \delta$. As mentioned at the beginning of this proof, we consider the stopping time

$$\sigma := \inf\{t \geq 0 : \|X_t - x_0\| > k\},$$

again, and set $k = \delta$. Thus,

$$q^{\xi_0,x_0}(X_t) > 0$$

on $[\![0,\sigma[\![$. We consider the existing limit in (3.8) with x_0 and ξ_0:

$$\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^{x_0}\left[\int_0^t \mathrm{Re}\left(Y_u^{\xi_0,x_0}\right)1_{[\![0,\sigma[\![}\,dS_u\right] = \lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}^{x_0}\left[\int_0^t q^{\xi_0,x_0}(X_u)1_{[\![0,\sigma[\![}\,dS_u\right].$$

Moreover $(S_t)_{t\geq 0}$ is a path-wise singular function. In order to establish the statement, we want to show that $(S_t)_{t\geq 0} \equiv 0$ (in this case X is an Itô process). Thus, we assume that there exists a set $M \in \mathcal{M}$ with $\mathbb{P}(M) > 0$ such that $S_\cdot(\omega) \neq 0$ for all $\omega \in M$. By Proposition A.2.6 for all $\omega \in M$ there exists a $t(\omega) \geq 0$ with

$$D_+ S_{t(\omega)} = \infty.$$

We define

$$\tau := \inf\{u \geq 0 : D_+ S_u = \infty\}$$

which is an $(\mathcal{M}_t)_{t\geq 0}$-stopping time since D_+S_u is $\mathcal{M}_{u+}$-measurable and the filtration is right-continuous. Thus,

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^{X_{\tau(\omega)}(\omega)}\left[\int_0^t \mathrm{Re}\left(Y_u^{\xi_0,x_0}\right) 1_{[\![0,\sigma[\![} \, dS_u\right] \tag{3.10}$$

exists for all $\omega \in M$.
By applying Fatou's lemma for the conditional expectation the following inequality holds true

$$\begin{aligned}
&\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^{X_{\tau(\omega)}(\omega)}\left[\int_0^{t\wedge\sigma} \mathrm{Re}\left(Y_u^{\xi_0,x_0}\right) \, dS_u\right]\\
=&\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^{x_0}\left[\int_0^{t\wedge(\sigma\circ\theta_\tau)} \mathrm{Re}\left(Y_u^{\xi_0,x_0}\circ\theta_\tau\right) \, dS_u\circ\theta_\tau \,\middle|\, \mathcal{M}_\tau\right]\\
\geq&\mathbb{E}^{x_0}\left[\liminf_{t\downarrow 0} \frac{1}{t}\int_0^{t\wedge(\sigma\circ\theta_\tau)} \mathrm{Re}\left(Y_u^{\xi_0,x_0}\circ\theta_\tau\right) \, dS_u\circ\theta_\tau \,\middle|\, \mathcal{M}_\tau\right]\\
\geq&\mathbb{E}^{x_0}\left[\liminf_{t\downarrow 0}\left(\inf_{u\in[0,t\wedge(\sigma\circ\theta_\tau)]} \mathrm{Re}\left(Y_u^{\xi_0,x_0}\circ\theta_\tau\right)\right) \frac{S_t\circ\theta_\tau}{t} \,\middle|\, \mathcal{M}_\tau\right]\\
=&\mathbb{E}^{x_0}\left[\liminf_{t\downarrow 0}\left(\inf_{u\in[0,t\wedge(\sigma\circ\theta_\tau)]} \mathrm{Re}\left(Y_u^{\xi_0,x_0}\circ\theta_\tau\right)\right) \frac{S_{\tau+t}-S_\tau}{t} \,\middle|\, \mathcal{M}_\tau\right]
\end{aligned}$$

where we have used the strong additivity and the perfectness of S in the last equation.
Since

$$\mathrm{Re}\left(Y_u^{\xi_0,x_0}\circ\theta_\tau\right) > \varepsilon$$

on $[\![0,\sigma\circ\theta_\tau[\![$ and $(S_{\tau+t}-S_\tau)/t \to \infty$ for $t \downarrow 0$ by the definition of τ this is a contradiction to the existence of the limit.
In the other case let there exist a $x_0 \in \mathbb{R}$ with $\ell(x_0) \neq 0$. We set

$$\begin{aligned}
q^{\xi,x}(z) :=& \,\mathrm{Im}\left(i\xi e^{i(z-x)\xi}b(z) - \frac{1}{2}\xi^2 e^{i(z-x)\xi}c(z) - \int_{\mathbb{R}} e^{i(z-x)\xi}\left(e^{i\xi y}-1-i\xi y\chi(y)\right) N(z,dy)\right)\\
=& \,\xi\sin\left((z-x)\xi\right) b(z)
\end{aligned}$$

which is continuous, and satisfies

$$q^{\xi_0,x_0}(x_0) = \xi_0 b(x_0) > 0$$

for $\xi_0 = \mathrm{sign}\,(b(x_0))$. The rest of the proof is analogous to the case above where $Q(x_0) \neq 0$ or $N(x_0,dy) \neq 0$ but we use(3.9) instead of (3.8). □

As the previous theorem shows, when considering Hunt semimartingales, the class of Itô processes is the largest class of processes for which the symbol exists. This aligns, to a certain extent, with the expectations.
Surprisingly, when leaving the Hunt framework behind, we are able to construct a Markov semimartingale that is no Itô process but still admits a symbol. To demonstrate this, we want to construct a deterministic, one-dimensional example process, or more accurately, a family of functions that satisfies the deterministic version of the Markov property (cf. [70] and [68]):

For a deterministic process, which is allowed to start in every point $x \in \mathbb{R}$, the symbol reduces to

$$-\lim_{t\downarrow 0}\left(\frac{\cos((X_t^x - x)\xi) - 1}{t} + i\frac{\sin((X_t^x - x)\xi)}{t}\right).$$

Furthermore, since (for $\xi \neq 0$) the function $y \mapsto \sin((y-x)\xi)$ is locally diffeomorphic around zero, the symbol exists if and only if the process is right-differentiable at zero in every starting point x. In this case the cosine-part becomes zero. Hence,

$$p(x,\xi) = -i\lim_{t\downarrow 0}\left(\frac{\sin((X_t^x - x)\xi)}{t}\right).$$

Example 3.1.8. For $y \in \mathbb{R}$ let $y = [y] + \{y\}$, where $[\cdot]$ denotes the floor function and $\{y\} \in [0,1[$. We define

$$X_t^x := [x] + \{x+t\}. \tag{3.11}$$

Starting from x the process drifts with constant speed upwards, but jumps to $[x]$ directly before reaching $[x]+1$. Admittedly, this process does (at first) not look Markovian, since in the Definition (3.11) the starting point x appears. In fact this is just to simplify notation. Being at time s at the point y, we know that at time $t > s$ we are at $[y] + \{y + (t-s)\}$, without any knowledge on the fact where the process has started at time zero.

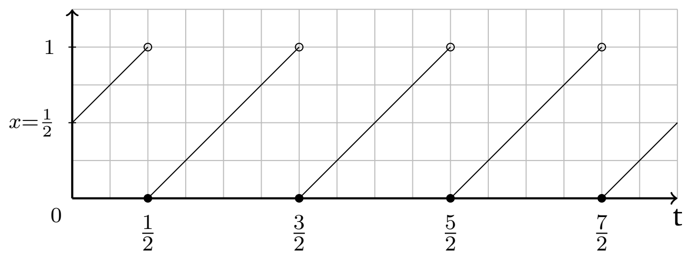

Fig. 3.1.: Sample path with starting point 1/2

The symbol of this process calculates as follows for $x, \xi \in \mathbb{R}$:

$$p(x,\xi) = -i\lim_{t\downarrow 0}\left(\frac{\sin(([x] + \{x+t\} - x)\xi)}{t}\right) = -i\xi$$

This means that the symbol is exactly the one of a deterministic drift with incline 0.5, that is, the most simple example of a Lévy process. Hence, from the point of view of the symbol, this process is a Lévy process (!) or in other words: The symbol does not contain information on the fixed times of discontinuity, making it practically useless for the analysis of the sample paths.

3.2. Blumenthal-Getoor Indices and Asymptotic Behavior of the Sample Path

The main objective of this section is to utilize the symbol of an Itô process to analyze properties related to the paths of the process such as conservativeness, asymptotic behaviour, strong γ-variation, Hausdorff-dimension and Hölder conditions.
Indeed, for α-stable processes, there exists a natural index associated with the symbol, which is linked to these properties. The analysis of the interplay between the stability index α and properties of this kind can be traced back to Bochner [9] and McKean [50]. After generalizing these results to the multivariate framework in [6], Blumenthal and Getoor [7] introduced the indices named after them in 1961. These indices allowed for the analysis of more general Lévy processes. In [55], Pruitt introduced another index γ that complements the aforementioned indices. Schilling [66] extensively extended all of these indices to the class of rich Feller processes satisfying (G) and (S). Considering homogeneous diffusion with jumps, Schnurr [69] was able to further generalize the indices. However, setting technical difficulties aside, the idea behind the aforementioned indices is quite similar: The behavior of the process's paths is governed by the behavior of the symbol in the variable ξ, which is expressed by the corresponding indices.
To define the generalized Blumenthal-Getoor indices, one utilizes the following quantities for $x \in \mathbb{R}^d$ and $R > 0$:

$$H(x,R) := \sup_{\|y-x\|\leq 2R} \sup_{\|\varepsilon\|\leq 1} \left| p\left(y, \frac{\varepsilon}{R}\right)\right| \tag{3.12}$$

$$H(R) := \sup_{y\in\mathbb{R}^d} \sup_{\|\varepsilon\|\leq 1} \left| p\left(y, \frac{\varepsilon}{R}\right)\right| \tag{3.13}$$

$$h(x,R) := \inf_{\|y-x\|\leq 2R} \sup_{\|\varepsilon\|\leq 1} \mathrm{Re}\left(p\left(y, \frac{\varepsilon}{4\kappa R}\right)\right) \tag{3.14}$$

$$h(R) := \inf_{y\in\mathbb{R}^d} \sup_{\|\varepsilon\|\leq 1} \mathrm{Re}\left(p\left(y, \frac{\varepsilon}{4\kappa R}\right)\right) \tag{3.15}$$

In (3.14) and (3.15) $\kappa = (4\arctan(1/2c_0))^{-1}$, where c_0 comes from the sector condition (S). In particular, $h(x,R)$ and $h(R)$ are only defined if (S) is satisfied.

Definition 3.2.1. The quantities (cf. [66] Definitions 4.2 and 4.5)

$$\beta_0 := \sup\left\{\lambda \geq 0 : \limsup_{R\to\infty} R^\lambda H(R) = 0\right\}$$

$$\underline{\beta_0} := \sup\left\{\lambda \geq 0 : \liminf_{R\to\infty} R^\lambda H(R) = 0\right\}$$

$$\overline{\delta_0} := \sup\left\{\lambda \geq 0 : \limsup_{R\to\infty} R^\lambda h(R) = 0\right\}$$

$$\delta_0 := \sup\left\{\lambda \geq 0 : \liminf_{R\to\infty} R^\lambda h(R) = 0\right\}$$

are called *indices of X at the origin*, while

$$\beta_\infty^x := \inf\left\{\lambda > 0 : \limsup_{R\to 0} R^\lambda H(x,R) = 0\right\}$$

$$\underline{\beta^x_\infty} := \inf\left\{\lambda > 0 : \liminf_{R\to 0} R^\lambda H(x,R) = 0\right\}$$
$$\overline{\delta^x_\infty} := \inf\left\{\lambda > 0 : \limsup_{R\to 0} R^\lambda h(x,R) = 0\right\}$$
$$\delta^x_\infty := \inf\left\{\lambda > 0 : \liminf_{R\to 0} R^\lambda h(x,R) = 0\right\}$$

are the *indices of X at infinity*.

The nomenclature of the indices seems counter-intuitive at first sight and in consideration that the growth of the sample paths at infinity is governed by the indices at the origin while the asymptotic behavior in zero is goverend by the indices at infinity (cf. Theorem 3.2.7 and 3.2.8). Indeed, here 'origin' refers to $\xi = 0$ relative to the symbol, and 'infinity' refers to $|\xi| \to \infty$.
Moreover, all indices are in $[0,2]$ because of (G).

Remark 3.2.2. The generalization of the Blumenthal-Getoor indices discussed in this chapter is not the only generalization of these indices. Indeed, in [2] Aït-Sahalia and Jacod propose so-called successive Blumenthal-Getoor indices to study the identification of the jump measure of a semimartingale X on a finite time interval $[0,T]$ at high frequency.

Remark 3.2.3. When considering an index at infinity, exemplified by β^x_∞, we observe that for $\lambda > \beta^x_\infty$ we have

$$\limsup_{R\to 0} R^\lambda\ H(x,R) = 0,$$

and for $\mu < \beta^x_\infty$

$$\limsup_{R\to 0} R^\mu\ H(x,R) = \infty.$$

For the indices in the starting point, considering for example β_0, we have for $\mu < \beta_0 < \lambda$

$$\limsup_{R\to\infty} R^\lambda\ H(x,R) = \infty$$
$$\limsup_{R\to\infty} R^\mu\ H(x,R) = 0.$$

Example 3.2.4. (a.) Let X be a Lévy process with characteristic exponent ψ for which (S) holds true. One sees that

$$\beta_0 = \overline{\delta_0}, \quad \underline{\beta_0} = \delta_0, \quad \beta_\infty = \beta^x_\infty = \overline{\delta^x_\infty} \text{ and } \delta_\infty = \delta^x_\infty = \underline{\beta^x_\infty}.$$

Moreover, in this case β_∞ is the original index introduced by Blumenthal and Getoor, and, we have (see Example 5.5 of [66])

$$\beta_\infty := \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\},$$
$$\beta_0 := \inf\left\{\lambda > 0 : \lim_{\|\xi\|\to 0} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\},$$
$$\delta_\infty := \inf\left\{\lambda > 0 : \liminf_{\|\xi\|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\},$$

$$\delta_0 := \inf\left\{\lambda > 0 : \liminf_{\|\xi\|\to 0} \frac{\operatorname{Re}\psi(\xi)}{\|\xi\|^\lambda} = 0\right\}.$$

(b.) For a symmetric α-stable Lévy process, i.e. the characteristic exponent of X is given by $\psi(\xi) = |\xi|^\alpha$, all indices coincide and equal α.

Before we are able to derive properties from the just defined indices, we need to establish the following maximal inequalities. Similar results were proved for Lévy processes by Pruitt in [55] and for rich Feller processes satisfying (G) and (S) by Schilling in [66]. The version presented here is taken from Schnurr [67]. One can find a throughout discussion of maximal inequalitys for various classes of stochastic processes in [45].

Theorem 3.2.5. *Let X be an Itô process such that the differential characteristics of X are continuous. In this case we have*

$$\mathbb{P}^x\Big((X_\cdot - x)_t^* \geq R\Big) \leq c_d \cdot t \cdot H(x,R) \tag{3.16}$$

for $t \geq 0$, $R > 0$ and a constant $c_d > 0$ which can be written down explicitly and only depends on the dimension d.
If (S) *holds in addition, we have*

$$\mathbb{P}^x\Big((X_\cdot - x)_t^* < R\Big) \leq c_\kappa \cdot \frac{1}{t} \cdot \frac{1}{h(x,R)} \tag{3.17}$$

for a constant c_κ only depending on the c_0 of the sector condition (S).

Remark 3.2.6. Similar to Theorem 3.1.4 of the previous section we can state Theorem 3.2.5 for homogeneous diffusions with jumps whose characteristics are locally bounded and finely continuous with the same proof. We omit this version since we are interested in the Markovian context.

This theorem plays a major role in analyzing various concepts related to the sample paths of the process, such as conservativeness (cf. [64], Theorem 5.5), asymptotic behaviour (cf. [69],Theorems 3.11 and 3.12), strong γ-variation (cf. [62] Corollary 5.10), Hausdorff-dimension (cf. [65], Theorem 4) and Hölder conditions [66].
Although being of significant importance in their own right, we will not state all of the aforementioned properties in detail in the following, but rather focus solely on the asymptotic behavior of the sample path. One can find a proof of the following in [69].

Theorem 3.2.7. *Let X be an Itô process such that the differential characteristics of X are continuous. Then we have*

$$\lim_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \textit{ for all } \lambda < \beta_0 \tag{3.18}$$

$$\liminf_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \textit{ for all } \beta_0 \leq \lambda < \underline{\beta_0}. \tag{3.19}$$

If the symbol p of the process X satisfies (S)*, then we have in addition*

$$\limsup_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \textit{ for all } \overline{\delta_0} < \lambda \leq \delta_0 \tag{3.20}$$

$$\lim_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \textit{ for all } \delta_0 < \lambda. \tag{3.21}$$

All these limits are meant $\mathbb{P}^x$-a.s with respect to every $x \in \mathbb{R}^d$.

Theorem 3.2.8. *Let X be an Itô process such that the differential characteristics of X are continuous. Then we have*

$$\lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \textit{ for all } \lambda > \beta_\infty^x \tag{3.22}$$

$$\liminf_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \textit{ for all } \beta_\infty^x \geq \lambda > \underline{\beta_\infty^x}. \tag{3.23}$$

If the symbol p of the process X satisfies (S) *then we have in addition*

$$\limsup_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \textit{ for all } \overline{\delta_\infty^x} > \lambda \geq \delta_\infty^x \tag{3.24}$$

$$\lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \textit{ for all } \delta_\infty^x > \lambda. \tag{3.25}$$

All these limits are meant $\mathbb{P}^x$-a.s with respect to every $x \in \mathbb{R}^d$.

At the end of this section, we want to return to Example 3.1.8: Naively, the existence of a symbol is sufficient for calculating the generalized Blumenthal-Getoor indices, and, therefore, for deriving the asymptotic behavior of the sample paths. However, the following example will illustrate that such a simplistic calculation is futile, as the symbol looses its applicability when dealing with processes which are no Itô processes:

Example 3.2.9. Again, let us consider a deterministic Markov process as in Example 3.1.8. Starting at time zero in zero, let us consider a drift with incline 1/2 disrupted in $\mathbb{N}$ by deterministic jumps of quadratic heights:

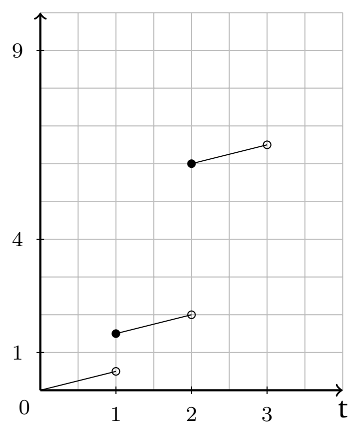

Fig. 3.2.: Sample path with starting point zero

The paths from other starting points are added in a time-homogeneous manner (cf. [68], [70]). Hence, the emerging process is a Markov semimartingale with characteristics

$$B_t = \frac{1}{2}t, \; C_t \equiv 0, \; \nu(dt, dy) = \sum_{s\in\mathbb{N}} \delta_s(dt)\delta_{s^2}(dy).$$

Although, this process is markovian by construction it is not quasi-left-continuous, and, therefore not a Hunt process. The symbol of the process calculates as

$$p(x,\xi) = -\lim_{t\downarrow 0} \mathbb{E}^x \left[\frac{e^{i(X_t^\sigma - x)'\xi} - 1}{t} \right] = -\frac{1}{2}\xi.$$

Moreover, we have

$$H(R) = \sup_{y\in\mathbb{R}} \sup_{\|\varepsilon\|\le 1} \left| p\left(y, \frac{\varepsilon}{R}\right) \right| = \frac{\ell}{R},$$

and, therefore,

$$\beta_0 := \sup \left\{ \lambda \geq 0 : \limsup_{R\to\infty} R^\lambda H(R) = 0 \right\} = 1.$$

Obviously,

$$\lim_{t\to\infty} t^{-1/\lambda} (X-x)_t^* = 0 \text{ for all } \lambda < \beta_0$$

is wrong for the process considered here, although, the symbol and the corresponding index do exist. In fact, one can mimic totally different kinds of behavior for $t \to \infty$ by using different jump structures.

4 The Time-Dependent Symbol and Blumenthal-Getoor Indices

As we have seen in the previous chapter, the (homogeneous) symbol $p(x, \xi)$ proves to be a crucial concept for deriving a wide range of properties of the underlying stochastic process. However, when leaving time-homogeneity behind one does not expect the symbol, being the right-hand side derivative of the characteristic functions corresponding to the one-dimensional marginal *at time zero*, to yield any information regarding the entire process. To overcome this, [59] proposed adding a time component to the symbol or more precisely

$$p(\tau, x, \xi) := -\lim_{h\downarrow 0} \mathbb{E}^{\tau,x}\left(\frac{e^{i(X^\sigma_{\tau+h}-x)'\xi}-1}{h}\right) \quad \text{for } x, \xi \in \mathbb{R}^d,\ \tau \geq 0. \tag{4.1}$$

The existence of such a *time-dependent probabilistic symbol* was shown for rich càdlàg Feller evolution processes, i.e., non-homogeneous càdlàg Markov processes such that

$$T_{\tau,t}u(x) := \mathbb{E}^{\tau,x}u(X_t)$$

for $0 \leq \tau \leq t$ and $u \in C_\infty(\mathbb{R}^d)$ forms a strongly continuous evolution system, i.e., a family of bounded linear operators such that for $0 \leq s \leq u \leq t$

(1.) $T_{s,t}1 = 1$,

(2.) $T_{s,t} = T_{s,u}T_{u,t}$ and $T_{s,s} = id$,

(3.) $\|T_{s,t}u\|_\infty \leq \|u\|_\infty$,

(4.) for $u \geq 0$ and $u \in C_b(\mathbb{R}^d)$ it holds that $T_{s,t}u \geq 0$,

(5.) $(s,t) \mapsto T_{s,t}$ is strongly continuous, i.e.,

$$\lim_{(s,t)\to(v,w)} \|T_{s,t}u - T_{v,w}u\|_\infty = 0$$

hold true. In addition, the domain of the *family of infinitesimal generators* $(A_\tau)_{\tau\geq 0}$,

$$A_\tau f := \lim_{h\downarrow 0} \frac{T_{\tau,\tau+h}f - f}{h}, \quad \tau > 0$$

given by

$$D(A_\tau) := \left\{ f : \lim_{h\downarrow 0} \frac{T_{\tau,\tau+h}f - f}{h} \text{ exists} \right\}$$

contains the test functions $C_c(\mathbb{R}^d)$. Theorem 4.5 of [59] shows that the generator $A_\tau|_{C_c^\infty}$ is a *pseudo-differential operator* with symbol $-q(\tau, x, \xi)$, i.e.,

$$A_\tau f(x) = -\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix'\xi} q(\tau, x, \xi) \hat{f}(\xi)\, d\xi$$

for $\tau \geq 0, x, \xi \in \mathbb{R}^d$ and $\hat{f}$ is the Fourier transform of f. Similar to the homogeneous case, it is shown that the symbol of the generator $q(\tau, x, \xi)$ and the time-dependent probabilistic symbol defined in (4.1) coincide if the symbol is continuous in x.
The content of this chapter is mostly based on [57], which is joint work with A. Schnurr.

4.1. The Time-Dependent Symbol of a Non-Homogeneous Itô Process

In this section, we prove the existence of the time-dependent symbol for non-homogeneous Itô processes (cf. Definition 4.1.2). We utilize this result to prove maximal-inequalities, the existence of non-homogeneous generalizations of the Blumenthal-Getoor indices and an exemplary selection of properties of such processes. For this chapter, let

$$X = (\Omega, \mathcal{M}, (\mathcal{M}_t^\tau)_{0 \leq \tau \leq t}, (X_t)_{t \geq 0}, (\theta_t)_{t \geq 0}, \mathbb{P}^{\tau,x})_{\tau \geq 0, x \in \mathbb{R}^d}$$

be a conservative, normal non-homogeneous Markov process as defined in Section 1.4. Moreover, let

$$\hat{X} := \left(\hat{\Omega}, \hat{\mathcal{F}}, (\mathcal{F}_t^\tau)_{0 \leq \tau \leq t}, (\hat{X}_t)_{t \geq 0}, (\hat{\theta}_t)_{t \geq 0}, \hat{\mathbb{P}}^{(\tau,x)}\right)_{(\tau,x) \in \mathbb{R}_+ \times \mathbb{R}^d}$$

be the associated space-time process.

Definition 4.1.1. We call a non-homogeneous Markov process X *non-homogeneuous Markov semimartingale* if for every $\mathbb{P}^{\tau,x}$, $\tau \geq 0, x \in \mathbb{R}^d$ the process $(X_t)_{t \geq 0}$ is a semimartingale on $[\tau, \infty)$.

Definition 4.1.2. We call a non-homogeneous Markov semimartingale *non-homogeneous Itô process* if its characteristics (B, C, ν) are of the form

$$\begin{aligned} B_t^{(i)} &= \int_\tau^t \ell^{(i)}(s, X_s)\, ds \quad \mathbb{P}^{\tau,x}\text{-a.s.}, \\ C_t^{(ij)} &= \int_\tau^t Q^{(ij)}(s, X_s)\, ds \quad \mathbb{P}^{\tau,x}\text{-a.s.}, \\ \nu(; dt, dx) &= dt N_t(X_t, dx) \quad \mathbb{P}^{\tau,x}\text{-a.s.} \end{aligned} \tag{4.2}$$

for $t \geq \tau \geq 0$, where $(\ell^{(i)}(s,x))_{1 \leq i \leq d} \in \mathbb{R}^d$, $(Q^{(ij)}(s,x))_{1 \leq i,j \leq d}$ is a symmetric positive semidefinite matrix, and $N_t(x, dy)$ is a measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{w \neq 0} (1 \wedge \|w\|_\infty^2)\, N_t(x, dy) < \infty$ for $t \geq \tau \geq 0, x \in \mathbb{R}^d$. Furthermore, under $\mathbb{P}^{\tau,x}$ we denote by dt the Lebesgue measure on $[\tau, \infty)$.

Definition 4.1.3. Let X be a non-homogeneous Markov process and let

$$\sigma := \sigma_R^{\tau,x} := \inf\{h \geq \tau : \|X_h - x\| > R\} = \tau + \inf\{h \geq 0 : \|X_{\tau+h} - x\| > R\} \tag{4.3}$$

be the first exit time from the ball of radius $R > 0$ after $\tau \geq 0$ and $\|\cdot\|$ the maximum norm. The function $p : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ defined by

$$p(\tau, x, \xi) := -\lim_{h\downarrow 0} \mathbb{E}^{\tau,x}\left(\frac{e^{i(X^{\sigma}_{\tau+h}-x)'\xi} - 1}{h}\right) \tag{4.4}$$

is called the *time-dependent probabilistic symbol* of the process, if the limit exists for every $\tau \geq 0$ and $x, \xi \in \mathbb{R}^d$ independently of the choice of R.

Example 4.1.4. (a.) Let $(X_t)_{t\geq 0}$ be an additive process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ in the sense of Definition 1.3.2. We define a family of probability measures on $(\Omega, \mathcal{F})$ by

$$\mathbb{P}^{\tau,x}(X_t \in B) := \mathbb{P}(X_t - X_\tau \in B - x)$$

for $B \in \mathcal{E}$ and $\tau \in \mathbb{R}_+, x \in E$. For this family it holds true that $\mathbb{P}^{\tau,x}(X_\tau = x) = 1$, and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, \mathbb{P}^{\tau,x})_{\tau\geq 0, x\in E}$ is a non-homogeneous Markov process.
Let, in addition, X be a semimartingale for all $\mathbb{P}^{\tau,x}$, which is quasi-left-continuous, and possesses the characteristics (B, C, ν). Theorem II.4.15 of [36] provides the existence of a version of (B, C, ν) that is deterministic. Hence, in the following, we assume (B, C, ν) to be deterministic. By Corollary II.4.18 of [36] X has no fixes times of discontinuity, i.e.,

$$\{t \geq 0 : \nu(\{t\} \times E) > 0\} = \emptyset.$$

Therefore, when calculating the time-dependent probabilistic symbol of X, Theorem II.4.15 of [36] provides the following:

$$\begin{aligned}
p(\tau, x, \xi) &= -\lim_{h\downarrow 0} \mathbb{E}^{\tau,x}\left(\frac{e^{i(X_{\tau+h}-x)'\xi} - 1}{h}\right) \\
&= -\lim_{h\downarrow 0} \mathbb{E}\left(\frac{e^{i(X_{\tau+h}-X_\tau)'\xi} - 1}{h}\right) \\
&= -\lim_{h\downarrow 0} \frac{e^{i\xi(B_{\tau+h}-B_\tau) - \frac{1}{2}\xi'(C_{\tau+h}-C_\tau)\xi + \int_{\mathbb{R}^d\setminus\{0\}} e^{i\xi' y} - 1 - i\xi' y\chi(y)\ \nu((\tau,\tau+h],dy)} - 1}{h}.
\end{aligned}$$

This limit exists if and only if

$$-\lim_{h\downarrow 0} \frac{i\xi(B_{\tau+h} - B_\tau) - \frac{1}{2}\xi'(C_{\tau+h} - C_\tau)\xi + \int_{\mathbb{R}^d\setminus\{0\}} e^{i\xi' y} - 1 - i\xi' y\chi(y)\ \nu((\tau, \tau+h], dy)}{h}$$

exists.
If $B^{(i)}$ and $C^{(ij)}$ are right-differentiable for all $i, j \in \{1, ..., d\}$ and if the function

$$\tau \mapsto \int_{\mathbb{R}^d\setminus\{0\}} e^{i\xi' y} - 1 - i\xi' y\chi(y)\ \nu((0, \tau], dy)$$

is right-differentiable the time-dependent symbol exists and is of the form

$$p(\tau, x, \xi) = i\xi\partial_+ B_\tau - \frac{1}{2}\xi'\partial_+ C_\tau \xi + \partial_+ \int_0^\tau \int_{\mathbb{R}^d\setminus\{0\}} e^{i\xi' y} - 1 - i\xi' y\chi(y)\ \nu(ds, dy).$$

(b.) Let $(X_t)_{t\geq 0}$ be a one-dimensional Brownian motion with variance function $\sigma^2(\cdot)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The process X is additive and a continuous semimartingale and

$$\mathbb{E}\left(e^{i(X_{t+\tau}-X_\tau)'\xi}\right) = \exp\left(-\frac{1}{2}(\sigma^2(t+\tau) - \sigma^2(\tau))\xi^2\right).$$

By the previous example the (non-homogeneous) probabilistic symbol exists if and only if the variance function is right-differentiable with right-derivative $\partial_+\sigma^2$. In this case we have

$$p(\tau, x, \xi) = -\frac{1}{2}\xi^2\partial_+\sigma^2(\tau).$$

At the beginning of this chapter we have seen that under some mild conditions for a rich càdlàg Feller evolution process the symbol of the generator $q(\tau, x, \xi)$ and the time-dependent probabilistic symbol $p(\tau, x, \xi)$ coincide. Additionally, Corollary 3.5 of [10] states that the symbol of the generator of the homogeneous space-time process $\hat{X}$ corresponding to X is given by

$$\hat{q}(\hat{x}, \hat{\xi}) = -i\xi_0 + q(\tau, x, \xi)$$

with $\tau \geq 0, x \in \mathbb{R}^d, \xi \in \mathbb{R}^d$, $\hat{x} = (\tau, x), \hat{\xi} = (\xi_0, \xi) \in \mathbb{R}^{d+1}$. Therefore, we expect the space-time process to be useful when calculating the symbol of a non-homogeneous Itô process, provided the characteristics of the space-time process are the ones of a homogeneous diffusion with jumps.

Lemma 4.1.5. *Let X be a non-homogeneous Markov semimartingale with characteristics (B, C, ν). In this case the space-time process $\hat{X}$ associated with X is a Markov semimartingale for all $\hat{\mathbb{P}}^{(\tau,x)}, (\tau, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and its characteristics $(\hat{B}, \hat{C}, \hat{\nu})$ are given by*

$$\hat{B}_t(c, \omega) = (t, B_{\tau+t}(\omega)), \quad \hat{\mathbb{P}}^{(\tau,x)}\text{-a.s.} \tag{4.5}$$

$$\hat{C}_t(c, \omega) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & C_{\tau+t}(\omega) \end{pmatrix}, \quad \hat{\mathbb{P}}^{(\tau,x)}\text{-a.s.} \tag{4.6}$$

$$\hat{\nu}((c, \omega); ds, du \times dy) = \nu(\omega; ds + \tau, dy)\delta_0(du), \quad \hat{\mathbb{P}}^{(\tau,x)}\text{-a.s.} \tag{4.7}$$

for $0 \leq \tau \leq t$ and $(c, \omega) \in \hat{\Omega}$.

Proof. Let X be a non-homogeneous Markov semimartingale. Our first objective is to evaluate whether the semimartingale property is preserved when transitioning to the space-time process $\hat{X} = (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq 0}, (\hat{X}_t)_{t\geq 0}, (\hat{\theta}_t)_{t\geq 0}, \hat{\mathbb{P}}^{(\tau,x)})_{(\tau,x)\in\mathbb{R}_+\times\mathbb{R}^d}$:
Let

$$X_t = X_\tau + M_t + V_t, \quad t \geq \tau,$$

be the semimartingale decomposition of X with respect to $\mathbb{P}^{\tau,x}$, i.e., M is a $\mathbb{P}^{\tau,x}$-local martingale on $[\tau, \infty)$ and $V \in \mathcal{V}$ on $[\tau, \infty)$. For $\hat{\omega} := (c, \omega) \in \hat{\Omega}$ we compute with (1.13) that

$$\begin{aligned}\hat{X}_t(\hat{\omega}) &= (t + c, X_{t+c}(\omega)) \\ &= (t + \tau, X_{t+\tau}(\omega)) \\ &= (t + \tau, X_\tau + M_{t+\tau} + V_{t+\tau}),\end{aligned} \tag{4.8}$$

for all $t \geq 0$, where all equalities are meant $\hat{\mathbb{P}}^{(\tau,x)}$-a.s.. We observe that $(t+\tau)_{t\geq 0}$ is a $\hat{\mathbb{P}}^{(\tau,x)}$-semimartingale, and, consequently, $\hat{X}$ is a $\hat{\mathbb{P}}^{(\tau,x)}$-semimartingale if $(M_{t+\tau})_{t\geq 0}$ is a $\hat{\mathbb{P}}^{(\tau,x)}$-local martingale.
Thus, let $(N_t := M_{t+\tau})_{t\geq 0}$ be a local martingale on $(\Omega, (\mathcal{M}^\tau_{\tau+t})_{t\geq 0}, \mathbb{P}^{\tau,x})$ with localization sequence $(T_n)_{n\in\mathbb{N}}$. We consider for $0 \leq s \leq t$ and $F_s \in \hat{\mathcal{F}}_s$:

$$\int_{F_s} N_t^{T_n(\omega)}(\omega)\, d\hat{\mathbb{P}}^{(\tau,x)}(c,\omega) = \int_{\pi_0^{-1}(\hat{\theta}_\tau^{-1}(F_s))} N_t^{T_n(\omega)}(\omega)\, d\mathbb{P}^{\tau,x}(\omega)$$

Since $\pi_0^{-1}(\hat{\theta}_\tau^{-1}(F_s)) \in \mathcal{F}^\tau_{\tau+s}$, the martingale property provides

$$\begin{aligned}\int_{F_s} N_t^{T_n(\omega)}(\omega)\, d\hat{\mathbb{P}}^{(\tau,x)}(c,\omega) &= \int_{\pi_0^{-1}(\hat{\theta}_\tau^{-1}(F_s))} N_s^{T_n(\omega)}(\omega)\, d\mathbb{P}^{\tau,x}(\omega)\\ &= \int_{F_s} N_s^{T_n(\omega)}(\omega)\, d\hat{\mathbb{P}}^{(\tau,x)}(c,\omega).\end{aligned}$$

Hence, the space-time process $\hat{X}$ is a $(d+1)$-dimensional semimartingale for all $\hat{\mathbb{P}}^{(\tau,x)}$, $(\tau,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.
We are now able to calculate the characteristics of $\hat{X}$ with respect to $\hat{\mathbb{P}}^{(\tau,x)}$. Let (B,C,ν) be the characteristics of X defined on $[\tau,\infty)$ with respect to $\mathbb{P}^{\tau,x}$ and the truncation function h. The space-time process $\hat{X}$ takes values in $\mathbb{R}_+ \times E$. Therefore, we consider a truncation function

$$\hat{h} : (\mathbb{R}_+ \times E) \to (\mathbb{R}_+ \times E); (c,x) \mapsto (h_0(c), h(x)),$$

where $h_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded and $h_0 = id$ in a neighborhood of zero.

(1.) In order to calculate the first characteristic, we compute

$$\begin{aligned}\dot{\hat{X}}(\hat{h})_t(c,\omega) = \dot{\hat{X}}(\hat{h})_t(\tau,\omega) : &= \sum_{s\leq t} \Delta\hat{X}_s(\tau,\omega) - \hat{h}(\Delta\hat{X}_s(\tau,\omega))\\ &= \sum_{s\leq t} (0, \Delta X_{\tau+s}(\omega)) - (0, h(\Delta X_{\tau+s}(\omega)))\\ &= \left(0, \sum_{\tau\leq s\leq \tau+t} \Delta X_s(\omega) - h(\Delta X_s(\omega))\right)\\ &= (0, \dot{X}(h)_{\tau+t}(\omega) - \dot{X}(h)_\tau(\omega)), \text{ and}\\ \hat{X}(\hat{h})_t(c,\omega) : &= \hat{X}_t(c,\omega) - \dot{\hat{X}}(\hat{h})_t(c,\omega)\\ &= \hat{X}_t(\tau,\omega) - \dot{\hat{X}}(\hat{h})_t(\tau,\omega)\\ &= (\tau+t, X(h)_{\tau+t}(\omega) - \dot{X}(h)_\tau(\omega)).\end{aligned}$$

Since $X(h)$ is a special semimartingale on $[\tau,\infty)$ with respect to $\mathbb{P}^{\tau,x}$, we see as above that $(X(h)_{\tau+t} - \dot{X}(h)_\tau)_{t\geq 0}$ is a special semimartingale with respect to $\hat{\mathbb{P}}^{(\tau,x)}$. Let $M_t(c,\omega) := \tau$ and $V_t \equiv t$, then M is a martingale with respect to $\hat{\mathbb{P}}^{(\tau,x)}$ and V is an increasing, predictable process. Thus, we conclude that the first characteristic of $\hat{X}$ is given by $((t, B_{\tau+t}))_{t\geq 0}$.

(2.) We consider $\hat{C}^{ij} := \langle \hat{X}^{i,c}, \hat{X}^{j,c}\rangle$. This is for $i,j \in \{2,...,d+1\}$

$$\hat{C}_t^{1j}(c,\omega) = \langle (\tau+t)^c, X_t(\omega)^{j,c}\rangle = \langle \tau, X_t(\omega)^{j,c}\rangle = 0$$

$$\hat{C}_t^{i1}(c,\omega) = \langle X_t(\omega)^{i,c}, (\tau+t)^c\rangle = \langle X_t(\omega)^{i,c}, \tau\rangle = 0,$$

since $\tau X_t(\omega)^{j,c}$ is a $\hat{\mathbb{P}}^{(\tau,x)}$-local martingale. Moreover, we have

$$\hat{C}_t^{ij}(c,\omega) = \langle X_{\tau+t}(\omega)^{i,c}, X_{\tau+t}(\omega)^{j,c}\rangle = C_{\tau+t}^{ij}(\omega).$$

(3.) Let μ^X be the integer-valued random measure of X and ν be the third characteristic of X, i.e., the compensator of μ^X. For $\hat{\omega} = (c,\omega) \in \hat{\Omega}, T \in \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$ we define

$$\hat{\nu}((c,\omega),T,B) := \nu(\omega, T+\tau, B_0),$$

where B_0 is the 0-slice of B. This yields for any non-negative predictable function $\hat{W}$ on $\hat{\Omega} \times \mathbb{R}_+ \times \hat{E}$ that

$$\begin{aligned}\hat{\mathbb{E}}^{(\tau,x)}\left[\hat{W} * \hat{\nu}_\infty(\hat{\omega})\right] &= \hat{\mathbb{E}}^{(\tau,x)}\left[\int_{\mathbb{R}_+\times\hat{E}} \hat{W}(\hat{\omega},s,\hat{x})\ \hat{\nu}(\hat{\omega};ds,d\hat{x})\right]\\ &= \mathbb{E}^{\tau,x}\left[\int_{\mathbb{R}_+\times\hat{E}} \hat{W}((\tau,\omega),s,\hat{x})\ \hat{\nu}((\tau,\omega);ds,d\hat{x})\right]\\ &= \mathbb{E}^{\tau,x}\left[\int_{[\tau,\infty)\times\mathbb{R}^d} \hat{W}((\tau,\omega),s-\tau,(0,x))\ \nu(\omega;ds,dx)\right]\\ &= \mathbb{E}^{\tau,x}\left[\int_{[\tau,\infty)\times\mathbb{R}^d} W(\omega,s,x)\ \nu(\omega;ds,dx)\right],\end{aligned}$$

where $W(\omega,s,x) := \hat{W}((\tau,\omega),s-\tau,(0,x))$ is a non-negative predictable function on $\Omega \times [\tau,\infty) \times E$. Since ν is the compensator of μ^X it holds true that

$$\begin{aligned}&\mathbb{E}^{\tau,x}\left[\int_{[\tau,\infty)\times\mathbb{R}^d} W(\omega,s,x)\ \nu(\omega;ds,dx)\right]\\ &= \mathbb{E}^{\tau,x}\left[\sum_{s\geq\tau} W(\omega,s,\Delta X_s(\omega))1_{\{\Delta X_s(\omega)\neq 0\}}\right]\\ &= \mathbb{E}^{\tau,x}\left[\sum_{s\geq\tau} \hat{W}((\tau,\omega),s-\tau,(0,\Delta X_s(\omega)))1_{\{(0,\Delta X_s(\omega))\neq 0\}}\right]\\ &= \hat{\mathbb{E}}^{(\tau,x)}\left[\sum_{s\geq 0} \hat{W}(\hat{\omega},s,\Delta\hat{X}_s(\hat{\omega}))1_{\{\Delta\hat{X}_s(\hat{\omega})\neq 0\}}\right].\end{aligned}$$

Thus, we see that $\hat{\nu}$ is the compensator of $\hat{X}$, and, therefore, the third characteristic of $\hat{X}$.

□

Before we state the main theorem of this section, the following lemma provides that non-homogeneous Itô process are indeed a generalization of rich Feller evolution processes:

Lemma 4.1.6. *Let $(X_t)_{t\geq 0}$ be a rich Feller evolution process on $C_\infty(\mathbb{R}^d)$ with the family of generators $(A_s)_{s\geq 0}$ and symbol of the generators $p : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that $p(\cdot, x, \xi)$ is continuous for all $x, \xi \in \mathbb{R}^d$. Then X is a non-homogeneous Itô process.*

Proof. Let $(X_t)_{t\geq 0}$ be a rich Feller evolution system on $C_\infty(\mathbb{R}^d)$. Analogously to the proof of Theorem 3.1 of [67], but using the non-homogeneous version of Dynkin's formula as mentioned in [59], we show that $(X_t)_{t\geq 0}$ is a semimartingale with characteristics (B, C, ν). Since it is well-known that X is a non-homogeneous Markov process, it suffices to show that the characteristics are of the form mentioned in Definition 4.1.2:
By Theorem 3.2 and Lemma 3.7 of [10] the space-time process $\hat{X}$ associated to X is a rich Feller process on $C_\infty(\mathbb{R}_+ \times \mathbb{R}^d)$. Hence, Theorem 3.10 of [67] provides that the characteristics $(\hat{B}, \hat{C}, \hat{\nu})$ of $\hat{X}$ are of the form

$$\begin{aligned}
\hat{B}_t^{(i)} &= \int_0^t \ell^{(i)}(\hat{X}_s)\, ds, \quad \hat{\mathbb{P}}^{(\tau,x)}\text{-a.s.} \\
\hat{C}_t^{(ij)} &= \int_0^t Q^{(ij)}(\hat{X}_s)\, ds, \quad \hat{\mathbb{P}}^{(\tau,x)}\text{-a.s.} \\
\hat{\nu}(\ ; dt, d\hat{x}) &= dtN(\hat{X}_t, d\hat{x}), \quad \hat{\mathbb{P}}^{(\tau,x)}\text{-a.s.,}
\end{aligned}$$

and, therefore, we conclude with equations (4.5) to (4.7) that

$$\begin{aligned}
B_t^{(i)} &= \int_0^{t-\tau} \ell^{(i)}(\hat{X}_s)\, ds = \int_0^{t-\tau} \ell^{(i)}(s+\tau, X_{s+\tau})\, ds = \int_\tau^t \ell^{(i)}(s, X_s)\, ds, \quad \mathbb{P}^{\tau,x}\text{-a.s.,} \\
C_t^{(ij)} &= \int_0^{t-\tau} Q^{(ij)}(\hat{X}_s)\, ds = \int_0^{t-\tau} Q^{(ij)}(s+\tau, X_{s+\tau})\, ds = \int_\tau^t Q^{(ij)}(s, X_s)\, ds, \quad \mathbb{P}^{\tau,x}\text{-a.s.,} \\
\nu(\ ; dt, dx) &= dtN(X_t, dx), \quad \mathbb{P}^{\tau,x}\text{-a.s.,}
\end{aligned}$$

for $t \geq \tau$. □

Theorem 4.1.7. *Let X be a non-homogeneous Itô process and let $\ell = (\ell^{(j)})_{1\leq j\leq d}$ and $Q = (q^{(ik)})_{1\leq j,k\leq d}$ be continuous, N be such that the function*

$$(s, x) \mapsto \int_{\{y\neq 0\}} (1 \wedge y^2)\ N_s(x, dy)$$

is continuous. In this case the time-dependent symbol exists and equals

$$p(\tau, x, \xi) = -i\ell(\tau, x)'\xi + \frac{1}{2}\xi' Q(\tau, x)\xi - \int_{y\neq 0} \left(e^{iy'\xi} - 1 - iy'\xi\chi(y)\right)\ N_\tau(x, dy).$$

Proof. Let X be a non-homogeneous Itô process, and let $\hat{X}$ be the associated space-time process. The characteristics of $\hat{X}$ are given by

$$\begin{aligned}
\hat{B}_t^{(1)} &= t = \int_0^t 1\, ds, \\
\hat{B}_t^{(i)} &= \int_\tau^{t+\tau} \ell^{(i-1)}(s, X_s)\, ds = \int_0^t \ell^{(i-1)}(s+\tau, X_{s+\tau})\, ds = \int_0^t \ell^{(i-1)}(\hat{X}_s)\, ds, \\
\hat{C}_t^{(1j)} &= \hat{C}_t^{(i1)} = \int_0^t 0\, ds,
\end{aligned}$$

$$\hat{C}_t^{(ij)} = \int_0^t Q^{(i-1,j-1)}(\hat{X}_s)\, ds,$$

for $i, j \in \{2, ..., d+1\}$. All of the equations above are meant $\hat{\mathbb{P}}^{(\tau,x)}$-a.s.. For $T \in \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(\hat{E})$ we have

$$\begin{aligned}
\hat{\nu}(\ ;T,B) &= \nu(\ ;T+\tau, B_0) \\
&= \int_{T+\tau} N_s(X_s, B_0)\, ds \\
&= \int_T N_{s+\tau}(X_{s+\tau}, B_0)\, ds \\
&= \int_T \hat{N}((s+\tau, X_{s+\tau}), B)\, ds \\
&= \int_T \hat{N}(\hat{X}_s, B)\, ds,
\end{aligned}$$

where $\hat{N}((c,\omega),B) = N_c(\omega, B_0)$ is a transition kernel from $\hat{\Omega} \times \hat{E}$ to $\mathbb{R}^d$. Consequently, $\hat{X}$ is an Itô process and we denote by $\hat{p}(\hat{x}, \hat{\xi})$ its (homogeneous) probabilistic symbol.
For the stopping time $\hat{\sigma} := \inf\{h \geq 0 : \|\hat{X}_h - (\tau, x)\| > R\}$ it holds true that

$$\begin{aligned}
\hat{\sigma}(\hat{\theta}_\tau(\pi_0(\omega))) &= \inf\{h \geq 0 : \|\hat{X}_h(\hat{\theta}_\tau(\pi_0(\omega))) - (\tau, x)\| > R\} \\
&= \inf\{h \geq 0 : \|\hat{X}_h(\tau, \omega) - (\tau, x)\| > R\} \\
&= \inf\{h \geq 0 : \|(\tau + h, X_{\tau+h}(\omega)) - (\tau, x)\| > R\} \\
&= \inf\{h \geq 0 : \|(h, X_{\tau+h}(\omega) - x)\| > R\} \\
&= R \wedge \inf\{h \geq 0 : \|X_{\tau+h}(\omega) - x\| > R\} \\
&= R \wedge (\sigma - \tau),
\end{aligned}$$

where $\pi_0 : \Omega \to \hat{\Omega}; \omega \mapsto (0, \omega)$. We compute for $\hat{\xi} = (0, \xi)$, $\hat{\ell} = (1, \ell)$, $\hat{Q} = (\hat{Q}^{(ij)})$ with $\hat{Q}^{(1,j)} = \hat{Q}^{(i,1)} = 0$ and $\hat{Q}^{(ij)} = Q^{((i-1),(j-1))}$ for $i, j \in \{1, ..., d+1\}$:

$$\begin{aligned}
p(\tau, x, \xi) &= -\lim_{h \downarrow 0} \mathbb{E}^{\tau,x} \left(\frac{e^{i(X^\sigma_{\tau+h} - x)'\xi} - 1}{h} \right) \\
&= -\lim_{h \downarrow 0} \mathbb{E}^{\tau,x} \left(\frac{e^{i(X_{\tau+(h\wedge(\sigma-\tau))} - x)'\xi} - 1}{h} \right) \\
&= -\lim_{\substack{h \downarrow 0 \\ h<R}} \mathbb{E}^{\tau,x} \left(\frac{e^{i(X_{\tau+(h\wedge(\sigma-\tau)\wedge R)} - x)'\xi} - 1}{h} \right) \\
&= -\lim_{\substack{h \downarrow 0 \\ h<R}} \mathbb{E}^{\tau,x} \left(\frac{e^{i(\hat{X}_{h\wedge(\hat{\sigma}\circ\hat{\theta}_\tau\circ\pi_0)} \circ\hat{\theta}_\tau\circ\pi_0 - (\tau,x))'\hat{\xi}} - 1}{h} \right) \\
&= -\lim_{\substack{h \downarrow 0 \\ h<R}} \hat{\mathbb{E}}^{(\tau,x)} \left(\frac{e^{i(\hat{X}^{\hat{\sigma}}_h - (\tau,x))'\hat{\xi}} - 1}{h} \right) \\
&= \hat{p}((\tau, x), \hat{\xi}) \\
&= -i\ell(\tau,x)'\hat{\xi} + \frac{1}{2}\hat{\xi}' c(\tau,x)\hat{\xi} - \int_{\hat{y}\neq 0} \left(e^{i\hat{y}'\hat{\xi}} - 1 - i\hat{y}'\hat{\xi}\chi(\hat{y}) \right)\ \hat{N}((\tau,x), dy)\delta_0(du)
\end{aligned}$$

$$= -i\ell(\tau,x)'\xi + \frac{1}{2}\xi' c(\tau,x)\xi - \int_{y\neq 0}\left(e^{iy'\xi} - 1 - iy'\xi\chi(y)\right)\, N_\tau(x,dy).$$

□

We have observed, for instance in Example 4.1.4 that a process does not necessarily need to be a non-homogeneous Itô process to possess a time-dependent symbol. However, the processes considered up to this point are quasi-left continuous. We will see in the subsequent example that, when dropping the assumption of quasi-left continuity, the time-dependent symbol does not contain the same information on the process as before. This is not unexpected, as we have encountered similar situations in the homogeneous case.

Example 4.1.8. Let us consider the following example:

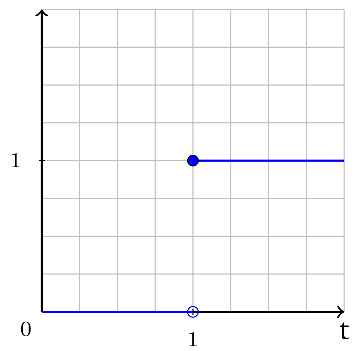

Fig. 4.1.: Sample path of a non-homogeneous Markov process starting in zero.

By adding all other starting points in a Markovian manner (c.f. [70]) and considering the truncation function $h(x) = x1_{\{|x|\leq 0.5\}}$ we receive a non-homogeneous Markov semimartingale with characteristics $B \equiv 0, C \equiv 0$ and $\nu(dt,dx) = \delta_1(dt)\delta_1(dx)$. In this case, the time-dependent symbol exists and is given by

$$p(\tau,x,\xi) = -\lim_{h\downarrow 0}\mathbb{E}^{\tau,x}\left(\frac{e^{i(X^\sigma_{\tau+h}-x)'\xi} - 1}{h}\right) \equiv 0.$$

Nevertheless, the symbol does not provide any information regarding the process.

4.2. Maximal Inequalities and Time-Dependent Blumenthal-Getoor Indices

Similar to the homogeneous case, we want to utilize the time-dependent symbol to derive maximal inequalities for a non-homogeneous Itô process X with time-dependent symbol p. As we have pointed out in Chapter 3, for homogeneous processes this is achieved by the Blumenthal-Getoor indices (cf. Definition 3.2.1) which rely on the symbol of the process. In this framework, the non-homogeneous equivalents of the growth (IG) and sector condition (IS) of the symbol play an important role:

$$\|q(s,x,\xi)\| \le c(1+\|\xi\|^2), \tag{IG}$$

$$|\operatorname{Im}(q(s,x,\xi)| \le c_0 \ \operatorname{Re}(p(s,x,\xi)) \tag{IS}$$

for every $s \ge 0, x, \xi \in \mathbb{R}^d$ and $c, c_0 > 0$.
Specifically, we want to use the maximal inequalities to examine the paths of the process, including the asymptotic behavior of the sample paths, the p-variation of the paths and the existence of the exponential moments of the process. The following indices generalize the Blumenthal-Getoor indices:

Definition 4.2.1. The quantities

$$\beta_0 := \sup\left\{\lambda \ge 0 : \limsup_{R\to\infty} R^\lambda H(R) = 0\right\}$$
$$\underline{\beta_0} := \sup\left\{\lambda \ge 0 : \liminf_{R\to\infty} R^\lambda H(R) = 0\right\}$$
$$\overline{\delta_0} := \sup\left\{\lambda \ge 0 : \limsup_{R\to\infty} R^\lambda h(R) = 0\right\}$$
$$\delta_0 := \sup\left\{\lambda \ge 0 : \liminf_{R\to\infty} R^\lambda h(R) = 0\right\}$$

are called *time-dependent indices of X in the starting point*, where

$$H(R) := \sup_{s\ge 0} \sup_{y\in\mathbb{R}^d} \sup_{\|\varepsilon\|\le 1} \left|p\left(s,y,\frac{\varepsilon}{R}\right)\right| \text{ and} \tag{4.9}$$

$$h(R) := \inf_{s\ge 0} \inf_{y\in\mathbb{R}^d} \sup_{\|\varepsilon\|\le 1} \operatorname{Re}\left(p\left(s,y,\frac{\varepsilon}{4\kappa R}\right)\right) \tag{4.10}$$

with $\kappa = (4\arctan(1/2c_0))^{-1}$, where c_0 comes from the sector condition (IS).

Definition 4.2.2. Let $\tau \in \mathbb{R}_+, x \in \mathbb{R}^d$ and $R > 0$. The quantities

$$\beta_\infty^{\tau,x} := \inf\left\{\lambda > 0 : \limsup_{R\to 0} R^\lambda H(\tau,x,R) = 0\right\}$$
$$\underline{\beta_\infty^{\tau,x}} := \inf\left\{\lambda > 0 : \liminf_{R\to 0} R^\lambda H(\tau,x,R) = 0\right\}$$
$$\overline{\delta_\infty^{\tau,x}} := \inf\left\{\lambda > 0 : \limsup_{R\to 0} R^\lambda h(\tau,x,R) = 0\right\}$$
$$\delta_\infty^{\tau,x} := \inf\left\{\lambda > 0 : \liminf_{R\to 0} R^\lambda h(\tau,x,R) = 0\right\}$$

are the *time-dependent indices of X at infinity*, where

$$H(\tau,x,R) := \sup_{\|y-x\|\le 2R} \sup_{\|\varepsilon\|\le 1} \left|p\left(\tau,y,\frac{\varepsilon}{R}\right)\right| \text{ and} \tag{4.11}$$

$$h(\tau, x, R) := \inf_{\|y-x\|\leq 2R} \sup_{\|\varepsilon\|\leq 1} \mathrm{Re}\left(p\left(\tau, y, \frac{\varepsilon}{4\kappa R}\right)\right) \tag{4.12}$$

with $\kappa = (4\arctan(1/2c_0))^{-1}$, where c_0 comes from the sector condition (IS).

The proofs of the previous section crucially rely on the space-time process to transfer properties of homogeneous Markov processes to the non-homogeneous framework. However, when deriving properties like the asymptotic behavior of sample paths or maximal inequalities, we do not expect the space-time process to be of much use. This is due to the fact that if we utilize the space-time process, a deterministic drift with slope 1 is added. This obscures the path-behavior of the original process.
For this section, we assume all characteristics encountered to be with respect to the truncation function $h = id \cdot \chi$, where $\chi \in C_c^\infty(\mathbb{R}^d)$ is a symmetric cut-off function with

$$1_{B_R(0)} \leq \chi \leq 1_{B_{2R}(0)}$$

where $R > 0$.

Theorem 4.2.3. *Let X be a non-homogeneous Itô process with characteristics as in Theorem 4.1.7. In this case we have*

$$\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq \tau+t} \|X_s - x\| \geq R\right) \leq c_d \cdot t \cdot \sup_{\tau< s\leq \tau+t} H(s, x, R) \tag{4.13}$$

for $t \geq 0$, $R > 0$ and a constant $c_d > 0$ which only depends on the dimension d. If, in addition, (IS) *holds true, we have*

$$\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq \tau+t} \|X_s - x\| < R\right) \leq c_k \cdot \frac{1}{t} \cdot \frac{1}{\inf_{\tau< s\leq \tau+t} h(s, x, R)} \tag{4.14}$$

where $c_k > 0$ only depends on c_0 of condition (IS).

Remark 4.2.4. Throughout the following proof we often consult Lemma 5.2 of [69] although it is a statement for time-homogeneous symbols. A closer consideration shows that the statement of Lemma 5.2 holds alike for the time-dependent probabilistic symbol.

Proof. The proof of (4.13) closely follows the proof of Proposition 3.10 of [69], but takes the time component of the time-dependent symbol into account. We omit the proof of (4.14) since it is generalized from the proof of Lemma 6.3 of [66] analogously to the following generalization. However, let us mention that one has to use Dynkin's formula for non-homogeneous processes as stated in [59] when Corollary 3.6 is utilized in [66].
Let X be a non-homogeneous Itô process such that the differential characteristics (ℓ, Q, ν) of X are continuous. We show that for $R > 0$ as before, $S > 0$ and

$$\sigma := \sigma_S^{\tau,x} := \inf\{t \geq \tau : \|X_t - x\| > S\}$$

we have

$$\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t} \|X_s^\sigma - x\| \geq 2R\right) \leq c_d \cdot (t-\tau) \cdot \sup_{\tau\leq s\leq t} \sup_{\|y-x\|\leq S} \sup_{\|\varepsilon\|\leq 1} \left|p\left(s, y, \frac{\varepsilon}{2R}\right)\right| \tag{4.15}$$

where $c_d = 4d + 16\widetilde{c}_d$ and $t \geq \tau$. We introduce the stopping time

$$\tau_R := \inf\{t \geq \tau : \|\Delta X_t^\sigma\| > R\},$$

as the first time after τ the jumps of X^σ exceed R and estimate the following

$$\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|X_s^\sigma - x\| \geq 2R\right) \leq \mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|X_s^\sigma - x\| \geq 2R, \tau_R > t\right) + \mathbb{P}^{\tau,x}\Big(\tau_R \leq t\Big). \tag{4.16}$$

Let us examine the first summand of the right-hand side:

We split the first term in (4.16) in order to get control over the big jumps. For $t \geq \tau$ we define

$$\dot{X}_t := X_t - \sum_{\tau\leq s\leq t} \Delta X_s(1 - \chi(\Delta X_s)).$$

The stopped process $\dot{X}^\sigma$ is a special semimartingale on $[\tau,\infty)$ with characteristics

$$\begin{aligned}
\dot{B}_t^{(i)} &= \int_\tau^{t\wedge\sigma} b^i(s, X_s)\,ds, \quad \mathbb{P}^{\tau,x}\text{-a.s.}\\
\dot{C}_t^{(ij)} &= \int_\tau^{t\wedge\sigma} c^{ij}(s, X_s)\,ds, \quad \mathbb{P}^{\tau,x}\text{-a.s.}\\
\dot{\nu}(;dt,dx) &= \chi(y)1_{[\![0,\sigma]\!]}(t)\;N(X_t,dy)\,dt, \quad \mathbb{P}^{\tau,x}\text{-a.s..}
\end{aligned}$$

Now let $u = (u_1, ..., u_d)' : \mathbb{R}^d \to \mathbb{R}^d$ be such that $u_j \in C_b^2(\mathbb{R}^d)$ is 1-Lipschitz continuous, u_j only depends on $x^{(j)}$ and is zero in zero for $j = 1, ..., d$. We define the auxiliary process $\dot{M}$ for $t \geq \tau$ by

$$\dot{M}_t := u(\dot{X}_t^\sigma - x) - \int_\tau^{t\wedge\sigma} \sum_{j=1}^d F_s^{(j)}\,ds$$

where

$$\begin{aligned}
F_s^{(j)} &= \partial_j u(\dot{X}_{s-} - x)\ell^{(j)}(s, X_{s-}) - \frac{1}{2}\partial_{jj}^2 u(\dot{X}_{s-} - x)Q^{(jj)}(s, X_{s-})\\
&\quad - \int_{\{z\neq 0\}} \Big(u(\dot{X}_{s-} - x + z) - u(\dot{X}_{s-} - x) - \chi(z)z^{(j)}\partial_j u(\dot{X}_{s-} - x)\Big)\chi(z)\;N_s(X_{s-}, dz).
\end{aligned} \tag{4.17}$$

$\dot{M}$ is a $\mathbb{P}^{\tau,x}$-local martingale on $[\tau,\infty)$, by [36] Theorem II.2.42. Applying Lemma 3.7 of [67] we obtain with (IG):

$$\left|F_s^{(j)}\right| \leq const \cdot \sum_{0\leq|\alpha|\leq 2} \|\partial^\alpha u\|_\infty$$

since $u_j \in C_b^2(\mathbb{R}^d)$. Let us mention that although Lemma 3.7 of [67] considers homogeneous diffusion with jumps the proof works alike for non-homogeneous Itô processes. In particular, since $\dot{M}$ is uniformly bounded, it is a L^2-martingale on $[\tau, t]$. We define

$$D := \left\{\omega \in \Omega : \int_\tau^{t\wedge\sigma(\omega)} \|F_s(\omega)\|\;ds \leq R\right\}$$

and obtain for $t \geq \tau$:

$$\begin{aligned} &\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|X_s^\sigma - x\| \geq 2R, \tau_R > t\right) \\ &\leq \mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|X_s^\sigma - x\| \geq 2R, \tau_R > t, D\right) + \mathbb{P}^{\tau,x}(D^c). \end{aligned} \tag{4.18}$$

For the first summand of (4.18) we derive for $t \geq \tau$:

$$\begin{aligned} &\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|u(X_s^\sigma - x)\| \geq 2R, \tau_R > t, D\right) \\ &= \mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|u(\dot{X}_s^\sigma - x)\| \geq 2R, \tau_R > t, D\right) \\ &\leq \mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|u(X_s^\sigma - x)\| - \int_\tau^{t\wedge\sigma} F_s\, ds \geq R, \tau_R > t, D\right) \\ &\leq \mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t}\|\dot{M}_{s\wedge\sigma}\| \geq R\right) \\ &= \mathbb{P}^{\tau,x}\left(\sup_{0\leq s\leq t-\tau}\|\dot{M}_{(s+\tau)\wedge\sigma}\| \geq R\right) \\ &\leq \frac{1}{R^2}\mathbb{E}^{\tau,x}\left(\left\|\dot{M}_t^\sigma\right\|^2\right) \\ &\leq \frac{1}{R^2}\sum_{j=1}^d \mathbb{E}^{\tau,x}\left(\left[\dot{M}^{(j)}, \dot{M}^{(j)}\right]_t^\sigma\right) \\ &\leq \frac{1}{R^2}\sum_{j=1}^d \mathbb{E}^{\tau,x}\left([\dot{X}^{(j)}, \dot{X}^{(j)}]_t^\sigma\right), \end{aligned}$$

where we use Doob's inequality for the martingale $\dot{M}^\sigma$ and the Lipschitz property of u in combination with Corollary II.3 of [54]. Since

$$\begin{aligned} &\mathbb{E}^{\tau,x}\left([\dot{X}^{(j)}, \dot{X}^{(j)}]_t^\sigma\right) \\ &= \mathbb{E}^{\tau,x}\left(\left\langle \dot{X}_\cdot^{(j),c}, \dot{X}_\cdot^{(j),c}\right\rangle_t^\sigma\right) + \mathbb{E}^{\tau,x}\left(\int_\tau^{t\wedge\sigma}\int_{\{z\neq 0\}} (z^{(j)})^2\chi(z)^2\, N_s(X_s, dz)\, ds\right) \end{aligned}$$

we obtain

$$\begin{aligned} &\mathbb{P}^{\tau,x}\Big(\sup_{\tau\leq s\leq t}\|u(X_s^\sigma - x)\| \geq 2R, \tau_R > t, D\Big) \\ &\leq \frac{1}{R^2}\sum_{j=1}^d \mathbb{E}^{\tau,x}\int_\tau^{t\wedge\sigma} Q^{(jj)}(s, X_s)\, ds + \mathbb{E}^{\tau,x}\int_\tau^{t\wedge\sigma}\int_{\{z\neq 0\}} \frac{\|z\|^2}{R^2}\chi(z)^2\, N_s(X_s, z)\, ds \\ &\leq 4\sum_{j=1}^d \mathbb{E}^{\tau,x}\int_\tau^{t\wedge\sigma}\left(\frac{e_j}{2R}'Q(s, X_s)\frac{e_j}{2R}\right) ds + 4^2\mathbb{E}^{\tau,x}\int_\tau^{t\wedge\sigma}\int_{\{z\neq 0\}}\left(\left\|\frac{z}{2R}\right\|^2 \wedge 1\right) N_s(X_s, dz) ds \\ &\leq 4\sum_{j=1}^d \mathbb{E}^{\tau,x}\sup_{\tau<s<t\wedge\sigma}\left(\frac{e_j}{2R}'Q(s, X_s)\frac{e_j}{2R}\right) + 4^2\mathbb{E}^{\tau,x}\sup_{\tau<s<t\wedge\sigma}\int_{\{z\neq 0\}}\left(\left\|\frac{z}{2R}\right\|^2 \wedge 1\right) N_s(X_s, dz) \end{aligned}$$

$$
\begin{aligned}
&\leq 4(t-\tau)\sum_{j=1}^{d} \sup_{\tau<s<t\wedge\sigma} \sup_{\|y-x\|\leq S} \operatorname{Re} p\left(s,y,\frac{e_j}{2R}\right) \\
&+ 4^2(t-\tau) \sup_{\tau<s<t\wedge\sigma} \sup_{\|y-x\|\leq S} \int_{\{z\neq 0\}} \left(\left\|\frac{z}{2R}\right\|^2 \wedge 1\right) N_s(y,dz) \\
&\leq 4(t-\tau)d \sup_{\tau<s<t\wedge\sigma} \sup_{\|y-x\|\leq S} \sup_{\|\varepsilon\|\leq 1} \left|p\left(s,y,\frac{\varepsilon}{2R}\right)\right| \\
&+ 4^2 t \sup_{\tau<s<t\wedge\sigma} \sup_{\|y-x\|\leq S} \widetilde{c}_d \sup_{\|\varepsilon\|\leq 1} \left|p\left(s,y,\frac{\varepsilon}{2R}\right)\right|
\end{aligned}
$$

where we apply Lemma 5.2 of [69] on the second term. By choosing a sequence $(u_n)_{n\in\mathbb{N}}$ of functions of the type described above which tends to the identity in a monotonous way, we obtain

$$
\begin{aligned}
&\mathbb{P}^{\tau,x}\left(\sup_{\tau\leq s\leq t} \|X_s^\sigma - x\| \geq 2R, \tau_R > t, D\right) \\
&\leq (4d + 4^2\widetilde{c}_d)(t-\tau) \sup_{\tau<s<t\wedge\sigma} \sup_{\|y-x\|\leq S} \sup_{\|\varepsilon\|\leq 1} \left|p\left(s,y,\frac{\varepsilon}{2R}\right)\right|.
\end{aligned} \tag{4.19}
$$

Now let us consider the term $\mathbb{P}^{\tau,x}(D^C)$ of (4.18). The Markov inequality provides

$$
\mathbb{P}^{\tau,x}(D^c) = \mathbb{P}^{\tau,x}\left(\int_\tau^{t\wedge\sigma} \|F_s\| \ ds > R\right) \leq \frac{1}{R}\sum_{j=1}^{d} \mathbb{E}^{\tau,x}\Big(\int_\tau^{t\wedge\sigma} \left|F_s^{(j)}\right| \ ds\Big)
$$

Again we choose a sequence $(u_n)_{n\in\mathbb{N}}$ of functions as described in (4.17), but this time it is important that the first and second derivatives are uniformly bounded. Since the u_n converge to the identity, the first partial derivatives tend to 1 and the second partial derivatives to 0. In the limit ($n\to\infty$) we obtain

$$
\frac{1}{R}\sum_{j=1}^{d} \mathbb{E}^{\tau,x}\Big(\int_\tau^{t\wedge\sigma} \left|F_s^{(j)}\right| \ ds\Big) \tag{4.20}
$$

$$
\leq \frac{1}{R}\sum_{j=1}^{d} \mathbb{E}^{\tau,x} \int_\tau^{t\wedge\sigma} \left| \ell^{(j)}(s,X_s) + \int_{\{z\neq 0\}} (-z^{(j)}\chi(z) + (\chi(z))^2 z^{(j)}) \ N_s(X_s,dz)\right| \ ds
$$

$$
\leq 2\sum_{j=1}^{d} \mathbb{E}^{\tau,x} \int_\tau^{t\wedge\sigma} \left| \frac{\ell^{(j)}(s,X_s)}{2R} + \int_{\{z\neq 0\}} \sin\left(\frac{z'e_j}{2R}\right) - \frac{z^{(j)}\chi(z)}{2R} \ N_s(X_s,dz)\right| \ ds \tag{4.21}
$$

$$
+ 2\sum_{j=1}^{d} \mathbb{E}^{\tau,x} \int_\tau^{t\wedge\sigma} \left| \int_{\{z\neq 0\}} \frac{(\chi(z))^2 z^{(j)}}{2R} - \sin\left(\frac{z'e_j}{2R}\right) \ N(X_s,dz)\right| \ ds. \tag{4.22}
$$

For term (4.21) we get

$$
\begin{aligned}
&2\sum_{j=1}^{d} \mathbb{E}^{\tau,x} \int_\tau^{t\wedge\sigma} \left| \frac{\ell(s,X_s)'e_j}{2R} + \int_{\{z\neq 0\}} \sin\left(\frac{z'e_j}{2R}\right) - \frac{z'e_j\chi(z)}{2R} \ N_s(X_s,dz)\right| \ ds \\
&\leq 2(t-\tau)\sum_{j=1}^{d} \sup_{\tau<s\leq t\wedge\sigma} \mathbb{E}^{\tau,x} \left| \frac{\ell(s,X_s)'e_j}{2R} + \int_{\{z\neq 0\}} \sin\left(\frac{z'e_j}{2R}\right) - \frac{z'e_j\chi(z)}{2R} \ N_s(X_s,dz)\right|
\end{aligned}
$$

$$\leq 2(t-\tau)d \sup_{\tau<s\leq t} \sup_{\|y-x\|\leq S} \sup_{\|\varepsilon\|\leq 1} \left|\operatorname{Im} p\left(s,y,\frac{\varepsilon}{2R}\right)\right| \tag{4.23}$$

and for term (4.22)

$$\begin{aligned}
&2\sum_{j=1}^{d} \mathbb{E}^{\tau,x} \int_{\tau}^{t\wedge\sigma} \left| \int_{\{z\neq 0\}} \frac{(\chi(z))^2 z' e_j}{2R} - \sin\left(\frac{z'e_j}{2R}\right) \ N_s(X_s,dz)\right| \ ds \\
&\leq 2\sum_{j=1}^{d} \mathbb{E}^{\tau,x} \int_{\tau}^{t\wedge\sigma} \left| \int_{B_{2R}(0)\setminus\{0\}} 1-\cos\left(\frac{z'e_j}{2R}\right) \ N_s(X_s,dz)\right| \\
&\quad + \left| \int_{B_{2R}(0)^c} 1 \ N_s(X_s,dz)\right| \ ds \\
&\leq 2(t-\tau)d \sup_{\tau<s\leq t} \sup_{\|y-x\|\leq S} \sup_{\|\varepsilon\|\leq 1} \operatorname{Re} p\left(s,y,\frac{\varepsilon}{2R}\right) &(4.24)\\
&\quad + 2^2(t-\tau)d \sup_{\tau<s\leq t} \sup_{\|y-x\|\leq S} \widetilde{c}_d \sup_{\|\varepsilon\|\leq 1} \left|p\left(s,y,\frac{\varepsilon}{2R}\right)\right|, &(4.25)
\end{aligned}$$

where we utilize Lemma 5.2 of [69] for the second term. It remains to deal with the second term of (4.16). Let $\delta > 0$ and $m : \mathbb{R} \to]1, 1+\delta[$ a strictly monotone increasing auxiliary function. Since $m \geq 1$ and since we have at least one jump larger than R on $\{\tau_R \leq t\}$, we obtain

$$\begin{aligned}
\mathbb{P}^{\tau,x}(\tau_R \leq t) &\leq \mathbb{P}^{\tau,x}\left(\int_{\tau}^{t}\int_{\|z\|\geq R} m(\|z\|) \ \mu^{X^\sigma}(\cdot;ds,dz) \geq m(R)\right) \\
&\leq \frac{1}{m(R)}\mathbb{E}^{\tau,x}\left(\int_{0}^{t}\int_{\|z\|\geq R} m(\|z\|)1_{[\![0,\sigma]\!]}(s) \ \mu^{X}(\cdot;ds,dz)\right) \\
&= \frac{1}{m(R)}\mathbb{E}^{\tau,x}\left(\int_{\tau}^{t}\int_{\{z\neq 0\}} m(\|z\|)1_{[\![0,\sigma[\![}(s)1_{B_R(0)^c}(z) \ N_s(X_s,dz) \ ds\right) \\
&\leq (1+\delta)(t-\tau) \sup_{\tau<s\leq t\wedge\sigma} \mathbb{E}^{\tau,x}(N_s(X_s,B_R(0)^c)) \\
&\leq (1+\delta)(t-\tau) \sup_{\tau<s\leq t} \sup_{\|y-x\|\leq S} N_s(y,B_R(0)^c) \\
&\leq (1+\delta)4(t-\tau) \sup_{\tau<s\leq t} \sup_{\|y-x\|\leq S} \int_{\{z\neq 0\}} \left(\left\|\frac{z}{2R}\right\|^2 \wedge 1\right) \ N_s(y,dz)
\end{aligned}$$

because $m(\|z\|)1_{[\![0,\sigma[\![}(s)1_{B_R(0)^c}(z)$ is in class F_p^1 of Ikeda and Watanabe (see [28], Section II.3). Since δ can be chosen arbitrarily small we obtain by Lemma 5.2 of [69]

$$\mathbb{P}^{\tau,x}(\tau_R \leq t) \leq 4(t-\tau) \sup_{\tau<s\leq t} \sup_{\|y-x\|\leq S} \widetilde{c}_d \sup_{\|\varepsilon\|\leq 1} \left|p\left(s,y,\frac{\varepsilon}{2R}\right)\right|. \tag{4.26}$$

Combining (4.19), (4.23), (4.25) and (4.26) we obtain (4.15). For the particular case $\sigma = \sigma^x_{3\widetilde{R}}$, we have

$$\left\{\sup_{\tau\leq s\leq t} \|u(X_s^\sigma - x)\| \geq 2\widetilde{R}\right\} = \left\{\sup_{\tau\leq s\leq t} \|u(X_s - x)\| \geq 2\widetilde{R}\right\},$$

and, therefore, for every $\widetilde{R} > 0$

$$\mathbb{P}^{\tau,x}\left(\sup_{\tau\le s\le t}\|u(X_s^\sigma - x)\| \ge 2\widetilde{R}\right) \le c_d\cdot(t-\tau)\cdot \sup_{\tau<s\le t}\ \sup_{\|y-x\|\le 3\widetilde{R}}\ \sup_{\|\varepsilon\|\le 1}\left|p\left(s,y,\frac{\varepsilon}{2\widetilde{R}}\right)\right|. \tag{4.27}$$

By setting $\widetilde{R} := (1/2)R$ we obtain (4.13). □

The maximal inequalities (4.13) and (4.14) allow for statements of the asymptotic behavior of the sample paths: First, we state a result concerning the behavior near the starting point $x\in\mathbb{R}^d$ at time $\tau\ge 0$ with respect to the measure $\mathbb{P}^{\tau,x}$. The second statement treats the same behavior at infinity. The proof of both statements is inspired by Theorem 4.3 and Theorem 4.6 of [66] but takes the time-component τ into account. Let us mention, that for a function $f:\mathbb{R}_+\to\mathbb{R}_+$ we denote by $f(t^+) := \limsup_{s\downarrow t} f(s)$ and $f(t_+) := \liminf_{s\downarrow t} f(s)$.

Lemma 4.2.5. *Let X be a non-homogeneous Itô process such that the differential characteristics of X are locally bounded and continuous. Then we have*

$$\lim_{t\to 0} t^{-1/\lambda}\sup_{\tau\le s\le\tau+t}\|X_s - x\| = 0 \textit{ for all } \lambda > \beta_\infty^{\tau^+,x} \tag{4.28}$$

$$\liminf_{t\to 0} t^{-1/\lambda}\sup_{\tau\le s\le\tau+t}\|X_s - x\| = 0 \textit{ for all } \beta_\infty^{\tau^+,x} \ge \lambda > \underline{\beta_\infty^{\tau^+,x}}. \tag{4.29}$$

If the symbol $p(\tau,x,\xi)$ of the process X satisfies (IS) *we additionally obtain*

$$\limsup_{t\to 0} t^{-1/\lambda}\sup_{\tau\le s\le\tau+t}\|X_s - x\| = \infty \textit{ for all } \overline{\delta_\infty^{\tau_+,x}} > \lambda \ge \delta_\infty^{\tau_+,x} \tag{4.30}$$

$$\lim_{t\to 0} t^{-1/\lambda}\sup_{\tau\le s\le\tau+t}\|X_s - x\| = \infty \textit{ for all } \delta_\infty^{\tau_+,x} > \lambda. \tag{4.31}$$

All these limits are meant $\mathbb{P}^{\tau,x}$-a.s with respect to every $\tau\in\mathbb{R}_+$ and $x\in\mathbb{R}^d$.

Proof. Here, we only prove (4.28) and (4.30) and omit the proofs of (4.29) and (4.31) since (4.28) and (4.31) and (4.29) as well as (4.30) are very similar, respectively.
Let $\varepsilon>0$, $\tau\in\mathbb{R}_+$ and $x\in\mathbb{R}^d$. We start by proving (4.28):
Let $\lambda > \sup_{\tau<s\le\tau+\varepsilon}\beta_\infty^{s,x}$ and choose $\sup_{\tau<s\le\tau+\varepsilon}\beta_\infty^{s,x} < \alpha_2 < \alpha_1 < \lambda$. For $t < T_0^\varepsilon$ with T_0^ε sufficiently small, (4.13) provides:

$$\begin{aligned}\mathbb{P}^{\tau,x}\left(\sup_{\tau\le s\le\tau+t}\|X_s - x\|\ge t^{1/\alpha_1}\right) &\le c_d\cdot t\cdot\sup_{\tau<s\le\tau+\varepsilon} H(s,x,t^{1/\alpha_1})\\ &\le c_d'\cdot t(t^{1/\alpha_1})^{-\alpha_2}\\ &= c_d' t^{1-(\alpha_2/\alpha_1)}.\end{aligned}$$

Now let $t_k := (1/2)^k$ for $k\in\mathbb{N}$. Since

$$\sum_{k=k_0^\varepsilon}^\infty \mathbb{P}^{\tau,x}\left(\sup_{\tau\le s\le\tau+t_k}\|X_s - x\|\ge (t_k)^{1/\alpha_1}\right)\le c_d'\sum_{k=k_0^\varepsilon}^\infty 2^{-k(1-(\alpha_2/\alpha_1))} < +\infty$$

where k_0^ε depends on T_0^ε, the Borel-Cantelli Lemma derives

$$\mathbb{P}^{\tau,x}\left(\limsup_{k\to\infty}\sup_{\tau\le s\le\tau+t_k}\|X_s-x\|\ge(t_k)^{1/\alpha_1}\right)=0,$$

and, hence, $\sup_{\tau\le s\le\tau+t_k}\|X_s-x\|<(t_k)^{1/\alpha_1}$ for all $k\ge k_1^\varepsilon(\omega)$ on a set of probability one. For fixed ω in this set, $t\in[t_{k+1},t_k]$ and $k\ge k_1^\varepsilon(\omega)\ge k_0^\varepsilon$, we have

$$\sup_{\tau\le s\le\tau+t}\|X_s(\omega)-x\|\le\sup_{\tau\le s\le\tau+t_k}\|X_s(\omega)-x\|\le t_k^{1/\alpha_1}\le 2^{1/\alpha_1}t^{1/\alpha_1}$$

and since $\lambda>\alpha_1$

$$t^{-1/\lambda}\sup_{\tau\le s\le\tau+t}\|X_s-x\|\le 2^{1/\alpha_1}t^{(1/\alpha_1)-(1/\lambda)}$$

which converges $\mathbb{P}^{\tau,x}$-a.s. to zero as $t\downarrow 0$. Since $\lambda>\sup_{\tau<s\le\tau+\varepsilon}\beta_\infty^{s,x}$ and $\varepsilon>0$ arbitrarily chosen, $\varepsilon\downarrow 0$ provides the statement.
In order to prove (4.30) we derive the following:
Let $\inf_{\tau<s\le\tau+\varepsilon}\overline{\delta_\infty^{s,x}}>\lambda'>\lambda$. Moreover, let $(t_k)_{k\in\mathbb{N}}$ be a sequence such that

$$\lim_{k\to\infty}(t_k)^{\lambda'}h(s,x,t_k)=\infty,\quad\forall s\in]\tau,\tau+\varepsilon].$$

Hence, the maximal inequality (4.14) provides for k sufficiently large

$$\begin{aligned}\mathbb{P}^{\tau,x}\left(\sup_{\tau\le s\le\tau+(t_k)^{\lambda'}}\|X_s-x\|<t_k\right)&\le c_d\cdot\frac{1}{(t_k)^{\lambda'}}\cdot\frac{1}{\inf_{\tau<s\le\tau+(t_k)^{\lambda'}}h(s,x,t_k)}\\&\le c_d\cdot\frac{1}{(t_k)^{\lambda'}}\cdot\frac{1}{\inf_{\tau<s\le\tau+\varepsilon}h(s,x,t_k)}\\&\underset{k\to\infty}{\longrightarrow}0.\end{aligned}$$

Fatou's Lemma implies

$$\begin{aligned}0&=\liminf_{k\to\infty}\mathbb{P}^{\tau,x}\left(\sup_{\tau\le s\le\tau+(t_k)^{\lambda'}}\|X_s-x\|<t_k\right)\\&=1-\limsup_{k\to\infty}\mathbb{P}^{\tau,x}\left(\sup_{\tau\le s\le\tau+(t_k)^{\lambda'}}\|X_s-x\|\ge t_k\right)\\&\ge 1-\mathbb{P}^{\tau,x}\left(\limsup_{k\to\infty}\left\{\sup_{\tau\le s\le\tau+(t_k)^{\lambda'}}\|X_s-x\|\ \ge t_k\right\}\right).\end{aligned}$$

Hence,

$$\mathbb{P}^{\tau,x}\left(\frac{1}{t_k}\sup_{\tau\le s\le\tau+(t_k)^{\lambda'}}\|X_s-x\|\ge 1,\text{ infinitely often}\right)=1,$$

and, therefore,

$$\limsup_{k\to\infty}\left(\frac{1}{t_k}\right)^{-\frac{1}{\lambda'}}\sup_{\tau\le s\le\tau+t_k}\|X_s-x\|\ge 1.$$

Since $\lambda < \lambda'$ we observe that

$$\limsup_{t\to 0} \left(\frac{1}{t}\right)^{-\frac{1}{\lambda}} \sup_{\tau\le s\le \tau+t} \|X_s - x\| = \infty.$$

Since $\varepsilon > 0$ is arbitrarily chosen, the statement follows. □

The proof of the following lemma parallels Lemma 4.2.5, and, hence, we omit details of the proof.

Lemma 4.2.6. *Let X be a non-homogeneous Itô process such that the differential characteristics of X are continuous. Then we have*

$$\lim_{t\to\infty} t^{-1/\lambda} \sup_{\tau\le s\le\tau+t} \|X_s - x\| = 0 \text{ for all } \lambda < \beta_0 \tag{4.32}$$

$$\liminf_{t\to\infty} t^{-1/\lambda} \sup_{\tau\le s\le\tau+t} \|X_s - x\| = 0 \text{ for all } \underline{\beta_0} > \lambda \ge \beta_0. \tag{4.33}$$

If the symbol p of the process X satisfies (IS) *then we additionally obtain*

$$\limsup_{t\to\infty} t^{-1/\lambda} \sup_{\tau\le s\le\tau+t} \|X_s - x\| = \infty \text{ for all } \overline{\delta_0} < \lambda \le \delta_0 \tag{4.34}$$

$$\lim_{t\to\infty} t^{-1/\lambda} \sup_{\tau\le s\le\tau+t} \|X_s - x\| = \infty \text{ for all } \delta_0 < \lambda. \tag{4.35}$$

All these limits are meant $\mathbb{P}^{\tau,x}$-a.s with respect to every $\tau \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$.

The time homogeneous version of the following result is Lemma 5.11 of [11].

Lemma 4.2.7. *Let X be a non-homogeneous Itô process such that the differential characteristics of X are continuous. Moreover, let* (IG) *hold true. Then*

$$\mathbb{E}^{\tau,x}\left(e^{X_t'\xi}\right) < \infty$$

for all $t \ge \tau \ge 0$ and $x, \xi \in \mathbb{R}^d$.

Proof. At first, let us reconsider the stopping time σ as defined in (4.3):

$$\sigma_R^{\tau,x} := \sigma := \inf\{h \ge \tau : \|X_h^{\tau,x} - x\| > R\}$$

and let p be the symbol of X given by

$$p(\tau,x,\xi) = -i\ell(\tau,x)'\xi + \frac{1}{2}\xi' Q(\tau,x)\xi - \int_{\{y\neq 0\}} \left(e^{iy'\xi} - 1 - iy'\xi\cdot\chi(y)\right) N_\tau(x,dy).$$

In order to apply Gronwall's Lemma we derive the following:

$$\begin{aligned}\mathbb{E}^{\tau,x}\left(e^{(X_t^\sigma - x)\xi}\right) = \mathbb{E}^{\tau,x}\Bigg[\int_\tau^{t\wedge\sigma} e^{(X_s^\sigma - x)\xi}\Bigg(&\xi\ell(s,X_s) - \frac{1}{2}\xi^2\, Q(s,X_s) \\ &+ \int_{\{y\neq 0\}} (e^{\xi y} - 1 - \xi y\chi(y))\, N_s(X_s,dy)\Bigg)\, ds\Bigg]\end{aligned}$$

$$\leq b(\xi) \int_\tau^t \mathbb{E}^{\tau,x} \left(e^{(X_s^\sigma - x)\xi} \right) \, ds$$

Condition (IG) provides (cf. Lemma 3.3 of [67]) the finiteness of the constant

$$b(\xi) := \sup_{s \geq 0, x \in \mathbb{R}^d} \left| \xi \ell(s,x) - \frac{1}{2}\xi^2 \, Q(s,x) + \int_{\{y \neq 0\}} (e^{\xi y} - 1 - \xi y \chi(y)) \, N_s(x, dy) \right|.$$

The application of Gronwall's Lemma yields

$$\mathbb{E}^{\tau,x} \left(e^{(X_s^\sigma - x)\xi} \right) \leq 1 + b(\xi) \int_\tau^t e^{b(\xi)(t-s)} \, ds = e^{b(\xi)(t-\tau)},$$

and with Fatou's Lemma it follows that

$$\mathbb{E}^{\tau,x} \left(e^{(X_s - x)\xi} \right) \leq \liminf_{R \to \infty} \mathbb{E}^{\tau,x} \left(e^{(X_s^\sigma - x)\xi} \right) \leq e^{b(\xi)(t-\tau)},$$

where the first inequality follows by Proposition 4.2.3. □

Finally, we generalize Theorem 2.10 of [49] which is a useful and applicable criterion for the finiteness of p-variation.

Lemma 4.2.8. *Let X be a non-homogeneous Itô process such that the differential characteristics of X are locally bounded and continuous. For every $t \geq \tau \geq 0$ and $p > \sup_{\tau \geq 0} \beta_\infty^{\tau,x}$ the following holds true:*

$$\sup_{\pi_n} \sum_{j=1}^n \| X_{t_j} - X_{t_{j-1}} \|^p < +\infty, \quad \mathbb{P}^{\tau,x}\text{-a.s.} \tag{4.36}$$

where the supremum is taken over all finite partitions $\pi_n = (t_i)_{i=1,\dots,n}$ with $\tau = t_0 < t_1 < \dots < t_n = t$. I.e. the p-variation of the paths of X are $\mathbb{P}^{\tau,x}$-a.s finite for $p > sup_{\tau \geq 0} \beta_\infty^{\tau,x}$.

Proof. Let $t, r > 0$ and $\lambda > p$. By using Theorem 4.27 we obtain the following calculation

$$\begin{aligned}
\alpha(t,r) &:= \sup \left\{ \mathbb{P}^{\tau,x}(\|X_s - x\| \geq r) : \tau \geq 0, s \in [\tau, \tau + t], x \in \mathbb{R}^d \right\} \\
&\leq \sup \left\{ \mathbb{P}^{\tau,x} \left(\sup_{\tau \leq s \leq \tau + t} \|X_s - x\| \geq r \right) : \tau \geq 0, x \in \mathbb{R}^d \right\} \\
&\leq c_d \cdot t \cdot \sup_{\tau \geq 0} \cdot \sup_{\tau \leq s \leq \tau + t} H(s,x,r) \\
&\leq c_d \cdot t \cdot \sup_{\tau \geq 0} H(\tau, x, r) \\
&\leq c_d \cdot t \cdot K \cdot r^{-\lambda}
\end{aligned}$$

for r small enough and $K > 0$. Hence, Theorem 1.3 of [48] yields the statement. □

5 Killing of Semimartingales

In the present chapter, we will take the theory of Markov processes, as a guideline to incorporate the concept of killing of Markov processes into the semimartingale framework. As we have seen, when dealing with Markov processes, working with the killing point ∂ is an integral part of the classical theory (cf. for example [8], [22]), and poses a natural way for the process to leave the state space. Nevertheless, in contrast to many other properties of Markov processes, the concept of killing was not included in the theory of semimartingales for a long time. This changed when Cheridito, Filipović and Yor [13] and Schnurr [71] dealt with this topic:
Cheridito et al. considered a stochastic process with values in a state space $(E \cup \{\partial\}, \mathcal{E}_\partial)$, where E is a closed subset of $\mathbb{R}^d$. They set $\|\partial\| := \infty$ and $T_\partial := \inf\{t \geq 0 : X_t = \partial \text{ or } X_{t-} = \partial\}$. The authors discovered that a transition to ∂ occurs either by a jump or by an explosion (see Definition 5.1.2 below). Hence, the main idea was to separate the space of paths depending on the kind of killing that occurs. Similarly, the killing state ∂ was separated into Δ and ∞, where Δ was reached by the paths killed by a sudden killing and ∞ by the paths killed through explosion. The state Δ was identified by $y \in \mathbb{R} \backslash E$. If $E = \mathbb{R}^d$, even an artificial new dimension was added in order to include Δ into the Euclidean space. To turn the process into a semimartingale the process was stopped before it explodes. Summing up, they dealt with two possible kinds of killing in different ways, but in each case they got rid of the points that are not in the Euclidean space. Afterwards, they treated the process as in the classical theory of semimartingales. Using this procedure, one loses the information on the killing. Moreover, one can not write down a representation of the whole process, and the new process is not canonical, because the point y can be chosen arbitrarily.
In [71], Schnurr utilized the idea to separate the process considering the two ways of killing. However, therein Δ remained a point outside the Euclidean space. Explosions were treated by stopping along an announcing sequence. A new characteristic, which describes the sudden killing, was introduced.
In the first section, we present the approach of [71] in detail. Moreover, we provide new insights together with generalizations of results known for classical semimartingales.
The second section treats the generalization of the probabilistic symbol in this context, and aims only to give a short introduction into this topic. Finally, the last section of this chapter provides a natural way in which the killing of semimartingales can occur. To this end, path-dependent killing by multiplicative functionals, known from the theory of Markov processes, is introduced and transferred into the semimartingale framework.
Let us note that the content of the first section is derived from Section 3 and the appendix of [72]. Furthermore, all proofs presented herein are due to the author of this work.

5.1. Generalized Semimartingales

In the following, let $E \subseteq \mathbb{R}^d$ be a closed set, and let $\mathcal{E}$ be the Borel σ-field on E. We equip E with the so-called *killing points*, namely, ∞ and Δ, lying outside $\mathbb{R}^d$. From a topological point-of-view $E\cup\{\infty\}$ is the Alexandrov compactification of E, afterwards one adds another singular point Δ to the (now compact) space. Let $\tilde{E} := E \cup \{\infty\} \cup \{\Delta\}$ and $\tilde{\mathcal{E}}$ be the smallest σ-field on $\tilde{E}$ containing $\mathcal{E}, \{\infty\}$, and $\{\Delta\}$. We consider a càdlàg stochastic process X on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with values in $\tilde{E}$. If not mentioned otherwise every function f on E is extended to $\tilde{E}$ by setting $f(\infty) = f(\Delta) = 0$. We establish the following calculation rules for the points ∞ and Δ:

- $\infty + r = \infty$ and $\Delta + r = \Delta$ for $r \in \mathbb{R}^d \cup \{\infty\}$ and $\Delta + \Delta = \Delta$, $\infty + \Delta = \Delta$.
- $\Delta - r = \Delta$ for $r \in \mathbb{R}^d \cup \{\infty\}$ and $\infty - r = \infty$ for $r \in \mathbb{R}^d$.
- $\Delta \cdot r = \Delta$ and $\infty \cdot r = \infty$ for $r \in \mathbb{R}^d \setminus \{0\}$ and $\Delta \cdot 0 = 0$, $\infty \cdot 0 = 0$.
- The norm $\|\cdot\|$ of ∞ and Δ equals $+\infty$.

Definition 5.1.1. Let X be a stochastic process. We call the increasing sequence of stopping times $(\sigma'_n)_{n\in\mathbb{N}}$ defined by

$$\sigma'_n := \inf\{t \geq 0 : \ ||X_t|| \geq n \text{ or } ||X_{t-}|| \geq n\}, \quad n \in \mathbb{N}$$

separating sequence of X.

As the name of $(\sigma'_n)_{n\in\mathbb{N}}$ suggests, it provides a distinction between the two different ways in which the process X is able to leave the conventional space E and takes the values ∞ or Δ. The subsequent definition illustrates this:

Definition 5.1.2. Let X be a stochastic process with separating sequence $(\sigma'_n)_{n\in\mathbb{N}}$. We define stopping times $\zeta^\partial, \zeta^\Delta, \zeta^\infty$ and σ_n as follows

$$\begin{aligned}
\zeta^\partial &:= \inf\{t \geq 0 : \ X_t \in \{\Delta, \infty\}\}, \\
\zeta^\Delta &:= \begin{cases} \zeta^\partial, & \text{if } \sigma'_n = \zeta^\partial \text{ for some } n \in \mathbb{N} \\ \infty, & \text{if } \sigma'_n < \zeta^\partial \text{ for all } n \in \mathbb{N} \end{cases}, \\
\zeta^\infty &:= \begin{cases} \zeta^\partial, & \text{if } \sigma'_n < \zeta^\partial \text{ for all } n \in \mathbb{N} \\ \infty, & \text{if } \sigma'_n = \zeta^\partial \text{ for some } n \in \mathbb{N} \end{cases}, \\
\sigma_n &:= \begin{cases} \sigma'_n, & \text{if } \sigma'_n < \zeta^\partial \\ \infty, & \text{if } \sigma'_n = \zeta^\partial \end{cases}.
\end{aligned}$$

The stopping time ζ^∂ marks the first time the process X leaves E. Furthermore, the stopping time σ'_n stops the process at the time where its norm exceeds n. Notably, when ζ^Δ is finite, it coincides with σ'_n for one n. Hence, one can think of ζ^Δ as a sudden killing. Moreover, by the definition of ζ^∞ the separating sequence never equals ζ^∂ (also when ζ^Δ is finite), and in this case one can think of ζ^∞ as some kind of explosion. Indeed, the explosion time is predictable. That is due to the fact, that $\sigma_n \wedge n$ converges a.s. to ζ^∞ from below. Such a sequence is called *announcing sequence* (cf. Definition I.2.16 of [36]). The following figures demonstrate these kind of killings.

Fig. 5.1.: Illustration of the explosion killing.

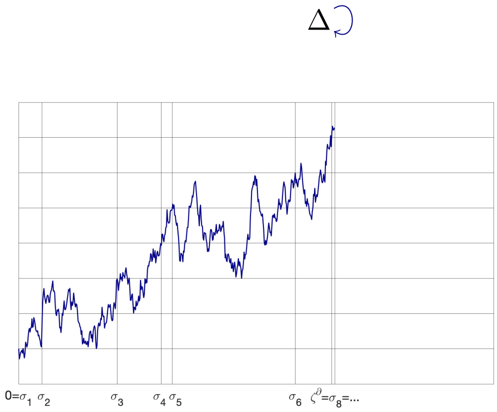

Fig. 5.2.: Illustration of the sudden killing.

Remark 5.1.3. At first sight, when encountering the killing time ζ^∂ it appears more canonical to separate into totally inaccessible and accessible (resp. predictable) stopping times. In fact, this leads nowhere. The only useful separation in this case is ‘explosion vs. everything else’. That is, because, intuitively speaking, considering the explosion not only yields a convergence ‘in time’ (of the announcing sequence to ζ^∞) but also a convergence ‘in space’ (of X^{σ_n} to $X^{\zeta^\infty-}$). This convergence allows for a pre-local treatment of the stopped process $X^{\zeta^\infty-}$, which proves to be handy for the subsequent theory.

In order to be more general, we allow a transition from ∞ to Δ, that is, we treat processes of the following kind:

Definition 5.1.4. Let X be a stochastic process on $(\Omega, \mathcal{F}^X, (\mathcal{F}^X_t)_{t\geq 0}, \mathbb{P})$ with values in $\tilde{E}$. Moreover, let ζ^∞ possess the announcing sequence $(\sigma_n \wedge n)_{n\in\mathbb{N}}$ and let ζ^Δ be as above. Then X is called a *process with killing*, if

$$X 1_{[\![0,\zeta^\infty[\![} \subseteq E,\ X 1_{[\![\zeta^\infty,\zeta^\Delta[\![} = \infty \text{ and } X 1_{[\![\zeta^\Delta,\infty[\![} = \Delta.$$

Thereby, we set $[\zeta^\infty(\omega), \zeta^\Delta(\omega)[= \emptyset$ if $\zeta^\infty(\omega) \geq \zeta^\Delta(\omega)$. In particular, if $\zeta^\Delta = +\infty$, we call X a *process with explosion*.

We now adapt the concept of killing to the class of semimartingales. The natural way to do so is to demand a process to fulfill the required properties before it leaves the state space E.

Definition 5.1.5. Let X be a process with killing and killing times $\zeta^\infty, \zeta^\Delta$, and let $(\tau_n)_{n\in\mathbb{N}}$ be the announcing sequence of ζ^∞. We define for every $n \in \mathbb{N}$ the stopping time

$$\alpha_n := \tau_n \wedge \zeta^\Delta,$$

and call the sequence $(\alpha_n)_{n\in\mathbb{N}}$ the *pre-explosion sequence* of X.

Without loss of generality, from now on we choose $\alpha_n = \zeta^\Delta$ for all $n \in \mathbb{N}$ if $\zeta^\Delta < \zeta^\infty$. This is since $\tau_n \uparrow \zeta^\infty$ for $n \to \infty$.

Definition 5.1.6. We call a stochastic process $\tilde{X}$ a *generalized semimartingale* if it possesses a decomposition of the form

$$\tilde{X}_t = X_t + K_t \tag{5.1}$$

where $(X_t)_{t\geq 0}$ is a process with explosion and $X^{\alpha_n -} \in \mathcal{S}$ for all $n \in \mathbb{N}$, and $(K_t)_{t\geq 0}$ defined by

$$K_t := \Delta \cdot 1_{[\![\zeta^\Delta, \infty[\![}(\cdot, t), \quad t \geq 0$$

is the so called *killing process*. Moreover, we denote by $\mathcal{S}^\dagger$ the set of all generalized semimartingales.

The class of generalized semimartingales is big enough to contain various examples like Lévy processes with killing, solutions of SDEs with locally Lipschitz coefficients and certain Markov processes defined by sub-Markovian kernels. Furthermore, this class can be treated in a similar way as classical semimartingales.

Definition 5.1.7. Let $\tilde{X}$ be a generalized semimartingale with killing times $\zeta^\infty, \zeta^\Delta$ and pre-explosion sequence $(\alpha_n)_{n\in\mathbb{N}}$ and let (B^n, C^n, ν^n) be the characteristics of the semimartingale $\tilde{X}^{\alpha_n -}$.
We call the processes B and C, and the random measure ν the *characteristics* of $\tilde{X}$ if they coincide with the characteristics (B^n, C^n, ν^n) of $X^{\alpha_n -}$ on $[\![0, \alpha_n[\![$ for every $n \in \mathbb{N}$.

Remark 5.1.8. (a.) Since the characteristics are unique up to an evanescent set, and $X^{\alpha_n -} = X^{\alpha_{n+1} -}$ on $[\![0, \alpha_n[\![$ the characteristics of a generalized semimartingale are well-defined.

(b.) By the previous definition, the characteristics of a generalized semimartingale are uniquely defined on $[\![0, \zeta^\Delta \wedge \zeta^\infty[\![$ only. Thus, we set

$$\begin{aligned} &C_t(\omega) = C_{(\zeta^\Delta(\omega)\wedge\zeta^\infty(\omega))-}(\omega) \quad \forall t \geq (\zeta^\Delta \wedge \zeta^\infty)(\omega), \\ &B_t(\omega) = B_{(\zeta^\Delta(\omega)\wedge\zeta^\infty(\omega))-}(\omega) \quad \forall t \geq (\zeta^\Delta \wedge \zeta^\infty)(\omega), \\ &\nu\left(\omega, [\zeta^\Delta \wedge \zeta^\infty(\omega), \infty[\times E\right) = 0 \quad \forall \omega \in \Omega. \end{aligned}$$

Example 5.1.9. We have seen in Remark 1.3.7 that the characteristic exponent Ψ of a Lévy process $(\tilde{L}_t)_{t\geq 0}$ with exponential killing is of the form

$$\phi(\xi) = a - i\ell'\xi + \frac{1}{2}\xi' Q\xi - \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y)\right) N(dy). \tag{5.2}$$

with $a > 0$ and Lévy triplet (ℓ, Q, ν). In the notation from above $(\tilde{L}_t)_{t\geq 0}$ is of the form

$$\tilde{L} = L + \Delta 1_{[\![\zeta^\Delta, \infty[\![}$$

where L is a Lévy process with characteristic exponent (ℓ, Q, ν) and ζ^Δ is exponentially distributed with parameter a and independent of L. The process $\widetilde{L}$ (with killing) behaves like L, but as soon as the killing time is reached, it jumps to Δ. Since the pre-explosion sequence equals ζ^Δ for all $n \in N$ we have

$$\widetilde{L}^{\alpha_n -} = L^{\zeta^\Delta -},$$

and, therefore, $\tilde{L}$ possess the characteristics

$$B_t = b(t \wedge \zeta^\Delta -), \quad C_t = c(t \wedge \zeta^\Delta -), \quad \nu(\cdot, ds, dx) = F(dx) 1_{[\![0,\zeta^\Delta[\![}(s) ds.$$

Remark 5.1.10. Let us shortly return to Remark 5.1.3. After defining the characteristics of a generalized semimartingale it becomes more obvious why the separation between explosion and sudden killing is favored to the separation into totally inaccessible and accessible killing. We consider the following deterministic example: Let $X_t = t 1_{[0,1[} + \Delta 1_{[1,\infty[}$. The killing time $\zeta^\Delta = 1$ is deterministic, and, therefore, predictable with announcing sequence $(\rho_n := 1 - 1/n)_{n \in \mathbb{N}}$. Of course the process $X_t^{\rho_n} = t \wedge (1 - 1/n)$ is a semimartingale for all $n \in \mathbb{N}$ with characteristics $(t \wedge (1 - 1/n), 0, 0)$ which converges to $t \wedge 1$. By stopping the process along the announcing sequence, one loses all information of the killing.

For a classical semimartingale X the random measure ν compensates the jumps of X. It is obvious that we are not able to use ν to compensate a jump with height ∞, namely the jump to Δ. Therefore, we are not able to use only the three characteristics (B, C, ν) to determine a generalized semimartingale $\tilde{X}$. We are in the need of a new characteristic, and to this end it seems natural not to compensate the jump to Δ itself, but to keep track, *when* this jump occurs.

Definition 5.1.11. Let $\tilde{X}$ be a generalized semimartingale with values in $\tilde{E}$ and stopping times $\zeta^\infty, \zeta^\Delta$. We define the process $(A_t)_{t \geq 0}$ to be the predictable compensator of the process $1_{[\![\zeta^\Delta,\infty[\![}$.
We call A the *fourth characteristic of a generalized semimartingale*, and, moreover, the quadruple (A, B, C, ν) the *characteristics of a generalized semimartingale*.

Remark 5.1.12. In the previous definition the compensator is meant with respect to the filtration $(\mathcal{F}_t^{\tilde{X}})_{t \geq 0}$. Indeed, taking the natural filtration of $1_{[\![\zeta^\Delta,\infty[\![}$ is not sufficient for some of the following theorems.

Example 5.1.13.

(1.) Let X be a semimartingale with explosion and killing time ζ^∞ and $K_t = \Delta 1_{[\![\zeta^\Delta,\infty[\![}$ be a killing process where the stopping time ζ^Δ is predictable and strictly positive. Then the fourth characteristic A of the generalized semimartingale $X + K$ equals $1_{[\![\zeta^\Delta,\infty[\![}$.

(2.) Let X be a semimartingale with explosion and killing time ζ^∞ and $K_t = \Delta 1_{[\![\zeta^\Delta,\infty[\![}$ be a killing process where the stopping time $\zeta^\Delta < \infty$ is totally inaccessible and finite. Then the fourth characteristic A of the generalized semimartingale $X + K$ is continuous.

(3.) Let $\tilde{L}$ be a Lévy process with exponential killing as defined in Example 5.1.9. For this process the fourth characteristic is given by

$$A_t := a(t \wedge \zeta^\Delta -).$$

(4.) Let $\left(Z^{(1)}, ..., Z^{(m)}\right)$ be a m-dimensional semimartingale with $Z_0^{(i)} = 0$ for all $i \in \{1, ..., m\}$ and $(f_j^i)_{i \le d, j \le m}$ be a $d \times m$ matrix of locally Lipschitz functions $f_j^i : \mathbb{R}^d \to \mathbb{R}$. Then there exists a process X with explosion and killing time ζ^∞ such that X is the unique solution of the stochastic differential equation

$$X_t = f(X_-) \bullet Z_t.$$

For more detail see Chapter 7, Theorem 38 of [54].

Proof. We only give a proof for (2.): The process $1_{[\![\zeta^\Delta,\infty[\![}$ is quasi-left-continuous since ζ^Δ is totally inaccessible by Proposition I.2.26 of [36]. Thus, we observe that

$$\begin{aligned}\{\Delta A \neq 0\} &= \left\{{}^p\left(\Delta 1_{[\![\zeta^\Delta,\infty[\![}\right) > 0\right\} \\ &= \left\{{}^p\left(\Delta 1_{\left\{\Delta 1_{[\![\zeta^\Delta,\infty[\![} \neq 0\right\}}\right) > 0\right\} \\ &= \emptyset.\end{aligned}$$

The first equation follows with Property I.3.21 of [36], and the last holds by Proposition I.2.35 of [36]. Furthermore, ${}^p(\Delta 1_{[\![\zeta^\Delta,\infty[\![})$ denotes the predictable projection of $1_{[\![\zeta^\Delta,\infty[\![}$ (cf. Theorem I.2.28 of [36]). □

The characteristics of $\tilde{X}$ are unique only up to an evanescent set, and, thus, it is possible to modify the characteristics on such a set, in order to obtain what we will call the 'good' version of (A, B, C, ν). The following theorem will provide this version, and is one of the main results of this section. Indeed, it is a generalization of Proposition II.2.9 of [36], and inspired by its proof.

Theorem 5.1.14. *Let $\tilde{X}$ be a generalized semimartingale with characteristics (A', B', C', ν'). Then there exists a version (A, B, C, ν) of (A', B', C', ν') satisfying the following conditions:*

$$A_t = \int_0^t a_s \, dF_s, \quad a.s. \tag{5.3}$$

$$B_t^{(i)} = \int_0^t b_s^{(i)} \, dF_s, \quad a.s. \tag{5.4}$$

$$C_t^{(ij)} = \int_0^t c_s^{(ij)} \, dF_s, \quad a.s. \tag{5.5}$$

$$\nu(\omega; dt, dx) = dF_t(\omega) K_{\omega,t}(dx), \quad a.s. \tag{5.6}$$

where we have

(i.) a predictable process F belonging to $\mathcal{A}^+_{loc}$,

(ii.) a predictable process a,

(iii.) a predictable process $b = (b^{(1)}, ..., b^{(d)})'$,

(iv.) a predictable process $c = (c^{(ij)})_{i,j \le d}$ taking values in the set of all symmetric, non-negative $d \times d$-matrices,

(v.) a transition kernel $K_{\omega,t}(dx)$ from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying

- $K_{\omega,t}(\{0\}) = 0$
- $\int(|x|^2 \wedge 1)\ K_{\omega,t}(dx) \leq 1$
- $\Delta F_t(\omega) K_{\omega,t}(\mathbb{R}^d) \leq 1$.

Furthermore, the upper 'good' version of (A', B', C', ν') *satisfies*

(1.) $(C_t^{(ij)} - C_s^{(ij)})_{i,j \leq d}$ *is a symmetric non-negative matrix for* $s \leq t$.

(2.) $(|x|^2 \wedge 1) * \nu \in \mathcal{A}_{loc}$ *and* $\nu(\{t\} \times \mathbb{R}^d) \leq 1$.

Proof. Let B^n, C^n, ν^n be the characteristics of the classical semimartingale $\tilde{X}^{\alpha_n -}$. By Proposition II.2.9 of [36] there exists an increasing, predictable process $F'^n \in \mathcal{A}_{loc}$ and $b^n, c^n, K^n_{\omega,t}(dx)$ with the properties mentioned in (iii.)-(v.) such that

$$\begin{aligned}
(B^n)^{(i)}_t &= \int_0^t (b^n_s)^{(i)}\ dF'^n_s, \quad a.s. \\
(C^n)^{(ij)}_t &= \int_0^t (c^n)^{(ij)}_s\ dF'^n_s \quad a.s. \\
\nu^n(\omega; dt, dx) &= dF'^n_t(\omega) K^n_{\omega,t}(dx), \quad a.s.,
\end{aligned}$$

and since $\tilde{X}^{\alpha_n -} = \tilde{X}^{\alpha_{n+1} -}$ on $[\![0, \alpha_n[\![$ we have

$$b^n = b^{n+1} 1_{[\![0,\alpha_n[\![}, c^n = c^{n+1} 1_{[\![0,\alpha_n[\![} \text{ and } K^n_{\omega,t}(dx) = K^{n+1}_{\omega,t}(dx) 1_{[\![0,\alpha_n[\![}.$$

Hence, we set

$$\begin{aligned}
F' &:= \sum_{n=0}^{\infty} F'^n 1_{[\![\alpha_{n-1},\alpha_n[\![} \\
b &:= \sum_{n=0}^{\infty} b^n 1_{[\![\alpha_{n-1},\alpha_n[\![} \\
c &:= \sum_{n=0}^{\infty} c^n 1_{[\![\alpha_{n-1},\alpha_n[\![} \\
K_{\omega,t}(dx) &:= \sum_{n=0}^{\infty} K^n_{\omega,t}(dx) 1_{[\![\alpha_{n-1},\alpha_n[\![}.
\end{aligned}$$

Additionally, we observe for the process

$$F := (F')^{(\zeta^\infty \wedge \zeta^\Delta)-} + A' \tag{5.7}$$

that $dF' \ll dF$ and $dA' \ll dF$ since F' and A are increasing. Therefore, Theorem I.3.13 of [36] provides the existence of predictable processes f' and a such that

$$\begin{aligned}
A' &= a \bullet F, \\
F' &= f' \bullet F.
\end{aligned}$$

The associativity of the stochastic integral then provides on $[\![0, \alpha_n[\![$

$$(B^n)^{(i)} = ((b)^{(i)} f') \bullet F$$

$$(C^n)^{(ij)} = ((c)^{(ij)} f') \cdot F$$
$$\nu^n(\omega; dt, dx) = dF_t(\omega) f'_t(\omega) K_{\omega,t}(dx),$$

The properties $(i.) - (2.)$ follow by Proposition II.2.9 of [36] on $[\![0, \zeta^\infty \wedge \zeta^\Delta[\![$. □

If we combine generalized semimartingales and strong Markov process, as we have done in Section 2.6, we are able to formulate the following stronger version of the previous theorem. It is based on Theorem 6.27 of [12]. Thus, let $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$ be a Markov process.

Remark 5.1.15. For a process Y with killing and killing times $\zeta^\infty, \zeta^\Delta$ to be an additive functional as defined in Definition 2.6.4 it is necessary for the killing times $\zeta^\infty, \zeta^\Delta$ to fulfill

$$\zeta^\infty = (\zeta^\infty \circ \theta_s - s) \wedge 0 \text{ and } \zeta^\Delta = (\zeta^\Delta \circ \theta_s - s) \wedge 0.$$

Lemma 5.1.16. *Let $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$ be a strong Markov process, and let Y be a generalized $\mathbb{P}^x$-semimartingale which is additive and quasi-left continuous. Then the characteristics (A, B, C, ν) of Y are of the form*

$$A_t = \int_0^t a(X_s)\, dF_s \tag{5.8}$$
$$B_t^{(i)} = \int_0^t b^{(i)}(X_s)\, dF_s \tag{5.9}$$
$$C_t^{(ij)} = \int_0^t c^{(ij)}(X_s)\, dF_s \tag{5.10}$$
$$\nu(\omega; dt, dx) = dF_t(\omega) \tilde{K}(X_t(\omega); dx) \tag{5.11}$$

where

(i.) *F is a continuous, additive functional and belongs to $\mathcal{V}^+(\mathbb{P}^x)$ for every $\mathbb{P}^x$,*

(ii.) *a is $\mathcal{B}(\mathbb{R})^d$-measurable,*

(iii.) *b is $\mathcal{B}(\mathbb{R})^d$-measurable,*

(iv.) *c is $\mathcal{B}(\mathbb{R})^d$-measurable, $d \times d$- dimensional, and takes values in the set of all symmetric non-negative matrices,*

(v.) *$\tilde{K}(\omega, t; dx)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{O})$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with $\tilde{K}(\{0\}) = 0$ and $\int (|x|^2 \wedge 1)\, \tilde{K}(dx) < \infty$.*

Proof. At first, we want to prove the additivity of the characteristics: Since Y is an additive functional it is an additive functional on $[\![0, \alpha_n[\![$ for all $n \in \mathbb{N}$. Thus, the processes B_n, C_n and ν_n are additive on these stochastic intervals by Theorem 6.24 (iv) of [12]. The additivity of B, C and ν follows. It remains to show that the fourth characteristic A is an additive functional. Therefore, we consider the one-point jump process $1_{[\![\zeta^\Delta,\infty[\![}$. A simple computation shows that

$$\begin{aligned}\Theta_s 1_{[\![\zeta^\infty,\infty[\![}(\omega, t) &= 1_{[\![s+\zeta^\Delta\circ\theta_s,\infty[\![}(\omega, t)\\ &= 1_{(0,t-s]}(\zeta^\Delta(\theta_s(\omega)).\end{aligned}$$

Equally easy, we state that

$$1_{[\![\zeta^\Delta,\infty[\![}(\omega,t) - 1_{[\![\zeta^\Delta,\infty[\![}(\omega,t\wedge s) = 1_{(s,t]}(\zeta^\Delta(\omega)).$$

Since Y is an additive functional, it is known that $\zeta^\Delta = \zeta^\Delta \circ \theta_s - s$ for $s < \zeta^\Delta$. Thus, (A,B,C,ν) are additive functionals.
Let us now state the proof of the statement: Since Y is quasi-left continuous B is continuous (see I.4.36 and II.2.9 (i.) in [36]) and ν is $\mathbb{P}^x$-quasi-left continuous (see I.2.35 in [36]). C is continuous by definition. Moreover, the quasi left-continuity of Y implies that ζ^Δ is totally inaccessible, and, therefore, the fourth characteristic A is continuous. We are now able to apply Theorem 6.19 of [12]: There exists a continuous process $F' \in \mathcal{V}^+_{ad}$ with respect to $(\mathcal{M}_t)_{t\geq 0}$ and a positive transition kernel $\tilde{K}'$ from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ into $(E,\mathcal{E})$, such that

$$\nu(\omega;dt;dx) = dF'_t(\omega)\tilde{K}'(X_t(\omega),dx).$$

Now, let

$$F := F' + \sum_{i\leq d} \mathrm{Var}(B^{(i)}) + \sum_{i,j\leq d} \mathrm{Var}(C^{i,j}) + A.$$

Here, $\mathrm{Var}(X)_t(\omega)$ denotes the variation process of $X \in \mathcal{V}^+$, i.e. the total variation of the function $s \mapsto X_s(\omega)$ on the interval $[0,t]$. The process F belongs to $\mathcal{V}^+$ and is continuous, and an additive functional. Moreover, we have:

$$dF' \ll dF,\ dB^{(i)} \ll dF,\ dC^{(ij)} \ll dF,\ dA \ll dF.$$

Theorem 3.55 of [12] provides the existence of $\mathcal{B}(\mathbb{R})^d$-measurable functions a,b,c such that

$$\begin{aligned} B &= b(X)\bullet F,\\ C &= c(X)\bullet F,\\ A &= a(X)\bullet F. \end{aligned}$$

The theorem follows analogously to the proof of 6.25 in [12]. □

Definition 5.1.17.

(i.) We denote the vector $(1,...,1)' \in \mathbb{R}^d$ by writing $\mathbf{1}$.

(ii.) Let $\tilde{X}$ be a generalized semimartingale with values in $\tilde{E}$ and characteristics (A,B,C,ν). Let $X^n := \tilde{X}^{\alpha_n-}$ possess the characteristics (B^n,C^n,ν^n). We define a complex-valued, predictable process $L^n(u)$ by

$$\begin{aligned} L^n(u)_t &:= e^{iu'\mathbf{1}\cdot 1_{[\![\zeta^\Delta,\infty[\![}} \bullet A_t - iu'B^n_t - \frac{1}{2}u'C^n_t u \\ &\quad + \int_{\mathbb{R}^d} (e^{iu'x} - 1 - iu'h(x))\,\nu^n([0,t]\times dx), \quad t\geq 0. \end{aligned}$$

The following proposition generalizes Proposition II.2.42 of [36].

Proposition 5.1.18. *Let $\tilde{X}$ be a process with killing, possessing killing times $\zeta^\infty, \zeta^\Delta$. Let X^n be defined as before and $H^n := X^n + \mathbf{1}\cdot 1_{[\![\zeta^\Delta,\infty[\![}$. Then the following statements are equivalent:*

(a.) $\tilde{X}$ *is a generalized semimartingale with characteristics* (A, B, C, ν).

(b.) The process

$$e^{iu'H^n} - e^{iu'H^n} \bullet A - e^{iu'X^n_-} \bullet L^n(u)$$

is a complex valued, local martingale for every $n \in \mathbb{N}$.

(c.) The process

$$\begin{aligned}
&f(H^n) - f(X_0) - \sum_{i=1}^{d} \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-) \right) \bullet (B^n)^{(i)} \\
&- \frac{1}{2} \sum_{i,j=1}^{d} \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(X^n_-) \right) \bullet (C^n)^{(ij)} \\
&- \left[f(X^n_- + x) - f(X^n_-) - \sum_{i=1}^{d} \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-) \right) h((X^n))^{(i)} \right] * \nu^n \\
&- \Delta f(H^n) \bullet A
\end{aligned}$$

is a local martingale for every $n \in \mathbb{N}$ *and every function* $f \in \mathcal{C}^2_b$.

Proof. $(a.) \Rightarrow (c.)$: Let $\tilde{X}$ be a generalized semimartingale with characteristics (A, B, C, ν). We defined H^n to be $X^n + \mathbf{1} \cdot 1_{[\![\zeta^\Delta, \infty[\![}$, where $X^n := \tilde{X}^{\alpha_n -}$, and $\mathbf{1} = (1, ..., 1)' \in \mathbb{R}^d$. Thus, the process H^n is a semimartingale, since it possesses the decomposition

$$H^n = X_0 + M^n + \left(A^n + \mathbf{1} \cdot 1_{[\![\zeta^\Delta, \infty[\![} \right),$$

where $M^n \in \mathcal{L}$ and $A^n + \mathbf{1} \cdot 1_{[\![\zeta^\Delta, \infty[\![} \in \mathcal{V}$ for every $n \in \mathbb{N}$.
Let now h be the truncation function belonging to the semimartingale X^n for every $n \in \mathbb{N}$. In order to evaluate the characteristics of H^n, we observe that

$$\begin{aligned}
\dot{H}^n(h)_t &:= \sum_{s \le t} (\Delta H^n_s - h(\Delta H^n_s)) \\
&= \sum_{s \le (t \wedge \alpha_n)} (\Delta H^n_s - h(\Delta H^n_s)) + (\mathbf{1} - h(\mathbf{1})) 1_{[\![\zeta^\Delta, \infty[\![}(t) \\
&= \sum_{s \le t} (\Delta X^n_s - h(\Delta X^n_s)) + (\mathbf{1} - h(\mathbf{1})) 1_{[\![\zeta^\Delta, \infty[\![}(t) \\
&= \dot{X}^n(h)_t + (\mathbf{1} - h(\mathbf{1})) 1_{[\![\zeta^\Delta, \infty[\![}(t),
\end{aligned}$$

and

$$\begin{aligned}
H^n(h)_t &:= H^n_t - \dot{H}^n(h)_t \\
&= X^n_t + \mathbf{1} \cdot 1_{[\![\zeta^\Delta, \infty[\![}(t) - \left(\dot{X}^n(h)_t + (\mathbf{1} - h(\mathbf{1})) 1_{[\![\zeta^\Delta, \infty[\![}(t) \right) \\
&= X^n_t - \dot{X}^n(h)_t + h(\mathbf{1}) 1_{[\![\zeta^\Delta, \infty[\![}(t) \\
&= X^n(h)_t + h(\mathbf{1}) 1_{[\![\zeta^\Delta, \infty[\![}(t).
\end{aligned}$$

We already know that $X^n(h)$ is a special semimartingale. Thus, the previous equality shows that $H^n(h)$ also is a special semimartingale with canonical representation

$$\begin{aligned} H^n(h) &= X^n(h) + h(\mathbf{1})1_{[\![\zeta^\Delta,\infty[\![} \\ &= X_0 + M^n(h) + \left(B^n(h) + h(\mathbf{1})1_{[\![\zeta^\Delta,\infty[\![}\right), \end{aligned}$$

where $M^n(h) + B^n(h) + X_0$ is the canonical representation of $X^n(h)$. This decomposition allows to determine the characteristics $\overline{B^n}$ and $\overline{C^n}$ of H^n:

$$\begin{aligned} \overline{B^n} &:= B^n(h) + h(\mathbf{1})1_{[\![\zeta^\Delta,\infty[\![}, \\ \overline{C^n}^{(ij)} &:= \langle (H^n)^{(i),c}, (H^n)^{(j),c} \rangle = \langle (X^n)^{(i),c}, (X^n)^{(j),c} \rangle = (C^n)^{(ij)} \end{aligned}$$

for $i, j \in \{1, ..., d\}$. Analogously to the proof of Theorem II.2.42 of [36] we apply Itô's formula to H^n and obtain

$$\begin{aligned} f(H_t^n) - f(X_0) &= \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_-^n)\right) \cdot \overline{M^n}_t^{(i)} + \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_-^n)\right) \cdot \overline{B^n}_t^{(i)} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(H_-^n)\right) \cdot \overline{C^n}_t^{(i,j)} \\ &\quad + \sum_{s \le t} \left[f(H_s^n) - f(H_{s-}^n) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_{s-}^n)\right) h((H_s^n)^{(i)}) \right] \\ &= \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_-^n)\right) \cdot \overline{M^n}_t^{(i)} + \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_-^n)\right) \cdot (B^n)_t^{(i)} \\ &\quad + \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_-^n)\right) \cdot \left(h(\mathbf{1})1_{[\![\zeta^\Delta,\infty[\![}\right) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(H_-^n)\right) \cdot \overline{C^n}_t^{(i,j)} \\ &\quad + \sum_{s \le t} \left[f(X_s^n) - f(X_{s-}^n) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X_{s-}^n)\right) h((X_s^n)^{(i)}) \right] \\ &\quad + \Delta f(H_{\zeta^\Delta}^n) 1_{[\![\zeta^\Delta,\infty[\![} - \sum_{i=1}^d \frac{\partial}{\partial x^{(i)}} f(H_{\zeta^\Delta -}^n) h(\mathbf{1}) 1_{[\![\zeta^\Delta,\infty[\![} \\ &= \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_-^n)\right) \cdot (M^n)_t^{(i)} + \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_-^n)\right) \cdot (B^n)_t^{(i)} \\ &\quad + \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(H_{\zeta^\Delta -}^n)\right) h(\mathbf{1}) 1_{[\![\zeta^\Delta,\infty[\![} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(H_-^n)\right) \cdot (C^n)_t^{(i,j)} \\ &\quad + \sum_{s \le t} \left[f(X_s^n) - f(X_{s-}^n) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X_{s-}^n)\right) h((X_s^n)^{(i)}) \right] \end{aligned}$$

$$+ \Delta f(H^n_{\zeta^\Delta})1_{[\![\zeta^\Delta,\infty[\![} - \sum_{i=1}^d \frac{\partial}{\partial x^{(i)}} f(H^n_{\zeta^\Delta -})h(\mathbf{1})1_{[\![\zeta^\Delta,\infty[\![}$$

We now use the fact, that B^n_t, M^n_t and C^n_t are constant for $t \geq \alpha_n(\omega)$:

$$\begin{aligned}
&= \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) \cdot (M^n)^{(i)}_t + \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) \cdot (B^n)^{(i)}_t \\
&\quad + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(X^n_-)\right) \cdot (C^n)^{i,j}_t \\
&\quad + \sum_{s \leq t} \left[f(X^n_s) - f(X^n_{s-}) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_{s-})\right) h((X^n_s)^{(i)}) \right] \\
&\quad + \Delta f(H^n_{\zeta^\Delta})1_{[\![\zeta^\Delta,\infty[\![} \\
&= \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) \cdot (M^n)^{(i)}_t + \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) \cdot (B^n)^{(i)}_t \\
&\quad + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(X^n_-)\right) \cdot (C^n)^{i,j}_t \\
&\quad + \left[f(X^n_- + x) - f(X^n_-) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) h((X^n))^{(i)} \right] * \mu^{X^n} \\
&\quad + \Delta f(H^n) \cdot 1_{[\![\zeta^\Delta,\infty[\![}
\end{aligned}$$

The above equality provides that

$$\begin{aligned}
& f(H^n) - f(X_0) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) \cdot (B^n)^{(i)} \\
& - \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} f(X^n_-)\right) \cdot (C^n)^{(i,j)} \\
& - \left[f(X^n_- + x) - f(X^n_-) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) h((X^n))^{(i)} \right] * \nu^n - \Delta f(H^n) \cdot A \\
&= \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) \cdot (M^n)^{(i)} \\
&\quad + \left[f(X^n_- + x) - f(X^n_-) - \sum_{i=1}^d \left(\frac{\partial}{\partial x^{(i)}} f(X^n_-)\right) h((X^n))^{(i)} \right] * \left(\mu^{X^n} - \nu^n\right) \\
&\quad + \Delta f(H^n) \cdot (1_{[\![\zeta^\Delta,\infty[\![} - A)_t
\end{aligned}$$

Since the right hand side belongs to $\mathcal{M}^d$, the statement follows.
$(c.) \Rightarrow (b.)$

Let $f : \mathbb{R}^d \to \mathbb{C}; x \mapsto e^{iu'x}$ with $u \in \mathbb{R}^d$. Obviously, f is bounded, and belongs to $\mathcal{C}^2(\mathbb{R}^d)$, and we have

$$\frac{\partial}{\partial x^{(j)}} f(x) = iu^{(j)} f(x), \text{ and}$$
$$\frac{\partial^2}{\partial x^{(j)} \partial x^{(k)}} f(x) = -u^{(k)} u^{(j)} f(x).$$

If we compute the expression in $(c.)$ for the function f, we obtain that

$$\begin{aligned}
&e^{iu'H_t^n} - \sum_{j=1}^{d} (iu^{(j)} e^{iu'X_{t-}^n}) \bullet (B^n)_t^{(j)} - \frac{1}{2} \sum_{j,k=1}^{d} u^{(j)} u^{(k)} (e^{iu'X_{t-}^n}) \bullet (C^n)_t^{jk} \\
&- \int_{[0,t]\times\mathbb{R}^d} e^{iu'X_-^n} (e^{iu'x} - 1 - iu'h(x)) \, \nu^n(ds \times dx) \\
&- \Delta e^{iu'H^n} \bullet A_t \\
=& e^{iu'H_t^n} - \sum_{j=1}^{d} (iu^{(j)} e^{iu'X_{t-}^n}) \bullet (B^n)_t^{(j)} - \frac{1}{2} \sum_{j,k=1}^{d} u^{(j)} u^{(k)} (e^{iu'X_{t-}^n}) \bullet (C^n)_t^{jk} \\
&- e^{iu'X_{t-}^n} \bullet \int_{\mathbb{R}^d} (e^{iu'x} - 1 - iu'h(x)) \, \nu^n([0,t] \times dx) \\
&- \left(e^{iu'H^n} - e^{iu'H_-^n} \right) \bullet A_t \\
=& e^{iu'H_t^n} - \sum_{j=1}^{d} (iu^{(j)} e^{iu'X_{t-}^n}) \bullet (B^n)_t^{(j)} - \frac{1}{2} \sum_{j,k=1}^{d} u^{(j)} u^{(k)} (e^{iu'X_{t-}^n}) \bullet (C^n)_t^{jk} \\
&- e^{iu'X_{t-}^n} \bullet \int_{\mathbb{R}^d} (e^{iu'x} - 1 - iu'h(x)) \, \nu^n([0,t] \times dx)) \\
&- \left(e^{iu'H^n} - e^{iu'X_-^n} e^{iu'\left(\mathbf{1} \cdot 1_{[\![\zeta^\Delta, \infty[\![}\right)} \right) \bullet A_t \\
=& e^{iu'H_t^n} - e^{iu'H^n} \bullet A_t - e^{iu'X_-^n} \bullet \left(e^{iu'\left(\mathbf{1} \cdot 1_{[\![\zeta^\Delta, \infty[\![}\right)} \bullet A_t - iu'B_t^n - \frac{1}{2} u' C_t^n u \right. \\
&\left. + \int_{\mathbb{R}^d} (e^{iu'x} - 1 - iu'h(x)) \, \nu^n([0,t] \times dx) \right)
\end{aligned}$$

is a local martingale for every $n \in \mathbb{N}$.
$(b.) \Rightarrow (a.)$
Let now $e^{iu'H_t^n} - e^{iu'H^n} \bullet A_t - e^{iu'X_-^n} \bullet L^n(u)$ be a local martingale for every $n \in \mathbb{N}$ and arbitrary $u \in \mathbb{R}^d$. The process

$$\begin{aligned}
&\left(e^{iu'H_t^n} - e^{iu'H^n} \bullet A_t - e^{iu'X_-^n} \bullet L^n(u) \right)^{\alpha_n -} \\
=& e^{iu'X^n} - e^{iu'X_-^n} \bullet \left(iu'B_t^n - \frac{1}{2} u' C_t^n u + \int (e^{iu'x} - 1 - iu'h(x)) \, \nu^n([0,t] \times dx) \right)
\end{aligned}$$

is a local martingale. Application of Theorem II.2.42 of [36] provides that X^n is a semimartingale with characteristics (B^n, C^n, ν^n). Thus, the generalized semimartingale $\tilde{X}$ possesses the characteristics (B, C, ν). It remains to show that the process A is the fourth characteristic of $\tilde{X}$. Let therefore be A' the fourth characteristic of $\tilde{X}$. We already know that implication $(a.) \Rightarrow (b.)$ holds. Let $L'^n(u)$ be the process mentioned in Definition 5.1.17 with (A', B^n, C^n, ν^n) such that

$$e^{iu'H_t^n} - e^{iu'H^n} \bullet A'_t - e^{iu'X_-^n} \bullet L'^n(u)$$

is a local martingale for every $n \in \mathbb{N}$. It follows that

$$\begin{aligned} &e^{iu'H_t^n} - e^{iu'H^n} \bullet A_t - e^{iu'X_-^n} \bullet L^n(u) - \left(e^{iu'H_t^n} - e^{iu'H^n} \bullet A'_t - e^{iu'X_-^n} \bullet L'^n(u)\right) \\ &= \left(e^{iu'H^n} + e^{iu'(X_-^n + \mathbf{1}\cdot 1_{[\![\zeta^\Delta,\infty[\![})}\right) \bullet A' - \left(e^{iu'H^n} + e^{iu'(X_-^n + \mathbf{1}\cdot 1_{[\![\zeta^\Delta,\infty[\![})}\right) \bullet A \end{aligned}$$

belongs to $\mathcal{M}^d$ for all $u \in \mathbb{R}^d$. Therefore, $A' - A$ also belongs to $\mathcal{M}^d$. Thus, A is the fourth characteristics of $\tilde{X}$. □

5.2. The Symbol of a Generalized Semimartingale

The generalization of semimartingales through the introduction of killing, as we have seen in the previous section, yields a rich theory generalizing the results known for classical semimartingales. Given its wide applicability, it is no surprise that we now aim to fit the probabilistic symbol as defined in Chapter 3 into this framework. To this end, we revisit Feller processes as prime example. Dealing with non-conservative rich Feller processes, which are generalized semimartingales after separating ∂ into Δ and ∞ (cf. Corollary 2.14 of [71]) the infinitesimal generator A is of the form

$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} q(x,y)\hat{u}(\xi)\, d\xi$$

for all $u \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, where the function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is given by

$$q(x,\xi) = a(x) - i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy'\xi} + i\xi' y\chi(|y|)\, \nu(x,dy).$$

where $a(x) \geq 0$ and $(\ell(x), Q(x), \nu(x,\cdot))$ is a Lévy triplet for all fixed $x \in \mathbb{R}^d$ as we have seen in (1.7). Since for conservative, rich Feller processes the probabilistic symbol and the symbol of the operator coincide, we expect a similar result in the non-conservative case, where a represents the differential characteristic of the fourth characteristic. Consequently, we begin with the generalization of homogeneous diffusions with jumps. Let $X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$ be a Markov process.

Definition 5.2.1. Let $\tilde{X}$ be a generalized semimartingale with characteristics (A, B, C, ν) and killing times $\zeta^\infty, \zeta^\Delta$. We call $\tilde{X}$ an *autonomous semimartingale* if the characteristics are of the form

$$A_t = \int_0^t a(\tilde{X}_s)\, ds, \tag{5.12}$$

$$B_t^{(i)} = \int_0^t \ell^{(i)}(\tilde{X}_s)\, ds, \tag{5.13}$$

$$C_t^{(ij)} = \int_0^t Q^{(ij)}(\tilde{X}_s)\, ds, \tag{5.14}$$

$$\nu(\omega; dt, dx) = N(\tilde{X}_s(\omega); dx)\ ds, \tag{5.15}$$

where a is positive $\mathcal{B}(\mathbb{R}_+)$-measurable, $\ell = (\ell^{(i)})_{1\le i\le d}$ is $\mathcal{E}$-measurable, $Q = (Q^{(ij)})_{1\le i,j\le d}$ is $\mathcal{E}$-measurable with values in the set of all symmetric non-negative matrices, $N(x, dy)$ is a Borel transition kernel satisfying $N(x, \{0\}) = 0$ and $\int(|y|^2 \wedge 1)\ N(x, dy) < \infty$. We call a, ℓ, Q and $n := \int_{\{w\neq 0\}}(1 \wedge \|w\|^2)\ N(\cdot, dw)$ the *differential characteristics* of the process.

Dealing with the symbol, we could work on E with its relative topology. We make things a bit easier by prolonging the process to $\mathbb{R}^d$ by setting $X_t := x$ for $x \in \mathbb{R}^d\backslash E$ and $t \ge 0$. Hence, from now on we assume that our processes live on $\mathbb{R}^d$ respectively on $\widetilde{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty, \Delta\}$.

Definition 5.2.2. Let X be a generalized semimartingale with respect to $\mathbb{P}^x$ for every $x \in \mathbb{R}^d$. Fix a starting point $x \in \mathbb{R}^d$ and let $K \subseteq \mathbb{R}^d$ be a compact neighborhood of x. Define σ to be the first exit time of X from K:

$$\sigma := \sigma_K^x := \inf\left\{t \ge 0 : X_t \in \widetilde{\mathbb{R}^d}\backslash K\right\}. \tag{5.16}$$

The function $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by

$$p(x, \xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x\left(e^{i(X_t^\sigma - x)'\xi} - 1\right)}{t} \tag{5.17}$$

is called the *(probabilistic) symbol of the process* if the limit exists for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$ independently of the choice of K.

Theorem 5.2.3. *Let $\tilde{X}$ be an autonomous semimartingale on $\widetilde{\mathbb{R}^d}$ such that the differential characteristics a, ℓ, Q and n are continuous for every $\mathbb{P}^x$, $x \in \mathbb{R}^d$. In this case the limit* (5.17) *exists and the symbol of X is*

$$p(x, \xi) = a(x) - i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0}\left(e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y)\right)\ N(x, dy). \tag{5.18}$$

for $x, \xi \in \mathbb{R}^d$.

Proof. See Theorem 2.18 of [71]. □

5.3. Subprocesses of Markov Semimartingales

The notion of generalized semimartingales as introduced in Definition 5.1.6 might initially seems to be artificial. In this section, we will show how a sudden killing arises naturally in the context of semimartingales. To this end, let us reconsider a Lévy process $\tilde{L}$ with exponential killing (cf. Example 5.1.9) and characteristic exponent

$$\Psi(\xi) = \lambda - i\ell'\xi + \frac{1}{2}\xi' Q\xi + \int_{\mathbb{R}^d\backslash\{0\}} (1 - e^{iy'\xi} + iy'\xi\chi(y))\ \nu(dy)$$

where (ℓ, Q, ν) is a Lévy triplet and $\lambda > 0$. For such a process, we have seen that the killing time ζ_Δ is exponentially distributed with parameter λ. Since the expected time of killing $\mathbb{E}(\zeta_\Delta)$ is given by $1/\lambda$, independently of the process L, λ can be viewed as the 'rate of the killing'. Keeping the Lévy-context in mind, we want to 'kill' a classical semimartingale X with a 'killing-rate' which depends on the path-wise development of the process. More precisely, for a certain time $t \geq 0$ we want to consider a killing rate of the form

$$\Lambda_t(\omega) := \frac{1}{t} \int_0^t a(X_s(\omega))\, ds$$

for a fitting function $a : \mathbb{R}^d \to \mathbb{R}_+$ taking the place of the parameter λ from the Lévy-context. In other words, we want the cumulative distribution function of the killing time to be given by

$$F(t) = 1 - \mathbb{E}\left(e^{-\int_0^t a(X_u) du}\right).$$

Conveniently, a generalization of such a concept already exists in the Markovian framework as we will point out in detail in the following subsection.

5.3.1. Multiplicative Functionals and Canonical Subprocesses

Let

$$X := (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$$

be a conservative, càdlàg Markov process.

Definition 5.3.1. A right-continuous stochastic process $M = (M_t)_{t\geq 0}$ on a measurable space $(\Omega, \mathcal{F})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ which takes values in $\mathbb{R}_+$ is called *multiplicative functional (MF)* associated to X if

(i.) $M_{t+s} = (M_s \circ \theta_t) \cdot M_t$ $\mathbb{P}^x$-a.s. for all $t, s \geq 0$ and $x \in E$,

(ii.) $0 \leq M_t(\omega) \leq 1$ for all $t \geq 0$ and $\omega \in \Omega$,

(iii.) $M_0 = 1$ $\mathbb{P}^x$-a.s. for all $x \in E$.

Remark 5.3.2. (a.) In the literature, a multiplicative functional is not always restricted to $[0, 1]$. However, for the purpose of constructing probability measures with the help of multiplicative functionals, as we will do in the following, this constraint seems appropriate.

(b.) By (i.) and (ii.) the function $M_\cdot(\omega)$ decreases to zero. It will be convenient to set $M_\infty := 0$.

Indeed, multiplicative functionals generalize the examples discussed in the introduction:

Example 5.3.3. (1.) The deterministic process $(M_t = e^{-\lambda t})_{t\geq 0}$ for $\lambda > 0$ is a multiplicative functional.

(2.) Let X be a Markov process, and let a be a positive, $\mathcal{B}(\mathbb{R})$-measurable function. Then the stochastic process defined by

$$M_t = e^{-\int_0^t a(X_s)\,ds} \tag{5.19}$$

for all $t \geq 0$ is a MF. M is continuous if the function a is bounded. But it does not have to be right-continuous if a is unbounded.
In case that the filtration $(\mathcal{F}_t)_{t\geq 0}$ of M is right-continuous the functional defined by

$$N_t := 1_{[0,\zeta)}(t)e^{-\int_0^t a(X_s)\,ds}$$

for $\zeta = \inf\left\{r \in \mathbb{R}_+ : \int_0^r a(X_u)\,du = \infty\right\}$ is right continuous (cf. [8] III.1.5).

We now utilize multiplicative functionals to add a 'killing' to the Markov process X: We construct a process $\tilde{X}$ that coincides with X up to a certain stopping time T and is Δ after T, for $\Delta \notin E$. We enlarge the underlying measurable space as follows: Let

$$\tilde{\Omega} := \Omega \times \mathbb{R}_+, \quad \widetilde{\mathcal{M}} := \mathcal{M} \otimes \mathcal{B}(\mathbb{R}_+).$$

Moreover, we need to consider a new family of probability measures on $(\tilde{\Omega}, \widetilde{\mathcal{M}})$ to obtain a positive probability to leave the state space E. Therefore, we consider the transition kernel $\kappa : (\Omega, \mathcal{M}) \times (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \to [0,1]$ defined by

$$\kappa(\omega, (t,\infty)) := M_t(\omega) \tag{5.20}$$

for each $\omega \in \Omega,\ t \geq 0$ and a multiplicative functional M of X. Utilizing κ, we are able to define a family of probability measures $(\tilde{\mathbb{P}}^x)_{x\in E}$ on $(\tilde{\Omega}, \widetilde{\mathcal{M}})$ as follows:

$$\widetilde{\mathbb{P}}^x(\tilde{\Lambda}) := \mathbb{E}^x\left(\kappa(\omega, \tilde{\Lambda}_\omega)\right)$$

for $t \geq 0$, $\tilde{\Lambda} \in \widetilde{\mathcal{F}}$ and $\tilde{\Lambda}_\omega := \{r \in \mathbb{R}_+ : (\omega, r) \in \tilde{\Lambda}\}$. If $\tilde{\Lambda}$ is of the form $\Lambda \times (t,\infty)$, this reads as

$$\widetilde{\mathbb{P}}^x(\Lambda \times (t,\infty)) = \int_\Lambda M_t\, d\tilde{\mathbb{P}}^x.$$

In addition, we choose $\mathbb{P}^\Delta$ to be a probability measure fulfilling $\mathbb{P}^\Delta(\Omega \times \{0\}) = 1$.
At this point, we want to mention the necessity of M taking values in $[0,1]$ and the convention of setting $M_0 = 1$. Otherwise $\tilde{\mathbb{P}}^x$ would not define a probability measure.
Let now $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ be defined by

$$\tilde{X}_t(\tilde{\omega}) = \tilde{X}_t(\omega, r) := \begin{cases} X_t(\omega) & \text{, if } t < r \\ \Delta & \text{, if } t \geq r \end{cases} \tag{5.21}$$

for $t \geq 0$ and $\tilde{\omega} := (\omega, r) \in \tilde{\Omega}$. Obviously, $\tilde{X}$ takes values in $E_\Delta = E \cup \{\Delta\}$, which is the Alexandrov compactification of E by Δ. Moreover, let $\mathcal{E}_\Delta$ be a σ-field on E_Δ with $\mathcal{E} \subset \mathcal{E}_\Delta$. Finally, we state a filtration to which $\tilde{X}$ is adapted. Hence, we define the σ-field

$$\widetilde{\mathcal{M}_t} := \left\{\tilde{\Lambda} \in \tilde{\mathcal{M}} \mid \exists \Lambda \in \mathcal{M}_t : \tilde{\Lambda} \cap (\Omega \times (t,\infty)) = \Lambda \times (t,\infty)\right\}, \tag{5.22}$$

for $t \geq 0$. Additionally $(\widetilde{\mathcal{M}_t})_{t\geq 0}$ forms a filtration (cf. Proposition III.3.1 of [8]). For $\tilde{\Lambda}_t \in \widetilde{\mathcal{M}_t}$ we call the set $\Lambda_t \in \mathcal{M}_t$ with $\tilde{\Lambda}_t \cap (\Omega \times (t,\infty)) = \Lambda_t \times (t,\infty)$ the *corresponding set*. Intuitively, $\widetilde{\mathcal{M}_t}$ includes all sets which are indifferent whether $\tilde{X}$ or X is considered when knowing that $\tilde{X}$ is not killed.
Summing up, $\tilde{X}$ is a stochastic process on the stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{M}}, (\widetilde{\mathcal{M}_t})_{t\geq 0}, \widetilde{\mathbb{P}}^x)_{x\in\mathbb{R}^d\cup\{\Delta\}}$ which possesses a killing.

Definition 5.3.4. Let M be a multiplicative functional of the Markov process X. We call the process $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{M}}, (\widetilde{\mathcal{M}_t})_{t\geq 0}, (\tilde{\theta}_t)_{t\geq 0}, \widetilde{\mathbb{P}}^x)_{x\in E_\Delta}$ constructed above the *canonical subprocess* of X and M.

Remark 5.3.5. Since the canonical subprocess $\tilde{X}$ is a process with killing, we take a closer look at the stopping time

$$T := \zeta^\Delta = \inf\{t \geq 0 : \tilde{X}_t = \Delta\}.$$

It holds true that $X = \tilde{X}$ on $[\![0, T[\![$, and $\tilde{X} = \Delta$ on $[\![T, \infty[\![$. In order to describe the killing time of $\tilde{X}$ more precisely, we consider the distribution function of T under $\widetilde{\mathbb{P}}^x$:

$$\begin{aligned} F_T^x(t) &= \widetilde{\mathbb{P}}^x(T \in [0,t]) \\ &= \widetilde{\mathbb{P}}^x(\tilde{X}_t = \Delta) \\ &= \widetilde{\mathbb{P}}^x(\Omega \times [0,t]) \\ &= 1 - \mathbb{E}^x\left(M_t\right). \end{aligned} \tag{5.23}$$

The last equality follows by the definition of $\widetilde{\mathbb{P}}^x$. When considering the prime Example 5.3.3 (2.), i.e.

$$M_t = e^{-\int_0^t a(X_u)du}, \quad t \geq 0,$$

it holds true that

$$F_T^x(t) = 1 - \mathbb{E}^x\left(e^{-\int_0^t a(X_u)du}\right).$$

Since

$$\begin{aligned} 1 - \mathbb{E}^x\left(e^{-\int_0^t a(X_u)du}\right) &= \mathbb{E}^x\left(1 - e^{-\int_0^t a(X_u)\ du}\right) \\ &= \mathbb{E}^x\left(\int_0^t a(X_s)e^{-\int_0^s a(X_u)\ du}\ ds\right), \end{aligned}$$

Fubini's theorem provides that $\mathbb{E}^x[a(X_s)\exp(-\int_0^s a(X_u)\ du)]$ is the Lebesgue-density function of T with respect to $\mathbb{P}^x$.

Lemma 5.3.6. *Let X be a Markov process and M be a multiplicative functional of X, then the canonical subprocess $\tilde{X}$ is a Markov process with*

$$\tilde{\mathbb{E}}^x(f(\tilde{X}_t)) = \mathbb{E}^x(f(X_t)M_t) \tag{5.24}$$

for all bounded $\mathcal{E}$-measurable functions f. Moreover, the shift-operator is defined by

$$\tilde{\theta}_t(\tilde{\omega}) = \tilde{\theta}_t(\omega, r) := (\theta_t(\omega), (r-t) \vee 0),$$

where $t \geq 0$ *and* $\tilde{\omega} = (\omega, r) \in \tilde{\Omega}$*.*

Proof. See Theorem III.3.3 of [8]. □

Corollary 5.3.7. *Let* X *be a Markov process and* M *be a multiplicative functional of* X*, and* $\tilde{X}$ *the canonical subprocess. The equality*

$$\int_{\tilde{\Lambda}} f(\omega)\, d\tilde{\mathbb{P}}^x(\omega, r) = \int \kappa(\omega, \tilde{\Lambda}_\omega) f(\omega)\, d\mathbb{P}^x(\omega)$$

holds true for $\tilde{\Lambda} \in \widetilde{\mathcal{F}}_t$*, a* $\mathcal{E}$*-measurable, bounded function* f *and* $\tilde{\Lambda}_\omega := \{r \geq 0 : (\omega, r) \in \tilde{\Lambda}\}$*.*

Proof. Let $F \in \mathcal{F}$ and $\tilde{\Lambda} \in \widetilde{\mathcal{F}}_t$. We compute for a $\mathcal{F}$-measurable, bounded function f that

$$\begin{aligned} \int_{\tilde{\Lambda}} 1_F(\omega)\, d\tilde{\mathbb{P}}^x(\omega, r) &= \tilde{\mathbb{P}}^x\left(\tilde{\Lambda} \cap (F \times \mathbb{R}_+)\right) \\ &= \int \kappa\left(\omega, (\tilde{\Lambda} \cap (F \times \mathbb{R}_+))_\omega\right)\, d\mathbb{P}^x(\omega) \\ &= \int \kappa\left(\omega, \tilde{\Lambda}_\omega \cap (F \times \mathbb{R}_+)_\omega\right)\, d\mathbb{P}^x(\omega) \\ &= \int \kappa\left(\omega, \tilde{\Lambda}_\omega\right) 1_F(\omega)\, d\mathbb{P}^x(\omega) \end{aligned}$$

where the last equality holds since $(F \times \mathbb{R}_+)_\omega = 1_F(\omega)$. The statement follows by a monotone class theorem. □

At the beginning of this section we already started to outline the role of a 'killing rate' in the context of MF. We want to make this consideration more clearly with the next remark.

Remark 5.3.8. Let us again consider the assumptions of Lemma 5.3.6. Furthermore, we consider the σ-algebra $\mathcal{F}^* := \mathcal{F} \otimes \{\emptyset, \mathbb{R}_+\}$ on $\tilde{\Omega}$. The σ-algebra $\mathcal{F}^*$ contains only information of the original process X. One easily computes that

$$\tilde{\mathbb{P}}^x\left[\tilde{X}_t \neq \Delta \mid \mathcal{F}^*\right] = M_t$$

for all $x \in E$. Thus, the probability of not being killed in time t provided the information of X is given by M_t. To this end, the fraction

$$\frac{\tilde{\mathbb{P}}^x\left[\tilde{X}_{t+h} = \Delta, \tilde{X}_t \neq \Delta \mid \mathcal{F}^*\right]}{\tilde{\mathbb{P}}^x\left[\tilde{X}_t \neq \Delta \mid \mathcal{F}^*\right]} = \frac{M_t - M_{t+h}}{M_t} = 1 - M_h \circ \theta_t$$

states, in some sense, the probability that the process is killed in $t + h$ provided it was not killed before T under the information of the process X. By dividing this expression by h and taking the limit to $h \to 0$, we obtain some kind of 'killing rate' at time t, provided the limit exists.
For the multiplicative functional $M_t = \exp(-\int_0^t a(X_u)\, du)$ this leads to the following killing rate:

$$\lim_{h \to 0} \frac{1 - M_h \circ \theta_t}{h} = \lim_{h \to 0} \frac{1 - e^{-\int_0^h a(X_{u+t}) du}}{h} = \lim_{h \to 0} a(X_{t+h}).$$

5.3.2. Subprocesses of Markov Semimartingales

For Markov processes, multiplicative functionals and the associated subprocesses pose a natural way to add a killing to a conservative Markov process while also maintaining the Markovian structure of the process. Hence, it seems natural, in order to establish a killing of a Markov semimartingale, to consider the canonical subprocess. To this end, we have to make sure that the semimartingale property is preserved when transitioning to the subprocess.

Lemma 5.3.9. *Let X be a Markov semimartingale. The canonical subprocess $\tilde{X}$ is a generalized semimartingale for all $\tilde{\mathbb{P}}^x$, $x \in E_\Delta$ with killing time $\zeta^\Delta = T$ and $\zeta^\infty = \infty$.*

Proof. Let X be a semimartingale on $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$, and let

$$X_t = X_0 + M_t + A_t, \quad t \geq 0$$

be the decomposition of X, where $M \in \mathcal{M}_{loc}$ and $A \in \mathcal{V}$ on $(\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in E}$. We have $A \in \mathcal{V}$ on $(\tilde{\Omega}, \widetilde{\mathcal{M}}, (\widetilde{\mathcal{M}_t})_{t\geq 0}, \widetilde{\mathbb{P}}^x)$. To show that M is a local martingale with respect to $(\tilde{\Omega}, \widetilde{\mathcal{M}}, (\widetilde{\mathcal{M}_t})_{t\geq 0}, \widetilde{\mathbb{P}}^x)$ we consider the following:
Let T_n be the localization sequence of M. The mapping $\tilde{\omega} = (\omega, r) \mapsto T_n(\omega)$ is a stopping time with respect to the new stochastic basis for all $n \in \mathbb{N}$. Corollary 5.3.7 provides for $\widetilde{\Lambda} \in \widetilde{\mathcal{M}}_s$

$$\begin{aligned}
\int_{\widetilde{\Lambda}} M_t^{T_n} \, d\widetilde{\mathbb{P}}^x &= \int \kappa(\omega, \widetilde{\Lambda}_\omega) M_t^{T_n} \, d\mathbb{P}^x \\
&= \int \kappa(\omega, \widetilde{\Lambda}_\omega) \mathbb{E}[M_t^{T_n} \mid \mathcal{F}_s] \, d\mathbb{P}^x \\
&= \int \kappa(\omega, \widetilde{\Lambda}_\omega) M_s^{T_n} \, d\mathbb{P}^x \\
&= \int_{\widetilde{\Lambda}} M_s^{T_n} \, d\widetilde{\mathbb{P}}^x.
\end{aligned}$$

Hence, M is a $(\tilde{\Omega}, \tilde{\mathcal{M}}, (\tilde{\mathcal{M}}_t)_{t\geq 0}, \widetilde{\mathbb{P}}^x)$ local martingale, and, therefore, X is a semimartingale with respect to the new stochastic basis.
It follows that X^{T-} is a semimartingale to the new filtration, and since

$$X^{T-} = \tilde{X}^{T-}$$

the process $\tilde{X}^{T-}$ is a semimartingale. By definition, $\tilde{X}$ is a generalized semimartingale with killing time T. □

In the previous proof, we have seen that for the subprocess being a generalized semimartingale if the original process is a classical semimartingale we did not need any properties of the multiplicative functional, except the ones we need for the kernel κ to be well-defined.

Example 5.3.10. (a.) Let $(B_t)_{t\geq 0}$ be a standard Brownian Motion starting in $x \in \mathbb{R}$. Moreover, let $(\tilde{B})_{t\geq 0}$ be the canonical subprocess related to B and the multiplicative functional, $M_t := \exp\left(-\int_0^t B_s + s^2\, ds\right)$ for $t \geq 0$. In order to determine the distribution function of the killing time T we calculate:

$$\begin{aligned}
F_T^x(t) &= \mathbb{E}^x\left(1 - e^{-\int_0^t B_s + s^2\, ds}\right)\\
&= \mathbb{E}\left(1 - e^{-\int_0^t B_s + x + s^2\, ds}\right)\\
&= \mathbb{E}\left(1 - e^{-\int_0^t B_s ds - xt - \frac{1}{3}t^3}\right)\\
&= \mathbb{E}\left(1 - e^{-tB_t + \int_0^t s dB_s - xt - \frac{1}{3}t^3}\right)\\
&= \mathbb{E}\left(1 - e^{-\int_0^t (t-s) dB_s - xt - \frac{1}{3}t^3}\right)\\
&= 1 - e^{-\frac{1}{3}t^3 - xt}\mathbb{E}\left(e^{-\int_0^t (t-s) dB_s}\right),
\end{aligned}$$

and since $\int_0^t (t-s)\, dB_s \sim \mathcal{N}(0, t^3/3)$ we conclude

$$\begin{aligned}
F_T(t) &= 1 - e^{-\frac{1}{3}t^3 - xt}\frac{1}{\sqrt{\frac{2}{3}\pi t^3}}\int e^{-s} e^{-\frac{s^2}{2/3t^3}}\, ds\\
&= 1 - e^{-\frac{1}{3}t^3 - xt} e^{\frac{1}{6}t^3}\\
&= 1 - e^{-\frac{1}{6}t^3 - xt}.
\end{aligned}$$

(b.) Let $(L_t')_{t\geq 0}$ be a Lévy process with characteristic exponent ϕ and $(L_t)_{t\geq 0}$ the corresponding Lévy process starting in $x \in \mathbb{R}$. The distribution function of the killing-time of the subprocess $(\tilde{L})_{t\geq 0}$ associated with L and the multiplicative functional $M_t := \exp\left(-\int_0^t L_s\, ds\right)$ is given by

$$F_T^x(t) := 1 - e^{-\int_0^t \phi(is) ds - tx}.$$

Proof. We show that the characteristic function of the random variable

$$M_t := \int_0^t L_s'\, ds$$

is given by

$$\varphi_t(\xi) = e^{-\int_0^t \phi(s\xi) ds}.$$

The statement follows by the following argumentation: Let $(t_j^n)_{j=0,\dots,k_n}$ be an equidistant partition of length $1/(tn)$ of the interval $[0, t]$. We already know that

$$\frac{1}{tn}\sum_{j=0}^{k_n} L_{t_j}' \overset{a.s.}{\to} \int_0^t L_s'\, ds$$

for $n \to \infty$. We consider the following

$$\mathbb{E}\left(\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n} L_{t_j}'\right)\right) = \mathbb{E}\left[\mathbb{E}\left(\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n} L_{t_j}'\right)\right) \mid \mathcal{F}_{t_{k_n - 1}}^{L'}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n-1} L'_{t_j} + i\xi\frac{1}{tn}L'_{t_{k_n}-1}\right)\mathbb{E}\left(\exp\left(\frac{1}{tn}i\xi(L'_{t_{k_n}} - L'_{t_{k_n}-1})\right) \mid \mathcal{F}^{L'}_{t_{k_n-1}}\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n-1} L'_{t_j} + i\xi\frac{1}{tn}L'_{t_{k_n}-1}\right)\mathbb{E}\left(\exp\left(\frac{1}{tn}i\xi(L'_{t_{k_n}} - L'_{t_{k_n}-1})\right)\right)\right] \\
&= \mathbb{E}\left(\exp\left(\frac{1}{tn}i\xi(L'_{t_{k_n}} - L'_{t_{k_n}-1})\right)\right)\mathbb{E}\left[\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n-1} L'_{t_j} + i\xi\frac{1}{tn}L'_{t_{k_n}-1}\right)\right] \\
&= \mathbb{E}\left(\exp\left(\frac{1}{tn}i\xi L'_{1/(tn)}\right)\right)\mathbb{E}\left[\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n-1} L'_{t_j} + i\xi\frac{1}{tn}L'_{t_{k_n}-1}\right)\right] \\
&= e^{-\frac{1}{tn}\phi\left(\frac{\xi}{tn}\right)}\mathbb{E}\left[\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n-1} L'_{t_j} + i\xi\frac{1}{tn}L'_{t_{k_n}-1}\right)\right].
\end{aligned}
$$

Iteration of this procedure provides that

$$
\mathbb{E}\left(\exp\left(i\xi\frac{1}{tn}\sum_{j=0}^{k_n} L'_{t_j}\right)\right) = \exp\left(-\frac{1}{tn}\sum_{j=0}^{k_n}\phi\left(\frac{j}{tn}\xi\right)\right)
$$

and, therefore,

$$
\varphi_t(\xi) = e^{-\int_0^t \phi(s\xi)ds}.
$$

We follow that:

$$
\begin{aligned}
F_T^x(t) &= 1 - \mathbb{E}^x\left(\exp\left(-\int_0^t L_s\, ds\right)\right) \\
&= 1 - \mathbb{E}\left(\exp\left(-\int_0^t L'_s + x\, ds\right)\right) \\
&= 1 - \mathbb{E}\left(\exp\left(-\int_0^t L'_s\, ds\right)\right)e^{-tx} \\
&= 1 - e^{-\int_0^t \phi(is)ds}e^{-tx}.
\end{aligned}
$$

□

Up to this point, we have observed that the canonical subprocess of a Markov semimartingale X is a generalized Markov semimartingale. Let X be an Itô process, i.e. X is a Markov semimartingale with characteristics (B, C, ν) of the form

$$
\begin{aligned}
B_t^{(i)} &= \int_0^t \ell^{(i)}(X_s)\, ds, \\
C_t^{(ij)} &= \int_0^t Q^{(ij)}(X_s)\, ds, \\
\nu(\omega; dt, dx) &= N(X_t(\omega); dx)\, ds.
\end{aligned}
$$

Since the canonical subprocess possesses no explosion killing its characteristics $(\tilde{B}, \tilde{C}, \tilde{\nu})$ are of the form

$$\begin{aligned}
\tilde{B}_t^{(i)} &= \int_0^{t\wedge T-} \ell^{(i)}(\tilde{X}_s)\, ds, \\
\tilde{C}_t^{(ij)} &= \int_0^{t\wedge T-} Q^{(ij)}(\tilde{X}_s)\, ds, \\
\tilde{\nu}(\omega; dt, dx) &= N(\tilde{X}_s(\omega); dx) 1_{[\![0,T[\![}(\omega, s)\, ds.
\end{aligned}$$

The next theorem provides a necessary and sufficient condition of T for the canonical subprocess of an Itô process to be an autonomous semimartingale.

Theorem 5.3.11. *Let M be a multiplicative functional of the Markov process X. Moreover, let T be the killing-time of the canonical subprocess $\tilde{X}$. Let $a : \mathbb{R}^d \to \mathbb{R}_+$ be measurable and locally bounded. Then the predictable compensator A of the stochastic process $1_{[\![T,\infty[\![}$ is of the form*

$$A_t(\tilde{\omega}) = A_t(\omega, r) = \int_0^{t\wedge T(\tilde{\omega})} a(X_u(\omega))du = \int_0^t a(X_u(\omega)) 1_{(u,\infty)}(r) du$$

if and only if the distribution function $F_T^x(t) = 1 - \mathbb{E}^x(M_t)$ of the stopping time T under $\mathbb{P}^x$ is absolutely continuous with Lebesgue-density function $f^x(u) = \mathbb{E}^x(a(X_u)M_u)$.

Proof. Let $A_t(\tilde{\omega}) = \int_0^{t\wedge T(\tilde{\omega})} a(X_u(\omega))du$ be the predictable compensator of $1_{[\![T,\infty[\![}$. Since the process $1_{[\![T,\infty[\![}$ is an uniformly integrable submartingale it holds for $0 \le s \le t$

$$\begin{aligned}
&\tilde{\mathbb{E}}^x \left[1_{[\![T,\infty[\![}(\tilde{\omega}, t) - \int_0^t a(X_u(\omega)) 1_{(u,\infty)}(r) du \mid \tilde{\mathcal{F}}_s \right] = 1_{[\![T,\infty[\![}(\tilde{\omega}, s) - \int_0^s a(X_u(\omega)) 1_{(u,\infty)}(r) du \\
\Leftrightarrow\ &\tilde{\mathbb{E}}^x \left[1_{[0,t]}(r) - \int_0^t a(X_u(\omega)) 1_{(u,\infty)}(r) du \mid \tilde{\mathcal{F}}_s \right] = 1_{[0,s]}(r) - \int_0^s a(X_u(\omega)) 1_{(u,\infty)}(r) du \\
\Leftrightarrow\ &\tilde{\mathbb{E}}^x \left[1_{(s,t]}(r) - \int_s^t a(X_u(\omega)) 1_{(u,\infty)}(r) du \mid \tilde{\mathcal{F}}_s \right] = 0.
\end{aligned}$$

In particular, for each $\Lambda_s \in \mathcal{F}_s$ and $\tilde{\Lambda}_s = \Lambda_s \times \mathbb{R}_+$ we derive with Tonelli's theorem and Corollary 5.3.7:

$$\begin{aligned}
\int_{\tilde{\Lambda}_s} 1_{(s,t]}(r)\, d\widetilde{\mathbb{P}}^x &= \int_{\tilde{\Lambda}_s} \int_s^t a(X_u(\omega)) 1_{(u,\infty)}(r) du\, d\widetilde{\mathbb{P}}^x \\
&= \int_s^t \int_{\tilde{\Lambda}_s} a(X_u(\omega)) 1_{(u,\infty)}(r) d\widetilde{\mathbb{P}}^x\, du \\
&= \int_s^t \int_{\Lambda_s} a(X_u(\omega)) M_u(\omega) d\mathbb{P}^x\, du \\
&= \int_{\Lambda_s} \int_s^t a(X_u(\omega)) M_u(\omega) du\, d\mathbb{P}^x.
\end{aligned}$$

The left hand side equals $\int_{\Lambda_s} M_s - M_t\, d\mathbb{P}^x$ by definition of $\widetilde{\mathbb{P}}^x$. It follows that

$$\mathbb{E}^x \left[M_s - M_t \mid \mathcal{F}_s \right] = \int_s^t a(X_u) M_u\, du$$

$$
\begin{aligned}
\Leftrightarrow \mathbb{E}^x\left[M_t \mid \mathcal{F}_s\right] &= M_s - \int_s^t a(X_u)M_u\, du \\
\Leftrightarrow \mathbb{E}^x\left[M_{s+(t-s)} \mid \mathcal{F}_s\right] &= M_s - \int_s^t a(X_u)M_u\, du \\
\Leftrightarrow \mathbb{E}^x\left[M_s(M_{t-s} \circ \theta_s) \mid \mathcal{F}_s\right] &= M_s - \int_s^t a(X_u)M_u\, du \\
\Leftrightarrow M_s\mathbb{E}^{X_s}\left[M_{t-s}\right] &= M_s - \int_s^t a(X_u)M_u\, du
\end{aligned}
$$

where we used the Markov-property in the last equation. By choosing $s = 0$, the previous equation provides

$$
\mathbb{E}^{X_0}\left[M_t\right] = 1 - \int_0^t a(X_u)M_u\, du
$$

for $t \geq 0$. Moreover, taking the expected value with respect to $\mathbb{P}^x$ we have

$$
\begin{aligned}
\mathbb{E}^x\left(\mathbb{E}^{X_0}(M_t)\right) &= 1 - \mathbb{E}^x\left(\int_0^t a(X_u)M_u\, du\right) \\
\Leftrightarrow \mathbb{E}^x\left(M_t\right) &= 1 - \int_0^t \mathbb{E}^x(a(X_u)M_u)\, du \\
\Leftrightarrow \mathbb{E}^x\left(1 - M_t\right) &= \int_0^t \mathbb{E}^x(a(X_u)M_u)\, du.
\end{aligned}
$$

For the converse, let F_T^x be absolutely continuous for every $x \in \mathbb{R}^d$, and let $f^x(u) = \mathbb{E}^x(a(X_u)M_u)$ be its Lebesgue-density function. That is, we have

$$
\mathbb{E}^x(1 - M_t) = \int_0^t \mathbb{E}^x(a(X_u)M_u)\, du.
$$

At first, we observe that for $0 \leq s \leq t$ the following holds true

$$
\begin{aligned}
&\tilde{\mathbb{E}}^x\left[\int_0^t a(X_u(\omega))1_{(u,\infty)}(r)du \mid \tilde{\mathcal{F}}_s\right] \\
&= \int_0^s a(X_u(\omega))1_{(u,\infty)}(r)\, du \;+\; \tilde{\mathbb{E}}^x\left[\int_s^t a(X_u(\omega))1_{(u,\infty)}(r)du \mid \tilde{\mathcal{F}}_s\right].
\end{aligned}
$$

With the Markov-property and Fubini's theorem we derive for $\tilde{\omega} = (\omega, r) \in \tilde{\Omega}$ that

$$
\begin{aligned}
\tilde{\mathbb{E}}^x\left[\int_s^t a(X_u(\omega))1_{(u,\infty)}(r)du \mid \tilde{\mathcal{F}}_s\right] &= \tilde{\mathbb{E}}^x\left[\int_s^t a(\tilde{X}_u(\tilde{\omega}))1_{\{\tilde{X}_u \neq \Delta\}}(\tilde{\omega})du \mid \tilde{\mathcal{F}}_s\right] \\
&= \tilde{\mathbb{E}}^x\left[\int_0^{t-s} a(\tilde{X}_{u+s}(\tilde{\omega}))1_{\{\tilde{X}_{u+s} \neq \Delta\}}(\tilde{\omega})du \mid \tilde{\mathcal{F}}_s\right] \\
&= \tilde{\mathbb{E}}^{\tilde{X}_s}\left[\int_0^{t-s} a(\tilde{X}_u(\tilde{\omega}))1_{\{\tilde{X}_u \neq \Delta\}}(\tilde{\omega})du\right] \\
&= \tilde{\mathbb{E}}^{\tilde{X}_s}\left[\int_0^{t-s} a(\tilde{X}_u(\tilde{\omega}))1_{(u,\infty)}(r)du\right] \\
&= \int_0^{t-s} \tilde{\mathbb{E}}^{\tilde{X}_s}\left(a(\tilde{X}_u(\tilde{\omega}))1_{(u,\infty)}(r)\right) du
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{t-s} \tilde{\mathbb{E}}^{X_s} \left(a(\tilde{X}_u(\tilde{\omega})) 1_{(u,\infty)}(r) \right) 1_{\{\tilde{X}_s \neq \Delta\}} du \\
&= 1_{\{\tilde{X}_s \neq \Delta\}} \int_0^{t-s} \mathbb{E}^{X_s} \left(a(X_u(\omega)) M_u(\omega) \right) du \\
&= \mathbb{E}^{X_s} \left(1 - M_{t-s}(\omega) \right) 1_{\{\tilde{X}_s \neq \Delta\}} \\
&= \tilde{\mathbb{E}}^{X_s} \left(1_{\{\tilde{X}_{t-s}(\tilde{\omega}) = \Delta\}} \right) 1_{\{\tilde{X}_s \neq \Delta\}} \\
&= \tilde{\mathbb{E}}^{\tilde{X}_s} \left(1_{\{\tilde{X}_{t-s}(\tilde{\omega}) = \Delta\}} \right) 1_{\{\tilde{X}_s \neq \Delta\}} \\
&= \tilde{\mathbb{E}}^{x} \left[1_{[\![T,\infty[\![}(\tilde{\omega}, t) - 1_{[\![T,\infty[\![}(\tilde{\omega}, s) \mid \tilde{\mathcal{F}}_s \right]
\end{aligned}
$$

In summation, we obtain

$$
\begin{aligned}
&\tilde{\mathbb{E}}^{x} \left[1_{[\![T,\infty[\![}(\tilde{\omega}, t) - \int_0^t a(X_u(\omega)) 1_{(u,\infty)}(r) du \mid \tilde{\mathcal{F}}_s \right] \\
&= 1_{[\![T,\infty[\![}(\tilde{\omega}, s) - \int_0^s a(X_u(\omega)) 1_{(u,\infty)}(r) du.
\end{aligned}
$$

Since a is locally bounded A is continuous and, therefore, predictable. □

The following corollary is immediate from the previous theorem and Theorem 5.2.3.

Corollary 5.3.12. *Let X be an Itô process with differential characteristics ℓ, Q and n and M a multiplicative functional with Lebesgue-density function $f^x(u) = \mathbb{E}^x(a(X_u)M_u)$, where $a : \mathbb{R}^d \to \mathbb{R}_+$ is measurable and locally bounded. In this case the canonical subprocess $\tilde{X}$ is an autonomous semimartingale and the symbol of $\tilde{X}$ is given by*

$$
p(x,\xi) = a(x) - i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left(e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y) \right) N(x,dy),
$$

where $x, \xi \in \mathbb{R}^d$.

Remark 5.3.13. For an Itô process X and the additive functional

$$
M_t = e^{-\int_0^t a(X_u) du}, \quad t \geq 0,
$$

where $a : \mathbb{R}^d \to \mathbb{R}$ is positive, measurable and locally bounded, we have observed that the distribution function of T with respect to $\mathbb{P}^x$ is given by $\mathbb{E}^x[a(X_s)\exp(-\int_0^s a(X_u)\, du)]$. Hence, the canonical subprocess is an autonoumous semimartingale.

6 Conclusion and Outlook

As we have pointed out on several occasions during this work, the probabilistic symbol plays a crucial role in analyzing various properties related to the underlying process. In Chapter 3, in particular in Theorem 3.1.4, we observed the existence of the symbol for Itô processes. We even pointed out its existence for homogeneous diffusions with jumps. Both of these classes contain Lévy processes and rich Feller processes. However, Theorem 3.1.7 demonstrated that the symbol of a Hunt semimartingale, whose differential characteristics fulfill certain properties, exists if and only if the considered process is already an Itô process. Furthermore, we noted that the applicability of the symbol might be lost for processes that are not Hunt semimartingales, even if the symbol exists. To illustrate this, we examined two Markov semimartingales in Examples 3.1.8 and 3.2.9, both of which admitted a symbol. However, it became evident that the symbol lost information about the respective processes. For the construction of these processes we added jumps at fixed times to an Itô process. Consequently, the processes, now possessing fixed times of discontinuity, are no Hunt semimartingales. Based on these observations, the question arises how one could modify the process or its characteristics without altering the symbol. Or to put it another way, which processes possess identical symbols provided they exist.
In Section 3.2 we introduced the generalized Blumenthal-Getoor indices as it has been done for Feller processes by Schilling in [66], and subsequently extended by Schnurr in [69] to homogeneous diffusions with jumps. In Remark 3.2.2 we pointed out that Aït-Sahalia and Jacod also provided a generalization of the classical Blumenthal-Getoor indices, but in a different manner as stated in this work. It would be intriguing to consider whether or not these different generalizations are connected and if the theory developed for these distinct generalizations can be applied to each other.

In the fourth chapter, we considered non-homogeneous processes. In doing so, we introduced the so-called time-dependent symbol. This is essentially the probabilisitc symbol but modified by a time-component. For non-homogeneous Itô processes we showed the existence of the time-dependent symbol using the space-time process. These consideration enabled us to derive maximal inequalities for such processes. Similar to the homogeneous case, these allow for the analysis of different properties of the underlying process. We extended only a selected amount of properties from the homogeneous to the non-homogeneous case as outlined below Theorem 3.2.5. It is worth mentioning that one could utilize the theory provided in this chapter to explore the additional properties.

The fifth chapter introduces generalized semimartingales, in order to integrate a 'killing point' to the classical theory. Moreover, we pointed out the necessity of a fourth semimartingale characteristic in this new context, and generalized several results known for semimartingales. The main objective of the last section of this chapter was to introduce a natural way in which the killing can occur.
It would be a fruitful topic for further research to investigate different ways in which the killing of a semimartingale can occur. One possibility would be to analyze the adjoint of

Lévy-type operators (cf. [63]) on $\mathbb{R}^d$. For example, for stable-like processes in the sense of Bass the generator of the semigroup is a Lévy type operator which formal adjoint possesses a killing term.

Appendix A

A.1. Notations from Analysis and Basic Probability Theory

At this point, we want to state some notations from analysis and basic probability theory which are used throughout this thesis and are mostly standard.
We denote by $\mathbb{N} := \{1, 2, ...\}$ the natural number and by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We write $\mathbb{R}$ for the real numbers, $\mathbb{R}_+ := [0, \infty[$ and $\mathbb{C}$ for the complex numbers. Every vector $x = (x^{(1)}, ..., x^{(d)})$ in $\mathbb{R}^d := \mathbb{R} \times ... \times \mathbb{R}$ is a column vector, and we denote by x' the transposed vector. Hence $x'y$ is the scalar product on $\mathbb{R}^d$. On $\mathbb{R}^d$ we write $\|\cdot\|_2$ for the Euclidean norm, while $\|\cdot\|_\infty$ denotes the maximum norm, and the sup-norm on an arbitrary function space. For a matrix $Q \in \mathbb{R}^{d\times d}$ the expression $\mathrm{tr}(Q)$ is defined as the trace of the matrix. On $\mathbb{R}$ the expression $|\cdot|$ denotes the absolute value. As commonly done, we use $a\wedge b := \min\{a, b\}$ and $a\vee b := \max\{a, b\}$ for the minimum resp. maximum of two real numbers a, b. The indicator function of a set A is defined by

$$1_A(y) := \begin{cases} 1, & \text{if } y \in A \\ 0, & \text{if } y \notin A \end{cases}.$$

For functions, 'increasing' and 'non-decreasing' are used equivalently. The j-th partial derivative of a differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is denoted by ∂_j, and the second derivative by $\partial_j^2 := \partial_j\partial_j$ and $\partial_{ij} := \partial_i\partial_j$ for $i, j = 1, ..., d$. By ∇ we denote the Nabla-operator and by ∇^2 the Hessian matrix, i.e. for a two-times differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ we have $\nabla u(x) = (\partial_1 f(x), ..., \partial_d f(x))$ and

$$\nabla^2 f(x) = \begin{pmatrix} \partial_{11} f(x) & ... & \partial_{1d} f(x) \\ \vdots & & \vdots \\ \partial_{d1} f(x) & ... & \partial_{dd} f(x) \end{pmatrix}.$$

From measure theory, we want to state the following notations: The σ-algebra $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra on $\mathbb{R}^d$. The expressions $\mathbb{P}$, $\mathbb{P}^x$ and $\mathbb{P}^{\tau,x}$ are mostly used to denote probability measures on an arbitrary measurable space, and $\mathbb{E}$, $\mathbb{E}^x$ and $\mathbb{E}^{\tau,x}$ are the corresponding expected values respectively. For a random variable X on a measurable space $(\Omega, \mathcal{F})$, the conditional expected value with respect to $\mathbb{P}$ is denoted by $\mathbb{E}[X \mid \mathcal{F}]$, and we define $\mathbb{P}[A \mid \mathcal{F}] := \mathbb{E}[1_A \mid \mathcal{F}]$ for a set $A \in \mathcal{F}$. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we say an event happens almost surely or a.s. if there exists a Null-set N such that the event holds for all $\omega \in N^c$. For the Dirac measure in x we write δ_x.

A.2. Advanced Analysis on the Real Line

For the readers convenience we recall some concepts and results on advanced calculus on the real line (cf. [38] and [53]).

Definition A.2.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called singular if it is non-constant and its derivative exists and is zero almost everywhere.

Almost everywhere is meant with respect to the Lebesgue sets, that is, we are working on the completion of the Borel sets with respect to the Lebesgue measure.

Definition A.2.2. Let $f : [a, b] \to \mathbb{R}$ be a function, and let $x_0 \in [a, b]$ for $a, b \in \mathbb{R}$ and $a < b$. Then

(a.) the *upper resp. lower right Dini derivative* of f at x_0 is defined by

$$D^+ f(t) := \limsup_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ resp. } D_+ f(t) := \liminf_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

(b.) the *upper resp. lower left Dini derivative* of f at x_0 is defined by

$$D^- f(t) := \limsup_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{resp. } D_- f(t) := \liminf_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The concept of singular functions is fundamental for the reasoning in the previous chapters. More specifically, in order to prove the main theorem of Chapter 3 we use the fact that a singular function possesses at least one point where a Dini derivative is infinite. Although, this seems to be clear since the most prominent examples like the Cantor function or Minkowski's question mark function on $[0, 1]$ possess infinite derivatives in *all* points where the derivative is not zero, it was shown in [75] that there exists a singular function with a derivative that takes non-zero finite values on an uncountable dense set whose intersection with any interval (a, b) possesses Hausdorff dimension one.

Let us first recall the following definition:

Definition A.2.3. The number c (finite or infinite) is called a *derived number* of the function f at the point x_0 if there exists a sequence $h_n \to 0$ with

$$\lim_{n \to \infty} \frac{f(x_0 + h_n) - f(x_0)}{h_n} = c.$$

The proof of Proposition A.2.6 uses two lemmas which we recall subsequently. We write λ for the Lebesgue measure defined on the Lebesgue sets, that is, we are working on the completion of the Borel sets w.r.t. the Lebesgue measure. λ^* denotes the corresponding outer measure. In fact, all sets on which we will use the following results will be measurable. The first result is [38] Lemma 1.2.3 while the second one is a combination of Lemma 1.2.3 and Lemma 1.2.5 of [38].

Lemma A.2.4. *Let f be a strictly increasing function on $[a,b]$ and let $p \geq 0$. If at every point x of a set $E \subseteq [a,b]$ there exists at least one derived number $Df(x)$ such that $Df(x) \leq p$, then*

$$\lambda^*(f(E)) \leq p \cdot \lambda^*(E).$$

Lemma A.2.5. *Let f be a strictly increasing function. If at every point x of the set $E \subseteq [a,b]$ there exists $f'(x) = p$, then*

$$\lambda^*(f(E)) = p \cdot \lambda^*(E).$$

The proof of the following proposition is due to A. Schnurr who communicated it to the author of this thesis.

Proposition A.2.6. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be an increasing, singular function which is not constantly zero. Then there exist at least one point where g possesses an infinite upper right (resp. upper left, lower right, lower left) Dini derivative.*

Proof. We prove the result only for the upper right Dini derivative D^+ since all other cases, i.e. , D_+, D^-, D_-, work analogously.
Let $t \in \mathbb{R}_+$ with $g(t) \neq 0$. We consider g on the interval $I := [t-0.5, t+0.5]$. Let us assume that g possesses a finite upper right Dini derivative in every point of I. For $x \in I$, we consider $f(x) := g(x) + x$ which is a strictly increasing, continuous function with $f'(x) = 1$ almost everywhere, i.e., in a set of measure 1. We define the disjoint sets

$$\begin{aligned} B &:= \{x \in I : f'(x) = 1\}, \\ E_j &:= \{x \in I : j \leq D^+ f(x) < j+1\} \text{ for } j \in \mathbb{N}, \end{aligned}$$

and obtain that $I = B \cup \left(\bigcup_{j=1}^{\infty} E_j\right)$. Moreover, the sets B and $(E_j)_{j\in\mathbb{N}}$ are measurable by Theorem 3.6.5 of [38], and since f is continuous and strictly increasing the sets $f(B), f(E_1), ...$ form a disjoint decomposition of $f(I)$ into measurable sets. The sets $(E_j)_{j\in\mathbb{N}}$ do all have Lebesgue measure zero. We obtain

$$\begin{aligned} \lambda(f(I)) &= \lambda(f(B \cup E_1 \cup ...)) \\ &= \lambda(f(B) \cup f(E_1) \cup ...) \\ &= \lambda(f(B)) \\ &= \lambda(B) \\ &= 1 \end{aligned}$$

where we used Lemma A.2.4 in the third and Lemma A.2.5 in the fourth equation. Thus, since f maps intervals to intervals, we obtain that $f(t-0.5) = c$ and $f(t+0.5) = c+1$ for a $c \in \mathbb{R}$, and conclude that $g \equiv 0$ on I by monotonicity. This is a contradiction. □

In order to prove this proposition one can also combine two results which can be found in [60], namely Theorem (4.6) in Chapter IX and Theorem (6.7) in Chapter VII.

Acknowledgements

At first, I am very thankful to my supervisor, Alexander Schnurr, who not only introduced me to the present research topic but whose expertise, support, and mentorship, along with many discussions, have been indispensable for this work. Thanks for being my mathematical Mr. Miyagi.

Anita Behme agreed to act as an appraiser for this thesis, which I highly appreciate.

Further thanks go to my colleagues at the University of Siegen. They have, over the last three years, provided a pleasant environment in which one liked to work but also offered distraction when needed. Special thanks go to Riko Kelter, Dustin Kremer, Robin Mey, Ines Nüßgen, and Angelika Silbernagel, who helped to make this work more accurate. I am profoundly thankful to Franziska Wollny, who accompanied me during most of my academic journey. Franziska was always a source of approval, good advice, and a healthy dose of rivalry.

Last but not least, I would like to thank my family and my girlfriend, Sarah, for their unconditional support and the willingness of the latter to listen to and being interested in (or at least looking so) my clearly too long talks about my mathematical research.

Bibliography

[1] Aït-Sahalia, Y. and Jacod, J. Testing for jumps in a discretely observed process. *Ann. Statist.*, **37**:184–222, 2009.

[2] Aït-Sahalia, Y. and Jacod, J. Identifying the successive Blumenthal-Getoor Indices of a discretely observed process. *Ann. Statist.*, **40**:1430–1464, 2012.

[3] Aït-Sahalia, Y. and Jacod, J. *High Frequency Financial Econometrics*. Princeton University Press, Princton, 2014.

[4] Bauer, H. *Probability Theory*. De Gruyter, Berlin, 1996.

[5] Bertoin, J. *Lévy Processes*. Cambridge University Press, Camebridge, 1998.

[6] Blumenthal, R. M. and Getoor, R. K. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, **95**:263–273, 1960.

[7] Blumenthal, R. M. and Getoor, R. K. Sample functions of stochastic processes with stationary independent increments. *J. Math Mech.*, **10**:493–516, 1961.

[8] Blumenthal, R. M. and Getoor, R. K. *Markov Processes and Potential Theory*. Academic Press, New York, 1968.

[9] Bochner, S. Stochastic processes. *Ann. Math.*, **48**:1014–1061, 1947.

[10] Böttcher, B. Feller Evolution Systems: Generators and Approximation. *Stochastics and Dynamics*, **4**:239–241, 2018.

[11] Böttcher, B., Schilling, R., and Wang, J. *Lévy Matters III. Lévy-Type Processes: Construction, Approximation and Sample Path Properties*, volume **2099** of *Lect. Notes in Math.* Springer, Berlin, 2013.

[12] Çinlar, E., Jacod, J., Protter, P., and Sharpe, M. J. Semimartingales and Markov Processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **54**:161–219, 1980.

[13] Cheridito, P., Filipović, D., and Yor, M. Equivalent and absolutely continuous measure changes for jump-diffusion processes. *The Annals of Applied Probability*, **15**:1713–1732, 2005.

[14] Cohen, S. and Elliot, R. *Stochastic Calculus and Applications*. Birkhäuser, New York, second edition, 2015.

[15] Courrège, P. Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. *Séminaire Théorie du potentiel*, 1965/66. Exposé 2, 38pp.

[16] Davies, E. B. *One-Parameter Semigroups*. Academic Press, London, 1980.

[17] Dellacherie, C. Un Survol de la théorie de l'intégrale stochastique. *Stochastic Processes and their Applications*, **10**:115–144, 1980.

[18] Dellacherie, Claude and Meyer, Paul-André. *Probabilities and Potential*. North-Holland Mathematics Studies. North-Holland, 1978.

[19] Doléans-Dadé, C. and Meyer, P. A. Intégrale stochastique par rapport aux martingales locales. *Séminaire de Probabilités IV*, **124**:77–107, 1970.

[20] Doob, J. L. *Stochastic Processes*. Wiley, New York, 1953.

[21] Dynkin, E. B. *Stochastic Processes. Volume I*, volume **121** of *Die Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin, 1965.

[22] Ethier, S. N. and Kurtz, T. G. *Markov Processes - Characterization and Convergence*. Academic Press, New York, 1986.

[23] Fisk, D. Quasi-Martingales. *Transactions of the American Mathematical Society*, **120**:369–389, 1965.

[24] Grigelionis, B. On the absolute continuity of measures corresponding to stochastic processes. *Lith. Math. J.*, **11**:783–794, 1971.

[25] Grigelionis, B. On nonlinear filtering theory and absolute continuity of measures, corresponding to stochastic processes. *Proceedings of the Second Japan-USSR Symposium on Probability Theory*, Lecture Notes in Mathematics 330:80–95, 1972.

[26] Gulisashvili, A. and van Casteren, J. A. *Non-Autonomous Kato Classes And Feynman-Kay Propagators*. World Scientific, Singapore, 2006.

[27] Hörmander, L. Pseudo-differential Operators and Non-elliptic Boundary Problems. *Annals of Mathematics*, **83**:129–209, 1966.

[28] Ikeda, N. and Watanabe, S. Stochastic Differential Equations and Diffusion Processes. *North-Holland Publishing Company*, 1981.

[29] Ito, K. and Watanabe, S. Transformation of Markov processes by multiplicative functionals. *Annales de l'Institut Fourier*, **15**(1):13–30, 1965.

[30] Itô, K. On a formula concerning stochastic differentials. *Nagoya Math. Journ.*, **3**:55–75, 1951.

[31] Jacob, N. Dirichlet forms and pseudo differential operators. *Expo. Math*, **6**:313–351, 1988.

[32] Jacob, N. Feller semigroups, Dirichlet forms, and pseudo differential operators. *Forum Math*, **4**:433–446, 1992.

[33] Jacob, N. Characteristic Functions and Symbols in the Theory of Feller Processes. *Potential Analysis*, **8**:61–68, 1998.

[34] Jacob, N. *Pseudo-Differential Operators and Markov Processes I. Fourier Analysis and Semigroups*. Imperial College Press/World Scientific, London, 2001.

[35] Jacod, J. and Mémin, J. Caractéristiques locales et conditions de continuité absolue pour les semi-martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **35**:1–37, 1976.

[36] Jacod, J. and Shiryaev, A. *Limit Theorems for Stochastic Processes*. Springer, Berlin, 1987.

[37] Jarrow, Robert and Protter, Philip. A short history of stochastic integration and mathematical finance: The early years, 1880-1970. *Lecture Notes-Monograph Series*, **45**:75–91, 2004.

[38] Kannan, R. and Krueger, C. K. *Advanced analysis on the real line*. Springer, New York, 1996.

[39] Khintchine, A. A new derivation of one formula by Levy P. *Bull. Moscow State Univ.*, **I**:1–5, 1937.

[40] Klenke, A. *Probability Theory - A Comprehensive Course*. Springer, Berlin, second edition, 2014.

[41] Kolmogorov, A. N. Sulla forma generale di un processo stocastico omogeneo (un probleme di B. de Finetti). *Atti Acad. Naz. Lincei Rend.*, **15**:805–808, 1932.

[42] Koshnevisan, D. and Schilling, R. *From Lévy-Type Processes to Parabolic SPDEs*. Birkhäuser, Berlin, 2016.

[43] Kunita, H. and Watanabe, S. On Square Integrable Martingales. *Nagoya Math. J.*, **30**:209–245, 1967.

[44] Kyprianou, A. E. Introductory Lectures on Fluctuations of Lévy Processes with Applications. *Springer*, 2006.

[45] Kühn, F. and Schilling, R. Maximal inequalities and applications. *Probability Surveys*, **20**:382–485, 2023.

[46] Lévy, P. Sur les intégrales dont les éléments sont des variables aléatoires indepen-dantes. *Ann. Scoula Norm. Pisa*, **3,4**:337–366 and 217–218, 1932.

[47] Ma, Z. and Röckner, M. Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. *Springer*, 1991.

[48] Manstavičius, M. p-variation of strong Markov processes. *Ann. Probab.*, **32**:2053–2066, 2004.

[49] Manstavičius, M. and Schnurr, A. Criteria for the Finiteness of the Strong p-Variation for Lévy-type Processes. *Stochastic Analysis and Applications*, **35**(5):873–899, 2017.

[50] McKean, H. P. Sample functions of stable processes. *Ann. Math.*, **61**:564–579, 1955.

[51] Meyer, P. A. Intégrales Stochastiques I. *Séminaire de Probabilités I*, Lecture Notes in Mathematics Vol. 39:72–94, 1967.

[52] Meyer, P. A. Intégrales Stochastiques II. *Séminaire de Probabilités I*, Lecture Notes in Mathematics Vol. 39:95–117, 1967.

[53] Natanson, I. P. *Theory of Functions of a Real Variable*. Dover Books on Mathematics, Mineola, 2016.

[54] Protter, P. *Stochastic Integration and Differential Equations*. Springer, Berlin, version 2.1 edition, 2005.

[55] Pruitt, W. E. The Hausdorff dimension of the range of a process with stationary independent increments. *Indiana J. Math.*, **19**:371–378, 1969.

[56] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*. Springer, Berlin, 1999.

[57] Rickelhoff, S. and Schnurr, A. The Time-Dependent Symbol of a Non-Homogeneous Itô Process and corresponding Maximal Inequalities. *J. Theor. Probab*, 2023. `https://doi.org/10.1007/s10959-023-01308-y`.

[58] Rickelhoff, S. and Schnurr, A. On the rôle of singular functions in extending the probabilistic symbol to its most general class. *Trans. Amer. Math. Soc.*, **377**(2):905–919, 2024.

[59] Rüschendorf, L., Schnurr, A., and Wolf, V. Comparison of time-inhomogeneous Markov processes. *Advances in Applied Probability*, **48**:1015–1044, 2016.

[60] Saks, S., Young, L.C., and Banach, S. *Theory of the Integral*. Monografie matematyczne. Hafner Publishing Company, 1937.

[61] Sato, K. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in advanced mathematics, Cambridge, 1999.

[62] Schilling, R. and Schnurr, A. The Symbol Associated with the Solution of a Stochastic Differential Equation. *Electronic Journal of Probability*, **15**:1369–1393, 2010.

[63] Schilling, R. and Wang, J. Lower bounded semi-dirichlet forms associated with lévy type operators. *Festschrift Masatoshi Fukushima, Interdiscip. Math. Sci.*, 17, 2011.

[64] Schilling, R. L. Conservativeness and Extensions of Feller Semigroups. *Positivity*, **2**:239–256, 1998.

[65] Schilling, R. L. Feller Processes Generated by Pseudo-Differential Operators: On the Hausdorff Dimension of Their Sample Paths. *J. Theor. Probab.*, **11**:303–330, 1998.

[66] Schilling, R. L. Growth and Hölder conditions for the sample paths of Feller processes. *Probab. Theory Rel. Fields*, **112**:565–611, 1998.

[67] Schnurr, A. *The Symbol of a Markov Semimartingale*. PhD thesis, TU Dresden, 2009. `https://nbn-resolving.org/urn:nbn:de:bsz:14-ds-1244626491491-70401`.

[68] Schnurr, A. A Classification of Deterministic Hunt Processes with Some Applications. *Markov Processes and Related Fields*, **17**(2):259–276, 2011.

[69] Schnurr, A. Generalization of the Blumenthal-Getoor Index to the Class of Homogeneous Diffusions with Jumps and some Applications. *Bernoulli*, **19**(5A):2010–2032, 2013.

[70] Schnurr, A. On Deterministic Markov Processes: Expandability and Related Topics. *Markov Processes and Related Fields*, **19**:693–720, 2013.

[71] Schnurr, A. The Fourth Characteristic of a Semimartingale. *Bernoulli*, **26**(1):642–663, 2020.

[72] Schnurr, A. and Rickelhoff, S. From Markov processes to semimartingales. *Probability Surveys*, **20**:568–607, 2023.

[73] Sharpe, M. *General Theory of Markov Processes*. Academic Press, 1988.

[74] Shiryayev, A. N. *On Analytical Methods In Probability Theory*, pages 62–108. Springer Netherlands, Dordrecht, 1992.

[75] Sánchez, J., Viader, P., Paradís, J., and Carrillo, M. A singular function with a non-zero finite derivative on a dense set with Hausdorff dimension one. *Journal of Mathematical Analysis and Applications*, **434**:713–728, 2016.

[76] van Casteren, J. A. *Markov Processes, Feller Semigroups and Evolution Equations*, volume 12 of *Series on Concrete and Applicable Mathematics*. World Scientific, Singapore, 2010.

[77] von Waldenfels, W. Fast positive Operatoren. *Berichte der Kernforschungsanlage Jülich*, 1964.

[78] von Waldenfels, W. Positive Halbgruppen auf einem n-dimensionalen Torus. *Archiv der Mathematik*, **15**:191–203, 1964.

List of Notations

General Notation

$\subsetneq$	Strict subset	20
Θ_s	The 'Big Shifts'	32
$\mathrm{tr}(Q)$	Trace of the matrix $Q \in \mathbb{R}^{d \times d}$	101
$\mathrm{Var}(A)$	Variation process of the process $A \in \mathcal{V}$	20

Classes of Functions

$B_b(E)$	Banach space of bounded, measurable functions on $(E, \mathcal{E})$	1
$C_c^\infty(\mathbb{R}^d)$	Infinitly often differentiable functions with compact support on $\mathbb{R}^d$	16
$C_\infty(E)$	Continuous, bounded functions which vanish at infinity on E	14
$C_\infty^k(\mathbb{R}^d)$	k-times continuously diff. functions which vanish in infinity on $\mathbb{R}^d$	15

Classes of Processes

$\mathcal{A}$	Class of processes in $\mathcal{V}$ with integrable variation process	21
$\mathcal{A}^+$	Class of integrable processes in $\mathcal{V}^+$	21
$\mathcal{C}_{ad}$	Processes belonging to a class of processes $\mathcal{C}$ and are additive	32
$\mathcal{V}^d$	$:= \mathcal{V} \times ... \times \mathcal{V}$	29
$\mathcal{L}^d$	$:= \mathcal{L} \times ... \times \mathcal{L}$	29
$\mathcal{L}$	Class of local martingales starting in zero	19
$\mathcal{M}_{loc}$	Class of local martingales	19
$\mathcal{M}$	Class of uniformly integrable martingales	7
$\mathcal{S}^d$	Class of all d-dimensional semimartingales	25
$\mathcal{S}^\dagger$	Class of generalized semimartingales	76
$\mathcal{S}$	Class of semimartingales	25
$\mathfrak{S}$	Class of simple predictable processes	26
$\mathcal{S}_p$	Class of special semimartingales	25
$\mathcal{H}_{loc}^2$	Localization of the class of square-integrable martingales	19
$\mathcal{V}$	Class of real, adapted, càdlàg processes with finite variation on intervals	20
$\mathcal{V}^+$	Class of real, adapted, càdlàg and increasing processes	20

List of Figures