Economic Theory of Culture

A Dynamic Approach

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1 Introduction

1.1 The role of culture in economic perspective

Consumption of culture has a major social component. We share the widespread presumption or conviction that the consumption of culture yields positive effects for society: creativity will be further developed, tolerance for others (race and gender) will be enhanced, crime will be reduced, and people's sense of identity will be strengthened.[1] Though economists have extensively discussed the concept of positive externality, they either claim an undersupply of culture without further specifying what the concrete link between culture and externality is like, or they enumerate various externalities and invoke them to justify public support for culture. Those arguments are not stringent, and none is demonstrated analytically, to our knowledge.

For example, Robbins (1963, p. 58) argues that "…the positive effects of the fostering of art and learning and the preservation of culture are not restricted to those immediately prepared to pay cash but diffuse themselves to the benefit of much wider sections of the community in much the same way as the benefits of the apparatus of public hygiene or of a well-planned urban landscape". In their pioneering contribution to cultural economics Baumol and Bowen (1966, p. 382n.) pay much attention to four types of general benefit which flow from the arts: national prestige, advantages cultural activity confers on business in its vicinity, benefit for future generations, contribution to education. They also point out that "…performing arts confer direct benefits on those who attend a performance but which also offer benefits to the community as a whole…". Peacock (1969, p. 328n.) also invokes intertemporal spillovers of culture and argues that even "… those who do not understand and appreciate music and drama may be glad to contribute towards making available their fruits to those who do, and to those whose tastes are not yet formed. Present generations may derive positive satisfaction from preserving live performance safe in the knowledge that they do not risk being accused of narrowing the range of choice of cultural activities for future generations through allowing arts to die". Netzer (1978, p. 22n) extends the list of external benefits by adding that the interdependence of art forms tends to support one another, and that innovation is fostered through artistic undertakings and business advantage through culture and arts. Fullerton (1992, p. 80) offers a new twist on the externality argument and justifies public support of the arts as fol-

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1 See e.g. "The First World Culture Report (1998) of UNESCO, Part one".

lows: If there are enough persons (external beneficiaries) who want others to attend art events and hence "…are willing to pay to see subsidized arts, and if similar individuals can "free ride" by enjoying others' donation, then none of them has sufficient incentive to give to the arts. It is possible for government, potentially, to make everybody better off by taxing those individuals who do value others' attendance, and then using the funds to subsidize the arts."

While all these positive externality arguments have some appeal in the cultural context, they are not made precise in formal intertemporal analysis and they do not explicitly account for the distinctive characteristics of culture. Though Ulibarri (2000) provides a dynamic framework to develop a theory of rational philanthropy in forming "cultural capital", he rather focuses on the interdependence between capital market opportunities and public funding for culture. The present analysis aims at capturing the specificities of culture in a dynamic stock-flow model by distinguishing - and focusing on the relations - between different aspects of culture which will be specified below as cultural goods, cultural services and cultural capital. It then models the social component of the consumption of culture via a process of accumulation (and depreciation) of cultural capital, the creation of new cultural goods to build up the stock of cultural goods, which in turn affects the individuals' well-being.

The notions of culture and related terms are applied in the literature in various often incompatible ways. We will refrain from surveying and comparing the major concepts comprehensively. Instead, we will define the terms culture, cultural goods, services and capital specifically for the purpose of the present investigation and will clarify these concepts by relating them briefly to other connotations and terms suggested in the literature.

In our analysis, it suffices to think of *culture* "…as being represented by the "cultural sector" of the economy" (Throsby 1995, p. 202). *Cultural goods*[2] are considered to be all items of cultural significance like heritage buildings, sites, locations, works of arts (e.g. paintings, sculptures), literature and music.[3] There is a stock of cultural goods inherited from the past, and there is an ongoing process of creating new cultural goods which are then added to the stock.

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2 The notion of cultural goods as introduced here is closely related to what is termed "cultural capital" by Throsby (1999), except that we do not link cultural goods with Thorsby's "cultural value". The latter is considered by Throsby (1999, p.6) as "…different from, though not unrelated to economic value", but Throsby does not specify how this value emerges.

3 Cultural goods are durable. They may be further differentiated according to whether they are tangible or intangible or whether they are private or public goods.

Following Throsby (1999, p. 7) we assume the cultural heritage to "…give rise to a flow of services that may be consumed as private and/or public goods entering final consumption immediately, and/or they may contribute to the production of future goods and services, including new cultural goods". Suppressing the role of these services as productive factors we focus on consumptive cultural services, called *cultural services*.

Cultural services are considered to be all cultural performances provided by cultural institutions. These cultural services may take many widely differing forms. Examples of (first-order) cultural services are (guided) tours to cultural sites, visits to museums, attendances of concerts or of opera and drama performances, reading books. Other cultural services (of second or higher order) are e.g. reading books about medieval paintings, about the Chinese terracotta army, or about the cultural treasures of Paris; watching broadcasted reports about these cultural goods, watching broadcasted opera performances or concerts; listening to broadcasted or recorded music; enjoying replica sculptures, paintings, or replica heritage buildings; watching a photo of the London parliament building in the show window of a travel agency, enjoying (own) photos or videos taken during tours to cultural sites.

This rather selective list of cultural services strongly suggests that the meaning and importance of cultural heritage for society is closely linked to the number and kinds of cultural services flowing from the stock of cultural goods.[4] The magnitude and the structure of those flows depend, in turn, to a large extent, on costs to provide them and on income and relative prices to consume them. Public cultural policies, intervention and regulation may have a great impact on these economic determinants. Consequently, the stock of cultural goods *can* facilitate the provision of cultural services, but there is no automatism in the cultural heritage "giving rise to a flow of cultural services".

Leaving the supply, demand and the pricing of cultural services and cultural goods to markets (*laissez-faire*) would be highly recommendable, if the consumption of cultural services and the creation of cultural goods were comparable to services like e.g. cutting hair or production like e.g. brewing beer which do not appear to have a major social component. Our principal hypothesis is that the continuous consumption of cultural services over time leads to an accumulation of *cultural capital* which, in turn, is positively valued by all members of society.

4 This is most clearly seen by considering the fictitious polar case in which not any cultural service flows from the cultural heritage. This would be a situation as if there exists no cultural heritage at all.

Following Becker (1998, p. 12n.) we conceive of cultural capital as an intangible and depreciable asset that is a form of social capital in the sense of Coleman (1990) who argues (ibidem, p. 317) that "...social capital [and hence cultural capital, as presently defined; the author] is an important resource for individuals and can greatly affect their ability to act and their perceived quality of life." He also maintains that "although it is a resource that has value in use, it cannot be easily exchanged. As an attribute of the social structure in which a person is embedded, social capital is not the private property of any of the persons who benefit from it" (ibidem, p. 315). According to Becker (1998) cultural changes over time may be slower than changes of other kinds of social capital but he rightly rejects the view "...that culture so dominates behavior that little room is left for choice" (ibidem, p. 16n.).

Introducing cultural goods and cultural capital as outlined above in dynamic setups implies that the greater is the stock of cultural goods, the greater is the probability that the flow of cultural services is broad, even though the link between both is not rigid; the more cultural services are consumed the more cultural capital is likely to be generated, after depreciation is accounted for, and the greater will be the external benefits provided for society. Though consumers may account for their own benefit from the increases in the stock of cultural goods/cultural capital brought about by their own creation of cultural goods and consumption of cultural services, they tend to ignore the beneficial impact which their own contribution to the generation of cultural goods/cultural capital has on their fellow citizens. When the number of consumers is very large, they may even neglect the enhancement of their own utility through the increases in cultural goods/cultural capital induced by their own (negligibly small) creation of cultural goods and consumption of cultural services. This myopic individual behavior gives rise to external cultural benefits.

1.2 Human capital vs. cultural capital

Related to the concept of cultural capital as perceived in the present analysis is the notion of human capital inspired by Becker (1964) and elaborated over the last decades in the context of endogenous growth theory surveyed, e.g., by Aghion and Howitt (1998, chapter 10). Human capital is accumulated by and "within" the individual consumer/worker either through education or through learning by doing. On an aggregate level, the stock of human capital enhances productivity and is therefore considered an important driving force for economic growth. In the model to be analyzed here, no individual consumer is supposed to build up her own "stock of cultural capital" in analogy to human capital. Consumers rather contribute to the accumulation of cultural capital (which is not the private property of any of the persons who benefit from it, as observed above) and thus enjoy not only a secondary benefit from their own investment in cultural capital but also a purely external benefit from the other consumers' investment in cultural capital. The individual's investment in cultural capital is not at all or not primarily her intention when consuming cultural services. In contrast, when consuming education the individual's main objective is the accumulation of human capital to improve her own market value. Human capital formation through learning-by-doing is closer, in spirit, to the process of cultural capital accumulation, but both are definitively distinct concepts. The positive externalities induced by cultural capital have been detailed above. The external benefits of human capital consist in the enhancement of productivity induced by the stocks of human capital built up by all individuals. On the other hand, the presence of external benefits is common to both approaches.

1.3 Characteristics of cultural goods, services and capital

Following the theoretical literature (Blümel et al. 1986), a good is denoted public if it is jointly consumable. Public goods may be non-exclusive and congestible or not. A good is denoted private if it is not jointly consumable. Cultural goods are public goods, if the producers of cultural services can jointly use those goods to produce cultural services. Examples are the Egyptian pyramids, the Chinese Wall, the Red Square in Moscow, the poems by Goethe and the lullaby by Brahms. Those historical heritages and artistic materials can be used jointly by producers of cultural services. Cultural goods are private goods, if they are not jointly consumable. An example is a Stradivarius violin that is played by a violinist acting as a cultural-services producer.[5] Cultural services are public goods, if they are jointly consumable. Examples are the TV broadcasting of a violinist's live concert, the display of all artifacts in the British museum. Cultural services are private goods, if they are not jointly consumable. An example is the violinist's private performance for a certain single person. Cultural capital is clearly and unambiguously a public good, since it is always jointly consumable.

5 Note, however, that if an individual original Stradivarius violin should be on display in a museum, it can be enjoyed by all visitors and therefore is a public good in that context.

Obviously, congestion is a realistic feature of many cultural services offered to an audience in rooms or halls such as opera houses, concert halls, museums etc. To keep the model tractable we assume that the cultural services are non-congestible throughout the following analysis. Important examples of non-congestible cultural services (so-called pure public goods) are broadcasted cultural programs. Another relevant distinction is whether public goods are excludable or not. As is well-known, this attribute is irrelevant when allocative efficiency is at issue in an institution-free world. However, it will play an essential role when alternative institutional arrangements and markets are investigated.

1.4 Outline of the analysis

Having discussed the alternatively possible properties of cultural goods and cultural services, we will treat them alternatively as public or private goods throughout the following study. In section 2 we build the general theoretical basis for all subsequent models and characterize as a benchmark, an efficient intertemporal allocation that is e.g. implemented by an omniscient benevolent social planner. The efficiency rules are shown to differ according to whether cultural goods and cultural services are public (model GM1) or private (model GM2). The general models are very useful in providing qualitative information about the socially optimal intertemporal allocation, but they do not answer the questions as to how the steady state of the economy is attained and what the determinants of the steady state allocation are.

To attain further insights into the intertemporal cultural process, some more restrictive assumptions are imposed on the general models. In section 3 we first model an economy whose stock of cultural goods is assumed to be constant so that cultural capital is left as the only state variable (model $S\bar{G}$). Two versions of this model are explored distinguished according to whether cultural goods and cultural services are public (model $S\bar{G}1$) or private (model $S\bar{G}2$). Next, the stock of cultural capital is assumed to be constant or, more precisely, irrelevant (model $S\bar{K}$) leaving the stock of cultural capital as the only state variable. Like the model $S\bar{G}$, the model $S\bar{K}$ comes in two versions: either cultural goods and cultural services are public (model $S\bar{K}1$) or private (model $S\bar{K}2$). For all these submodels the efficient intertemporal allocation is characterized.

Section 4 proceeds to answer the central question whether - or under which conditions - the optimal intertemporal allocation can be attained by the (competitive) market mechanism. First we invoke Lindahl's thought experiments and set up the fictitious market concept for public goods that is dual to the concept of perfectly competitive markets for private goods in the sense that the role of prices and quantities is interchanged. This set-up turns out to imply that under the condition that all agents reveal their willingness-to-pay for public goods truthfully, the market mechanism can indeed implement the intertemporal optimal allocation. The Lindahl markets hence serve as the benchmark for later reference. As before we discuss in section 4 two versions of the Lindahl-markets model (model BM) where cultural goods and cultural services are either public (model BM1) or private (model BM2).

Lindahl markets are highly artificial since they are based on the problematic assumption that the agents truthfully reveal their willingness-to-pay for public goods. Yet agents have an incentive to underreport their willingness-to-pay (free riding) and this is why Lindahl markets don't emerge in real market economies. For that reason we proceed in section 5 on the assumption that Lindahl markets do not exist. The corresponding markets economies are called *laissez-faire* economies in the absence of any cultural policy. As expected the market allocation in such economies is shown to be inefficient, and this market failure gives rise to an investigation of corrective cultural policies in form of appropriate Pigouvian tax-subsidy schemes. Of course, such policies have to account for whether cultural goods and cultural services are public (model BL1) or private (model BL2) and hence these two versions of the market economy without Lindahl markets need to be explored one at a time. Each of the models BL1 and BL2 is further differentiated by distinguishing two different types of the individual's behavior: ignorant behavior or Nash behavior. The associated submodels are denoted BLI1, BLI2 and BLN1, BLN2. Yet those submodels are still too general and complex to allow for a detailed characterization of their intertemporal allocations by using the phase-diagram technique. Therefore we reuse the procedure employed in section 3, namely to reduce the generality of the models BLI1, BLI2 and BLN1, BLN2 firstly by setting constant the cultural-goods stock, which yields the submodels $\mathrm{BLI\bar{G}1}$, $\mathrm{BLI\bar{G}2}$ and $\mathrm{BLI\bar{K}1}$, $\mathrm{BLI\bar{K}2}$, and secondly by setting constant cultural capital, which provides us with the submodels $\mathrm{BLN\bar{G}1}$, $\mathrm{BLN\bar{G}2}$ and $\mathrm{BLN\bar{K}1}$, $\mathrm{BLN\bar{K}2}$.

Our study will be concluded in section 6, in which we will summarize the principle findings and discuss some possible extensions of the present models. Table 1.1 shows the outline of our analysis.

Table 1.1 Outline of the alternative approaches

		Public goods			Private goods		
Allocative efficiency		**GM1**	*g*: constant *k*: free	$S\overline{G}1$	**GM2**	*g*: constant *k*: free	$S\overline{G}2$
			g: free *k*: no impact	$S\overline{K}1$		*g*: free *k*: no impact	$S\overline{K}2$
Complete set of markets including Lindahl markets for all public goods		**BM1**			**BM2**		
As above, but all Lindahl markets for consumers are absent	Type of model	**BL1**			**BL2**		
	Model with Nash behavior	**BLN1**	*g*: constant *k*: free	$BLN\overline{G}1$	**BLN2**	*g*: constant *k*: free	$BLN\overline{G}2$
			g: free *k*: no impact	$BLN\overline{K}1$		*g*: free *k*: no impact	$BLN\overline{K}2$
	Model with ignorant behavior	**BLI1**	*g*: constant *k*: free	$BLI\overline{G}1$	**BLI2**	*g*: constant *k*: free	$BLI\overline{G}2$
			g: free *k*: no impact	$BLI\overline{K}1$		*g*: free *k*: no impact	$BLI\overline{K}2$

2 The general model

Consider an economy in which a single given composite resource serves as an input to produce three goods: a private consumer good, new cultural goods and cultural services. Society consists of n_c individuals who are not only consumers but also creators of new cultural goods and hence are producers in that capacity. In traditional economic modeling the individuals' roles as consumers and producers are neatly separated because production is assumed to have an instrumental value only. As a consumer, the individual derives additional utility only from passively receiving "an additional quantity of certain goods" (Pareto, 1971, p. 112). According to Slutsky (1970, p. 28) "the utility of a combination of goods is a quantity, which has the property that its value is the greater the more the given combination is [passively, Sao-Wen Cheng] desired by the individual"[6]. Becker (1998, p. 26) hence calls the consumer-individual a "passive maximizer of utility" in the traditional set-up. However, when cultural goods are produced by consumer-individuals, this separation is inadequate because her creation of cultural goods gives herself satisfaction. Hence she is an "active maximizer of utility" (in Becker's sense), and her pattern of consumption is thus influenced[7] by active maximization utility. Furthermore the "creator" may have certain rights to the commercialization (e. g. copyright) of her own cultural goods. Therefore we will model individuals as *consumer-artists*[8]. As a consumer-artist, individual *i* has the utility function[9]

$$u_i = U^i \underset{+\ \ +\ \ +\ \ +\ \ +}{\left(g_i, k_i, s_i, v_i, y_i \right)} \qquad i = 1, \ldots, n_c \; . \tag{2.1}$$

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6 Similarly, Gossen (1853), Jevons (1871), Edgeworth (1881), Debreu (1959) or Uzawa (1960) also adopted the framework that the household is a "passive maximizer of utility" in the market place (in Becker's sense), to investigate the consumer's behavior in their works.

7 Lancaster (1966, p. 281) termed this influence on the pattern of consumption as "consumption technology".

8 The notion of "artist" is used here in its most general sense as a synonym for "creator of new cultural goods".

9 This notion of active maximizing behavior is similar to Becker's (1965), Michael-Becker's (1973), Lancaster's (1971) or even Stigler-Becker's (1977) approaches. The present analysis follows the route taken by Stigler-Becker (1977). However, those authors innovate upon traditional consumer theory and rather illuminate that the taste is stable (exogenous), the household using "time" as input factor (hence active) for producing their own "appreciation" of certain (addictive) behaviors, thus the increase in factor time, the more productive in consumption. In our approach, consumer-artists are rather more "creative" in producing some (cultural) commodities, which give satisfaction to individuals. We do not consider the argument of stability of taste, and we do not take into account the input factor of time for "active" consumption either.

g_i and k_i represent consumer-artist *i*'s (passive) demand for the stock of cultural goods and cultural capital which will be specified and interpreted further below. Throughout the subsequent analysis we apply the demand-supply scheme to those stocks. s_i is her consumption of cultural services (passive as well), v_i is the amount of new cultural goods created by her (active behavior), and y_i is her consumption of the consumer good (passive).

As an artist, individual *i* possesses cultural and technical skills to produce new cultural goods using the production function

$$v_i = V^i \underset{+\ \ +}{\left(r_{vi}, k_i\right)} \qquad i = 1,\ldots, n_c, \tag{2.2}$$

where r_{vi} is the resource input to produce the amount v_i of new cultural goods, given the input k_i of cultural capital. A painter needs water colors and painting paper as resource input r_{vi} for his aquarelle, a sculptor needs different wooden, metal or stone materials as input for his sculptures, a composer needs music sheets[10]. The stock of cultural capital, k_i, as an argument of the production function (2.2) expresses the hypothesis that a diffusing creation-stimulating atmosphere in the society stimulates the artists to create more artworks[11]. Note that the variable v_i is an argument in the utility function U^i from (2.1), i. e. individual *i* is positively affected by her own creation and thus may, perhaps, have an incentive to create cultural goods irrespective of whether v_i is of interest to any other individual or whether the production activity (2.2) yields a profit to its creator. This hypothesis builds on Friedman and Kuznets (1945), who indicate that there are some groups of professions - academics, re-

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10 As a distinguished economist and hobby-composer Peacock and his co-author Weir (1975) investigate the economic characteristics of musical composition and find particular features of a composition as a product distinguished from other goods. Those features include the employment of musical performance, while the cost of such performance could be almost reduced to zero owing to reproduction or broadcasting. Hence the problem of property rights emerges.

11 According to Wickert (1998, p. 131) Albert Einstein had quite similar ideas: "Zwischen Individuum und Gesellschaft kommt es zu einem Wechselgeschehen. Der schöpferische Einzelne nimmt auf, was er aus Tradition und Gesellschaft empfängt, und gibt dies an den "Nährboden" der Gemeinschaft zurück, bereichert um den eigenen individuellen Beitrag". (An exchange process evolves between the individuals and society. The creative individual absorbs what she receives from tradition and society, and gives it back to the "fertile soil" of the community enriched with her own individual contribution). (Translated by the author.)

searchers, experimental scientists and artists - who derive satisfaction from the process of their work itself and not just from the professional incomes they earn[12].

New cultural goods, v_i, constitute investments in the stock of cultural goods, *g*. This stock changes over time according to the investment function

$$\dot{g} = \sum_{i=1}^{n_c} v_i - \alpha_g g \ . \qquad (2.3)$$

The state variable, *g*, is the stock of cultural goods created by all artists and known to exist at time *t*. We refer to *g* as the stock of cultural goods supplied at time *t*. Cultural goods are here considered to consist of a broad range of tangible or intangible items of cultural significance like heritage buildings and sites, works of art, literature and music. Examples are the Mona Lisa painting by Leonardo da Vinci, the 9th Symphony by Ludwig van Beethoven, and the ancient Greek marble statue of Laocoon, the Colosseum in Rome or the movie film Casablanca etc[13]. The stock of cultural goods is inherited from the past (cultural heritage), and there is an ongoing process of degradation at an exogenous positive rate of depreciation α_g. The depreciation of the stock of cultural goods reflects the observation that some fraction of cultural goods gets lost over time either physically or in the memory of the artists and society at large. Examples are the disappearance of the Maya culture in South America, the destruction of the historical statues of the Bamiyan Buddhas in Afghanistan under the Taliban regime, the loss of the blueprint of the Chinese earthquake-forecasting device invented by Chang-Ih during the Han dynasty. As shown in (2.3), the depreciation of the stock of cultural goods, $\alpha_g g$, is countervailed by "actively" adding all newly created cultural goods to it,

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12 To reinforce our hypothesis see UNESCO (2002): "All human beings have a need and a capacity to create. From weaving to websites, they seek outlets for artistic self-expression and for contributing to the greater community. The encouragement of creativity from an early age is one of the best guarantees of growth in a healthy environment of self-esteem and mutual respect, critical ingredients for building a culture of peace."
Cowen and Tabarrok (2000, p. 232 - 253) provide other informative data for the artist's labor supply and their satisfaction related to pecuniary benefits (usually connected with low culture) and non-pecuniary benefits (usually connected with high culture).

13 UNESCO (2000) describes cultural goods as follows: "Cultural goods…are the result of individual or collective creativity, include printed matter and literature, music, visual arts, cinema and photography, radio and television, games and sporting goods". In spirit this description is quite close to the term we use here.

$\sum_{i=1}^{n_c} v_i$, such that the net increment of the stock of cultural goods may be positive or negative[14]. On the other hand, the stock of cultural goods can be preserved, so that the cultural policy may aim at decelerating the (speed of) depreciation. In the technical term in (2.3) the rate of depreciation α_g may itself be a policy variable. To determine the socially optimal depreciation rate of cultural-goods α_g is highly interesting, since the preservation of that stock may give satisfaction to people. On the other hand, it takes up social resources. Navrud and Ready (2002, p.5) thus point out: "But what is the right amount of cultural heritage goods? We live in a world with limited resources, and must make tradeoffs among competing objectives". Surely, these economic tradeoffs play a very significant role in debates on cultural policy not only at the regional and national, but also at the international level. To simplify our exposition in the subsequent analysis we solely focus on the "active" investment effort in the cultural-goods stock, and we ignore totally the preservation activity (in the sense of determining the socially optimal rate α_g)[15].

As expressed in (2.1), consumers are affected by the stock of cultural goods (or by parts of it) in two different ways[16]:

(i) The mere existence of cultural goods may give satisfaction to people, it may make them happy and/or proud. This existence value or passive-use value of cultural goods is captured in (2.1) through $U^i_{gi} > 0$ [17].

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14 The stock of cultural goods as defined in the present study is denoted "cultural capital" by some other authors, e.g. Bourdieu (1983) and Throsby (1999). As will be clarified below we use the terms "cultural capital" here in an entirely different way.

15 Numerous authors have discussed the determination of the socially optimal depreciation rate, α_g . E.g. Mossetto (1994, p.81-96) uses various concepts of conservations (re-use, restoration, preservation) to study the optimization conditions of heritage preservation; Throsby (2001, p. 75-92) combines the economic and cultural appraisals by invoking the criteria of sustainability in the assessment of heritage decisions; Navrud and Ready (2002) resort to non-market environmental valuation techniques and attempt to answer questions like "Should we raise taxes to increase spending on cultural heritage, or should we divert resources away from some other worthy cause such as education, health care, or aid to the poor?" or "What is the proper level of expenditure on cultural heritage?". Morey and Rossmann (2003, p. 215-229) invoke discrete-choice random-utility models to estimate the willingness-to-pay for preserving Marble Monuments in Washington D.C.

16 Riganti et al. (2002) use the same terminology "cultural goods" as in the present analysis, but they decompose it into "use" and "non-use" value: "People consume cultural goods as visitors to cultural and historic sites (use value), and may be willing to pay along with non-users to ensure their continued existence and availability for future generations (non-use value)."

17 Peacock (1997, p. 195) argues that the heritage is "an intangible service increasing the utility of consumers, in which historic buildings and artifacts are inputs".

(ii) Individuals make use of cultural goods actively through consuming cultural services: The Mona Lisa painting is enjoyed by visiting the Louvre in Paris or by looking at one of its photos, prints or replicas; Beethoven's 9th is consumed by attending a live concert or by listening to a radio broadcast or the CD-player; literature is consumed by reading a book, where the reader supplies a cultural service to herself.[18] The benefit derived from consuming cultural services is captured in (2.1) through $U^i_{si} > 0$.[19]

Clearly, $U^i_{gi} > 0$ reflects the consumer's passive interest in cultural goods, whereas $U^i_{si} > 0$ means that she is an active consumer of culture. $U^i_{gi} = 0$ and $U^i_{si} = 0$ portrays a person completely uninterested in and unaffected by culture. To further illustrate the important distinction between passive use (*g*) and active use (*s*) consider the Chinese terracotta army hidden underground for some 2500 years. Before it was (re)discovered in the 1970s it did not belong to the stock of (known) cultural-goods, *g*. It was then added to that stock through reports in the media. But beyond basic information about its existence people were eager to learn more about it, and this demand was satisfied through restoration and the supply of various cultural services ranging from art books, replicas and access to the site. To see that such services are a source of satisfaction in their own right (in addition to the existence value of the cultural good) consider a wealthy art collector who owns an outstanding piece of art but does not release photos or reproductions and denies all people access to that piece of art. Suppose further, the outside world knows about its existence and appreciates its existence, yet cultural

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18 Benhamou (2003, p. 256) points out that "… the existence of substitutes for (cultural) goods … present the advantage of including services offered through new technologies, provided that the consumer considers a visit through new technologies (CDRom, Internet) a satisfying substitute for a "real" use".

19 Various authors discuss the "consumption technology" (in Lancaster's sense) of cultural services in more detail, e.g. Marshall (1962, p.94): "There is however an implicit condition in this law (of diminishing marginal utility) which should be made clear. It is that we do not suppose time to be allowed for any alternation in the character or tastes of the man himself. It is therefore no exception to the law that the more good music a man hears, the stronger is his taste for it likely to become", in addition: "Much that is of chief interest in the science of wants, is borrowed from the science of efforts and activity"; McCain (1981, p. 332-334) applied the Becker-Lancaster approach to analyze the process of cultivation of taste and the demand for works of art. The essence in his model is the utility function:

$$U = f\left(Ax, z\right), \; U_x > 0, U_z > 0,$$

where *x* and *z* are the consumption flows of two goods. *A* is a parameter which changes as taste is cultivated. More specifically, *A* depends on the individual's history of consumption of *x* and *z*. *A* is changed over time by the flow of consumption of *x* as follows:

$$\dot{A} = G\left(x, A\right).$$

This set-up goes beyond our modeling of consumer-artists as passive maximizers of utility. It has many interesting implications, of course, but we need to sacrifice realism to avoid a diversion from our present focus.

services relating to that piece are not supplied (by assumption) and hence cannot be consumed even if consumers had a high willingness-to-pay for such services.

In our model, cultural services are produced by n_s firms with the help of the production functions

$$s_j = \underset{+\ +}{S^j\left(r_{sj}, g_{sj}\right)} \qquad j = 1,...,n_s, \tag{2.4}$$

where s_j are cultural services produced by firm[20] *j* with resource input r_{sj} and cultural-goods input g_{sj} [21]. An art gallery exhibiting Leonardo da Vinci's Mona Lisa needs to possess that painting as an input. If Beethoven's 9th symphony is performed in a concert hall, the musicians need to have the scores of that symphony etc[22]. These examples clarify that the use of cultural goods g_{sj} as an input for providing cultural services does not imply their destruction. Sometimes the original artefact is needed to provide the service as in case of services consisting of access to museums or cultural heritage sites[23]. Other services only need copies of the (original) piece of culture, e. g. when music, operas or plays etc. are performed. Moreover, in the latter case many different performances of the same piece of music, opera or play can be

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20 We use the term "firm" for convenience to include all kinds of cultural institutions and installations irrespective of whether they are operated as public or private enterprises. The modeling of individuals as "consumer-performers" is ruled out in the subsequent study for the benefit of analytical simplicity. Interesting discussions about the labor-supply decision of "consumer-performers" can be found, e.g. in Singer (1981, p.341-346), Pommerehne and Frey (1993, p. 152-187), Throsby (1994, p. 69-80), Towse (2001, p. 47-78), Baumol and Baumol (2002, p. 167-184).

21 Other authors combine the terms "cultural services" and "cultural goods" used here as "cultural goods". E.g. Towse (2003, p. 2) argues that "cultural goods are tangible objects, such as an artwork or a book; others are intangible services, like a musical performance or a visit to museum". Such a view totally ignores the productive effects of cultural goods on cultural services and therefore doesn't provide a solid basis for rigorous analysis, in our opinion.

22 Such services are perishable (Peacock, 1975, p. 16). Adam Smith (The Wealth of Nations, Book 2, Chapter 3) characterized a musical performance as "an activity which does not fix or realize itself in any permanent subject or vendible commodity which endures after the labour is past."

23 It is acknowledged that not only the "man-made" stock of cultural goods, *g*, can be used as an input for producing cultural services, but also the not man-made natural heritage. Consider, e. g., the amenities of the Grand Canyon that can be enjoyed by taking a helicopter ride over the canyon, offered by some cultural-services firms. The importance to preserve both cultural and natural heritage is emphasized in the "World Heritage Convention" (an international agreement signed by 175 states and adopted by UNESCO) (cf. the website of UNESCO, 2003), whose primary mission is to conserve the cultural *and* natural heritage. Throsby (2001, p. 51n) analyses the parallels between cultural and natural capital in more detail. Without doubt, the natural heritage plays an important role in cultural life, but in the present study our focus will be confined to the problem of intertemporal allocation of man-made cultural heritage (cultural goods).

offered at the same time indicating that the cultural good used as an input for these services is a public good (to be discussed below in more detail).

The specification of the production functions (2.2) and (2.4) appears to be quite plausible although it is conceded that more complex hypotheses about productive factors would make the setting more realistic. As for the production of new cultural goods it may be argued, for example, that in addition to the inputs r_{vi} and k_i in (2.2), g_j is also a factor stimulating the individuals' creativity in generating new cultural goods ($V^i_{gi} > 0$). Likewise, concerning the production of cultural services it may be assumed for good reasons, that with given inputs r_{sj} and g_{sj} the amount of cultural services produced increases with the stock of cultural capital ($S^j_k > 0$). Yet in what follows we will stick to the simpler production functions as specified in (2.2) and (2.4) to avoid unreasonable analytical complexity.

Cultural capital is conceived of as an intangible and depreciable asset that is built up by consuming cultural services. Similar to (2.3) the formation of cultural capital, *k*, is modeled as a dynamic process:

$$\dot{k} = \sum_{i=1}^{n_c} s_i - \alpha_k k \,, \tag{2.5}$$

where α_k is an exogenous positive rate of depreciation accounting for the observation that some fraction of the stock of the cultural capital gets lost over time. For example, the Chinese Cultural Revolution during the 1960s greatly diminished the Chinese society's cultural capital implying that, on the one hand, the external benefits provided by cultural capital declined and that, on the other hand, the Chinese culture became less and less valued by the Chinese society.

The distinction between, and separate consideration of, the stock of cultural goods and the stock of cultural capital in the model and as arguments in the consumer's utility function is motivated by the observation that the existence of cultural goods per se is not an appropriate indicator of a society's intensity of cultural life and its cultural atmosphere. The stock of cultural goods needs to be "activated" to create a cultural atmosphere or - as we call it - cultural capital, which is achieved through the supply and consumption of cultural services (which, in turn, are based on cultural goods as an essential input). Therefore the stock of cultural goods has an impact on the accumulation of cultural capital, (2.5), only indirectly through (2.4). In other words, the (aggregate) amount of cultural services consumed is related to but is not uniquely determined by the size of the (aggregate) stock of cultural goods: Societies with a rather small cultural heritage (low g) may be culturally very active (high s) and vice versa. Hence it is not the stock of cultural goods per se that determines the cultural atmosphere or climate in society but primarily the volume and richness of cultural services through which the existing stock of cultural goods is used by the members of society. Our principal hypothesis is that the continuous consumption of cultural services leads to an accumulation of cultural capital which, in turn, is positively valued by all members of society.

So far, we specified the production of new cultural goods and cultural services with their links to the stock of cultural goods and to cultural capital, respectively. It remains to introduce the production of a private consumer good that is produced by a single (aggregate) firm using the technology

$$y = Y\big(\underset{+}{r_y}\big), \tag{2.6}$$

where y is the amount of the consumer goods produced by the resource input r_y. The reason for treating the consumer goods industry as an aggregate is to simplify the exposition. Since the firms in that industry neither generate nor are affected by externalities or public goods, disaggregation would only complicate notation without providing additional insights.

The description of our model will now be completed by listing all supply constraints:

$$y > \sum_{i=1}^{n_c} y_i \,, \tag{2.7}$$

$$\sum_{i=1}^{n_c} \bar{r}_i \geq r_y + \sum_{j=1}^{n_s} r_{sj} + \sum_{i=1}^{n_c} r_{vi} \,, \tag{2.8}$$

$$k \geq k_i \qquad \text{for all} \qquad i = 1,\ldots,n_c \,, \tag{2.9}$$

$$g \geq g_i \qquad \text{for all} \qquad i = 1,\ldots,n_c \,, \tag{2.10}$$

$$g \geq g_{sj} \qquad \text{for all} \qquad j = 1,\ldots,\ n_s\,, \tag{2.11}$$

$$\sum_{j=1}^{n_s} s_j \geq s_i \quad \text{for all} \qquad i = 1,\ldots,\ n_c\,. \tag{2.12}$$

The constraints (2.7) and (2.8) are conventional supply constraint for private goods and hence straightforward. In (2.8) $\overline{r_i}$ denotes consumer-artist *i*'s constant resource endowment that is the $\left(1/n_c\right)th$ part of the aggregate resource endowment, while the aggregate resource endowment, irrespective of the varying number of consumer-artists, keeps always constant: $\sum_{i=1}^{n_c} \overline{r_i} \equiv \overline{\overline{r}}$. The constraints (2.9) - (2.12) characterize cultural capital, the stock of cultural goods and cultural services, respectively, as public goods.

Since the distinction between private and public goods is decisive for our subsequent analysis, some remarks on the concept of and the literature on public goods is in order. Unfortunately, Bonus' (1980, p. 50-81) observation still holds that there is extreme disagreement of what public goods are all about. Despite numerous surveys, e.g. Blümel et al. (1986, p. 241-309) and the literature quoted there, some part of the contemporary literature still contributes to confusion rather than enlightenment. To be sure, defining the concept of public goods for the purpose of economic theory is not about capturing, or failing to capture, the "essence" of some class of goods. Every researcher is free to come up with her own definition that she considers most appropriate and useful in relation to the problem to be tackled. However, it is beyond the scope of the present study to discuss the merits and drawbacks of different approaches to the concept of public goods put forward in the literature. Such a survey is not necessary, either, because theory-oriented economists have remarkably little dissent about how to define public goods.

In fact, the mainstream contemporary approach to the theory of public goods is still the one originated by Samuelson's (1954, p. 387-389) seminal paper "The Pure Theory of Public-goods": A good is said to be public (nonrival, indivisible) if it is ***jointly consumable*** in the sense that a unit of the good can be consumed by one individual without detracting from the consumption opportunities still available for the others from that same unit. Following Blümel et al. (1986) we take joint consumability as the only constitutive defining criterion, while subsets of public goods may be further distinguished according to whether they are excludable, congestible or rejectable. Excludability will play some role later in our study but all public goods considered here are assumed to be non-rejectable and non-congestible. In the literature such goods are often referred to as pure public goods.

It appears natural to assume that all consumer-artists jointly enjoy the cultural capital as expressed in (2.9) and also jointly enjoy the extant stock of cultural goods, (2.10). As our preceding discussion shows it is also reasonable to characterize (or approximate) as public goods some subset of cultural goods (in their role as inputs in the production of cultural services) and cultural services. But there is another subset of these good for which joint consumability is not characteristic. In that case the constraints (2.11) and (2.12) need to be replaced by

$$g \geq \sum_{j=1}^{n_s} g_{sj} \ , \tag{2.13}$$

$$\sum_{j=1}^{n_s} s_j \geq \sum_{i=1}^{n_c} s_i \ . \tag{2.14}$$

Note that the constraints (2.13) and (2.14) are alternative specifications, and therefore we deal with two variants of the model which we will address as General Model 1 (GM1) and General Model 2 (GM2) in the subsequent analysis. GM1 consists of the equations (2.1) - (2.11) and is also referred to as the “public-goods model” while GM2 is constituted by the equations (2.1) - (2.10), (2.13) and (2.14) and will also be denoted “private-goods model”.

The terms “public-goods model” and “private-goods model” are introduced to avoid clumsy phrases. Both sub-models entail public goods, of course, but in the “private-goods model” two goods that are public in GM1 are assumed to be private in GM2, namely cultural services and cultural goods as inputs in the production of cultural services. The distinction is shown in Table 2.1.

Table 2.1 The defining characteristics of the models

		GM1	GM2
Cultural capital *k*		Public	public
Cultural goods *g*	as productive factor	Public	private
	for consumer-artists' passive use	Public	public
Cultural services *s*		Public	private

At first glance, the characterization of the stock of cultural goods in GM2 appears to be inconsistent. As in GM1, for consumers it is a public good due to (2.10), but in GM2, cultural-goods that are used as an input for producing cultural services are assumed to be private goods in (2.13). However, this differential treatment is not contradictory. Consider, for example, the Eiffel tower in Paris. It is a famous architectonic cultural landmark which the French people (and others) are proud of (hence $U_g^i > 0$ for all i). Nevertheless, the tower itself is a private good, and as such it is also used as an input to produce services, e.g. the service "using the staircase or the elevator to get on top of it".[24]

We now turn to the case of cultural services as private goods. Fullerton (1992, p. 74) maintains that: "art itself is not categorically a public good because consumption can be charged for (at least at museums) (excludable) and is limited (rival)."[25] For reasons that will soon become apparent, it is not easy to provide convincing examples of private cultural services. Fullerton (ibidem, p. 74) also concedes that "although not always relevant and perhaps a bit strained, the example is not unlike a movie theater where empty seats imply that one more viewer has no social cost. The best exhibits can get very crowded, so one more visitor does impose a cost on others. Travel for the exhibition to reach additional visitors can be very expensive." As mentioned before, since we take "joint consumability" as the only defining criterion to distinguish public and private goods, and therefore do not share Fullerton's view. We rather want to set up our rigorous theoretical analysis with a more pragmatic interpretation of the (otherwise unchanged) theoretical distinction between private and public goods.

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24 Other services are pictures of the Eiffel tower on postcards, souvenir replicas etc. Whether such services should be best considered private or public goods are not always easy to decide.

25 In contrast to Fullerton, we don't consider non-excludability as a defining attribute of public goods. See also Blümel et. al (1986, p. 248-249).

To be more specific, take the Nobel-prize laureate Amartya Sen as an example who gives a public lecture in India. Due to his reputation and popularity in India, his ideas are well-accepted and people are eager to listen to his speech. His lecture can be heard not only by the audience in the lecture hall but live broadcasting makes it possible that in India hundreds of millions of people follow (consume) his presentation at the same time. Hence this lecture is not only jointly consumable, and therefore is a public good without doubt, but it is actually also jointly consumed by a large number of people. Back to Trinity College, Cambridge, Sen gives his regular university lectures that may be attended by less than a hundred students only. The degree of ***joint*** consumption is hence very much reduced.

To provide another example, quite different from the former one, suppose now Sen is asked by the Indian prime minister for his advice on fighting poverty in India. Although Sen's oral advice could be presented to a large audience (of government officials), usually only a very small group of top administrators or even only the prime minister himself actually listens to Sen. Although joint consumability can be acknowledged (as a theoretical attribute) the number of jointly consuming people is usually very small. As a consequence for modeling purposes it appears sensible to treat such a consulting service as a private good ignoring the purists' objection that, "in principle", this service is jointly consumable. Summing up, jointly consumable (and hence public) goods will be analytically treated as private good whenever the number of people who actually do jointly consume that good is usually small.

We hence take this approximation route to characterize the property of cultural services. If the level of joint consumption of jointly consumable cultural services is very low the cultural services are treated as private goods. However, our objective in the present study is not to capture the "essence" of specific classes of goods. Irrespective of whether cultural services are private or public, our principal hypothesis is that the continuous consumption of cultural services (be they public or private) leads to the accumulation of cultural capital.

The structure of the model is conveniently summed up in a non-technical way with the help of Table 2.2.

Table 2.2 General structure of the model

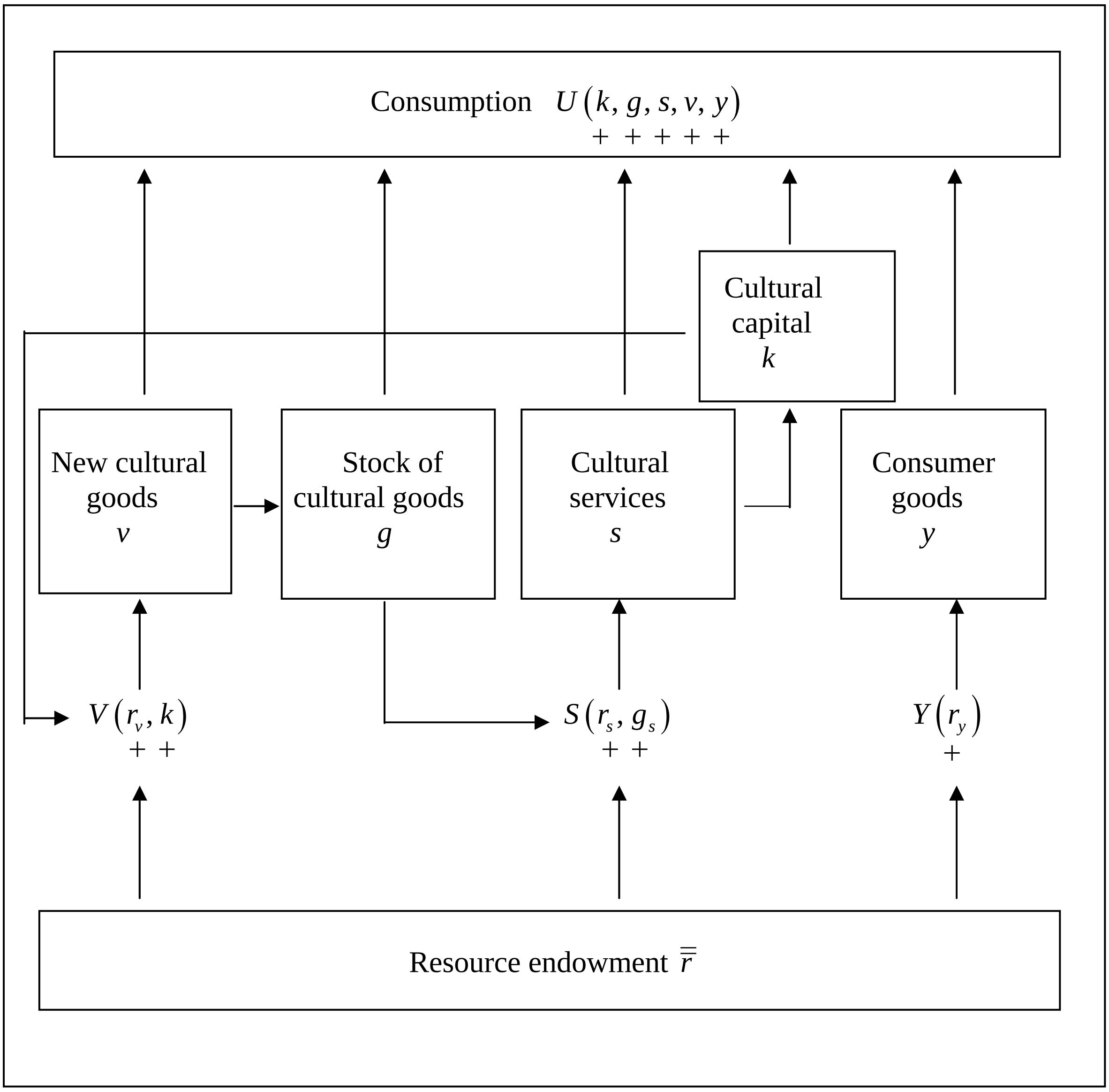

Table 2.2 shows that the economy's given resource endowment is used to produce three different types of goods: an ordinary (private) consumer goods, y, new cultural goods, v, and cultural services, s. While consumer goods are produced with resources as the only input, extant cultural goods are an essential input in the process of producing cultural services. Regarding the creation of new cultural goods it is assumed that cultural capital has a productivity enhancing effect. All three kinds of goods produced are demanded by consumers. In addition, consumers derive satisfaction from both cultural capital and the prevailing stock of cultural goods. These two stocks are not "produced" in a technical sense but they accumulate (or deplete) over time according to some stock-flow relationships modeled in (2.3) and (2.5), respectively. The driving force for the accumulation of cultural goods is the creation of new

cultural goods by all consumer-artists, while the accumulation of cultural capital is determined by the aggregate consumption of cultural services.

Table 2.2 also demonstrates how pervasive the interdependencies and effects of cultural variables are throughout the economy. On the one hand, “culture” (in its aspects *g*, *k*, *s* and *v*) is demanded and hence “demand driven” following the standard postulate of consumer sovereignty. But on the other hand, the prevailing level of cultural capital also feeds back to stimulate the creation of new cultural goods which in turn enhances the incentives to raise the supply of cultural services through increasing the stock of cultural goods. The allocative impact of these circular effects will depend, in a significant way, on whether some of the cultural variables involved are public (GM1) or private (GM2). To specify and elaborate these divergences in the subsequent sections it will be necessary to carry out our analysis for both versions of our model, GM1 and GM2.

2.1 The social planner’s optimization problem in the public-goods model (GM1)

With cultural goods as productive factor and cultural services being public goods, the social planner aims at maximizing the Utilitarian welfare function

$$\int_0^\infty e^{-\delta t} \sum_{i=1}^{n_c} U^i \left(g_i, k_i, s_i, v_i, y_i \right) dt \,, \quad \text{subject to (2.2) - (2.12),} \qquad (2.15)$$

where δ is a positive and constant social discount rate. Hence the planner has to solve a problem of optimal control[26] where the time path of the state variables *g* and *k* is guided by the control variables $g_i, g_{sj}, k_i, s_i, s_j, v_i, r_{sj}, r_{vi}, r_y, y_i$ and y. To characterize the socially optimal intertemporal allocation, consider the following Hamiltonian associated to the social planner’s optimization problem:

[26] On the methods and economic interpretations of the theory of optimal control see Dorfman’s seminal work (1969, p. 817-831); Feichtinger and Hartl (1986); Barro and Sala-I-Martin (1995).

$$H = \sum_{i=1}^{n_c} U^i\left(g_i, k_i, s_i, v_i, y_i\right) + \mu_g\left(\sum_{i=1}^{n_c} v_i - \alpha_g g\right) + \mu_k\left(\sum_{i=1}^{n_c} s_i - \alpha_k k\right) + \lambda_y\left[Y\left(r_y\right) - y\right]$$

$$+\sum_{j=1}^{n_s} \lambda_{sj}\left[S^j\left(r_{sj}, g_{sj}\right) - s_j\right] + \sum_{i=1}^{n_c} \lambda_{vi}\left[V^i\left(r_{vi}, k_i\right) - v_i\right] + \lambda_r\left(\sum_{i=1}^{n_c} \overline{r_i} - r_y - \sum_{j=1}^{n_s} r_{sj} - \sum_{i=1}^{n_c} r_{vi}\right)$$

$$+\lambda_c\left(y - \sum_{i=1}^{n_c} y_i\right) + \sum_{j=1}^{n_s} \lambda_{gj}\left(g - g_{sj}\right) + \sum_{i=1}^{n_c} \lambda_{\sigma i}\left(\sum_{j=1}^{n_s} s_j - s_i\right) + \sum_{i=1}^{n_c} \lambda_{gi}\left(g - g_i\right)$$

$$+\sum_{i=1}^{n_c} \lambda_{ki}\left(k - k_i\right), \tag{2.16}$$

where μ_g and μ_k are co-state variables, and where $\lambda_c, \lambda_{gi}, \lambda_{gj}, \lambda_{ki}, \lambda_r, \lambda_{sj}, \lambda_{\sigma i}, \lambda_{vi}$ and λ_y are Lagrange multipliers. Restricting our attention to an interior solution, the pertinent marginal conditions are

$$\frac{\partial H}{\partial s_i} = U_s^i + \mu_k - \lambda_{\sigma i} = 0 \qquad i = 1, \ldots, n_c, \tag{2.17}$$

$$\frac{\partial H}{\partial y_i} = U_y^i - \lambda_c = 0 \qquad i = 1, \ldots, n_c, \tag{2.18}$$

$$\frac{\partial H}{\partial v_i} = U_v^i + \mu_g - \lambda_{vi} = 0 \qquad i = 1, \ldots, n_c, \tag{2.19}$$

$$\frac{\partial H}{\partial s_j} = -\lambda_{sj} + \sum_{i=1}^{n_c} \lambda_{\sigma i} = 0 \qquad j = 1, \ldots, n_s, \tag{2.20}$$

$$\frac{\partial H}{\partial r_y} = \lambda_y Y_r - \lambda_r = 0, \tag{2.21}$$

$$\frac{\partial H}{\partial y} = -\lambda_y + \lambda_c = 0, \tag{2.22}$$

$$\frac{\partial H}{\partial r_{sj}} = \lambda_{sj} S_r^j - \lambda_r = 0 \qquad j = 1, \ldots, n_s, \tag{2.23}$$

$$\frac{\partial H}{\partial g_{sj}} = \lambda_{sj} S_g^j - \lambda_{gj} = 0 \qquad j = 1, \ldots, n_s, \tag{2.24}$$

$$\frac{\partial H}{\partial r_{vi}} = \lambda_{vi} V_r^i - \lambda_r = 0 \qquad i = 1,\ldots, n_c\,, \tag{2.25}$$

$$\frac{\partial H}{\partial g_i} = U_g^i - \lambda_{gi} = 0 \qquad i = 1,\ldots, n_c\,, \tag{2.26}$$

$$\frac{\partial H}{\partial k_i} = U_k^i + \lambda_{vi} V_k^i - \lambda_{ki} = 0 \qquad i = 1,\ldots, n_c\,, \tag{2.27}$$

$$\dot{\mu}_g = \delta\mu_g - \frac{\partial H}{\partial g} = \delta\mu_g + \alpha_g \mu_g - \sum_{i=1}^{n_c} \lambda_{gi} - \sum_{j=1}^{n_s} \lambda_{gj}\,, \tag{2.28}$$

$$\dot{\mu}_k = \delta\mu_k - \frac{\partial H}{\partial k} = \delta\mu_k + \alpha_k \mu_k - \sum_{i=1}^{n_c} \lambda_{ki}\,. \tag{2.29}$$

Inspection of (2.17) - (2.29) reveals:

(i) (2.18) implies $U_y^i = U_y^h$ for all $i, h = 1,\ldots, n_c$,

(ii) (2.20) implies $\lambda_{sj} = \lambda_{sl}$ for all $j, l = 1,\ldots, n_s$,

(iii) (2.23) and (ii) imply $S_r^j = S_r^l$ for all $j, l = 1,\ldots, n_s$.

Whenever it is convenient in the following analysis to indicate that U_y^i is the same for all *i* we write U_y^* for U_y^i. Correspondingly, we set λ_{s*} and S_r^*. With this notation we now rearrange (2.17) - (2.29) in various ways to elicit the characteristics of the optimal path.

First, (2.17) - (2.27) indicate that at every point in time the decision variables should be selected so that the marginal gains are in balance with the value of the marginal contributions to the accumulation of the cultural-goods stock and cultural capital. Second, (2.28) and (2.29) require that the cultural-goods stock and the cultural capital to depreciate at the same rate that they contribute to build up the social welfare.

The next step is to determine the characteristics of the optimal time path in more detail by combining (2.18), (2.21), (2.22) and (2.25) to rewrite (2.19) as

$$\frac{\mu_g}{U_y^*} = \frac{Y_r}{V_r^i} - \frac{U_v^i}{U_y^*} \qquad \text{or} \qquad \frac{\mu_g}{U_y^* Y_r} = \frac{1}{V_r^i} - \frac{U_v^i}{U_y^* Y_r} \quad , \tag{2.30}$$

μ_g / U_y^* is the shadow price of cultural goods produced by consumer-artists in terms of the consumer good and $\mu_g / U_y^* Y_r$ denotes that shadow price in terms of the resource. In the subsequent analysis we choose to calculate all costs and benefits in terms of the resource thus focusing on the second equation in (2.30). According to that equation the shadow price of cultural goods equals the difference between individual *i's* marginal cost of producing new cultural goods (i.e. her marginal investment cost) and *i's* marginal benefit from creating new cultural goods. Ceteris paribus, the shadow price goes up when the investment is successively increased and vice versa.

From (2.30) we infer that

$$\frac{Y_r}{V_r^i} \gtreqless \frac{U_v^i}{U_y^*} \qquad \Leftrightarrow \qquad \mu_g \gtreqless 0 . \tag{2.31}$$

Our conjecture is that $\mu_g > 0$ and hence $\left(Y_r / V_r^i\right) > \left(U_v^i / U_y^*\right)$ due to (2.31). But that conjecture cannot be validated by checking the equations (2.17) - (2.29). To see that, observe that if $\lambda_{vi} > 0$ is easily established (in fact, all Lagrangean multipliers are strictly positive except for $\lambda_{\sigma i} \geq 0$). However, $U_v^i + \mu_g = \lambda_{vi} > 0$ from (2.19) allows for any sign of μ_g since $U_v^i > 0$.

Invoking (2.20), (2.21), (2.22), (2.23) and (2.24) we elicit more information about the optimal path:[27]

$$\frac{\lambda_{gj}}{U_y^i Y_r} = \sum_{j=1}^{n_S} \frac{S_g^j}{S_r^j} = \sum_{i=1}^{n_C} \frac{U_g^i S_g^j}{U_y^i Y_r} \qquad \text{for all} \quad j = 1, \ldots, n_s , \tag{2.32}$$

$$[1] \quad = \quad [2] \quad = \quad [3]$$

[27] For convenience of referring to individual terms of (2.32) in the text, the second line in (2.32) repeats the first line by assigning $[1] = \frac{\lambda_{gj}}{U_y^i Y_r}$, $[2] = \sum_{j=1}^{n_s} \frac{S_g^j}{S_r^j}$ and $[3] = \sum_{i=1}^{n_c} \frac{U_g^i S_g^j}{U_y^i Y_r}$. This procedure will also be applied in subsequent equations.

[*1*] is the strictly positive shadow price of cultural goods used as an input in the production of cultural services by producer *j*. This shadow price must be equal to the sum of the marginal rates of technical substitution ([*1*] = [*2*]), where S_g^l / S_r^l reflects the marginal value of cultural goods for producer *l* ($l = 1,\ldots, n_s$) in terms of the resource. The summation sign in [*2*] is due to the public-good property of cultural goods as productive inputs. [*3*] represents the aggregate marginal willingness of all consumers to pay for the cultural goods that are indirectly consumed through the consumption of cultural services. The summation sign in [*3*] is due to the public-good property of cultural services as a consumer good. Hence [*1*] = [*2*] = [*3*] means essentially that the shadow price of using cultural goods as an input must equal its marginal cost in production of cultural services [*2*] which, in turn, must equal its marginal (indirect) benefit in consumption [*3*]. It is worth reemphasizing that (2.32) accounts for the public-good property of both cultural goods and cultural services. Note also that λ_{gj} is the same for all $j = 1,\ldots, n_s$.

We now proceed to characterize the optimal time path by turning to the differential equation (2.28). First we rearrange (2.28) to obtain:

$$\frac{\dot{\mu}_g}{\left(\delta+\alpha_g\right)U_y^*Y_r} = \frac{\mu_g}{U_y^*Y_r} - \left[\frac{\sum_{i=1}^{n_c}\frac{U_g^i}{U_y^*Y_r}}{\left(\delta+\alpha_g\right)} + \frac{\sum_{j=1}^{n_s}\frac{\lambda_{gj}}{U_y^*Y_r}}{\left(\delta+\alpha_g\right)}\right]. \tag{2.33}$$

Plugging (2.30) and [*2*] from (2.32) into (2.33) yields:

$$\frac{\dot{\mu}_g}{\left(\delta+\alpha_g\right)U_y^*Y_r} = -\left\{\left[\frac{\sum_{i=1}^{n_c}\frac{U_g^i}{U_y^*Y_r}}{\left(\delta+\alpha_g\right)} + \frac{\sum_{j=1}^{n_s}\frac{S_g^j}{S_r^j}}{\left(\delta+\alpha_g\right)}\right] - \left(\frac{1}{V_r^i} - \frac{U_v^i}{U_y^*Y_r}\right)\right\}. \tag{2.34}$$

[*4*] = -{([*5*] + [*6*]) - ([*7*] - [*8*])}.

In (2.34), [*4*] is the present value of the change in time of the shadow price of cultural goods (in terms of the resource). [*5*] is the present value of the consumers' aggregate marginal passive-use benefits from cultural goods. [*6*] is the present value of the aggregate marginal productivity effect of cultural goods in the production of cultural services. ([*7*] - [*8*]) (derived in (2.30)) is the marginal social cost of cultural goods while ([*5*] + [*6*]) is their marginal social benefit.

Closer inspection of (2.34) yields

$$\dot{\mu}_g \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad \Leftrightarrow \quad \left([5]+[6]\right)\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}\left([7]-[8]\right) \quad \Leftrightarrow \quad \left([5]+[6]+[8]\right)\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}[7].$$

The direction of change over time of the shadow price of cultural goods depends on the sign of the difference between marginal benefit ([*5*] + [*6*]) and marginal cost ([*7*] - [*8*]).

In view of (2.32) and (2.33), [*6*] in (2.34) can be replaced by $[3]/\left(\delta + \alpha_g\right)$ which is the present value of the consumers' aggregate marginal active-use benefit from cultural goods. Hence in (2.34), [*5*] + [*6*] = [*5*] + [*3*]/ $\left(\delta + \alpha_g\right)$ is the consumers' total marginal benefit from the existing stock of public goods.

After having investigated the optimal time path for cultural goods, we now turn to the characteristics of the optimal path for cultural capital. We consider (2.18), and (2.20) - (2.23) to transform (2.17) into

$$\frac{\mu_k}{U_y^* Y_r} = \frac{1}{n_c}\left(\frac{1}{S_r^*} - \sum_{i=1}^{n_c} \frac{U_s^i}{U_y^* Y_r}\right). \tag{2.35}$$

The shadow price of cultural capital (in terms of the resource) has a similar structure as that of cultural goods. It is the $\left(1/n_c\right)th$ part of the difference between the marginal resource cost of production and the consumers' aggregate marginal willingness-to-pay for cultural services. From the viewpoint of cultural-capital formation, the marginal benefits of cultural services accruing to consumers ($U_s^i > 0$) constitute a positive externality. Hence the aggregate marginal willingness-to-pay for cultural services reduces the marginal social costs of cultural capital.

Next we focus on the differential equation (2.29). Transforming (2.29) in a similar way as (2.28) gives us

$$\frac{\dot{\mu}_k}{(\delta+\alpha_k)U_y^*Y_r} = -\left\{\left[\frac{\sum_{i=1}^{n_c}\frac{U_k^i}{U_y^*Y_r}}{(\delta+\alpha_k)} + \frac{\sum_{j=1}^{n_v}\frac{V_k^j}{V_r^j}}{(\delta+\alpha_k)}\right] - \frac{1}{n_c}\left(\frac{1}{S_r^*} - \sum_{i=1}^{n_c}\frac{U_s^i}{U_y^*Y_r}\right)\right\}. \tag{2.36}$$

$$[9] = -\{([10]+[11]) - \frac{1}{n_c}([12]-[13])\}.$$

In (2.36), [*9*] is the present value of the change in time of the shadow price of cultural capital (in terms of the resource). [*10*] is the present value of the consumers' aggregate marginal benefits from cultural capital. [*11*] is the present value of the aggregate marginal productivity effect of cultural capital in the production of new cultural goods. ([*12*] - [*13*]) (derived in (2.35)) is the $(1/n_c)th$ part of the marginal social cost of cultural capital while ([*10*] + [*11*]) is their marginal social benefit. (2.36) implies

$$\dot{\mu}_k \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \iff ([10]+[11]) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{1}{n_c}([12]-[13]) \iff \left([10]+[11]+\frac{1}{n_c}[13]\right) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{1}{n_c}[12].$$

As above, the sign of shadow price depends on the sign of the difference between marginal social benefit ([*10*] + [*11*]) and marginal social cost ([*12*] - [*13*]).

A steady state of the socially optimal time path is defined by $\dot{k}=0, \dot{\mu}_k=0$ and $\dot{g}=0, \dot{\mu}_g=0$. In view of (2.3), (2.5), (2.34) and (2.36) it is straightforward to characterize such a steady state by

$$\sum_{i=1}^{n_c} v_i = \alpha_g g\,, \tag{2.37}$$

$$\sum_{i=1}^{n_c} s_i = \alpha_k k\,, \tag{2.38}$$

$$[5]+[6]+[8]=[7] \quad \text{or} \quad \frac{\sum_{i=1}^{n_c}\frac{U_g^i}{U_y^*Y_r}}{(\delta+\alpha_g)} + \frac{\sum_{j=1}^{n_s}\frac{S_g^j}{S_r^j}}{(\delta+\alpha_g)} + \frac{U_v^i}{U_y^*Y_r} = \frac{1}{V_r^i} \qquad i=1,\dots,n_c\,, \tag{2.39}$$

$$[10]+[11]+\frac{1}{n_c}[13]=\frac{1}{n_c}[12] \quad \text{or} \quad \frac{\sum_{i=1}^{n_c}\frac{U_k^i}{U_y^* Y_r}}{(\delta+\alpha_k)}+\frac{\sum_{j=1}^{n_v}\frac{V_k^j}{V_r^j}}{(\delta+\alpha_k)}+\frac{\sum_{i=1}^{n_c}\frac{U_s^i}{U_y^* Y_r}}{n_c}=\frac{1}{n_c S_r^*} \qquad i=1,\dots,n_c\,,$$

or, equivalently,

$$n_c\,([10]+[11])+[13]=[12] \quad \text{or} \quad n_c\left[\frac{\sum_{i=1}^{n_c}\frac{U_k^i}{U_y^* Y_r}}{(\delta+\alpha_k)}+\frac{\sum_{j=1}^{n_v}\frac{V_k^j}{V_r^j}}{(\delta+\alpha_k)}\right]+\sum_{i=1}^{n_c}\frac{U_s^i}{U_y^* Y_r}=\frac{1}{S_r^*} \qquad i=1,\dots,n_c\,. \quad (2.40)$$

According to (2.37), new cultural goods created by consumer-artists constituting the investments in the stock of cultural goods must equal the depreciation of that stock, and according to (2.28) the increment of cultural capital through the consumption of cultural services must equal the loss of cultural capital through depreciation in the steady state. The interpretation of (2.39) and (2.40) is obvious: The marginal production costs of new cultural goods and cultural services, respectively, on the right side of these equations are exactly matched by the respective marginal benefits of these goods. The summation signs in (2.39) reflect the stock of cultural goods being a public consumer good [*5*] and a public productive factor [*6*]. In that respect, (2.39) is a modified version of the famous summation condition of Samuelson (1954, p. 387-389) for the optimal allocation of public goods. Correspondingly, the summation signs in (2.40) result from cultural capital being a public consumer good [*10*] as well as a public factor of producing new cultural goods [*11*] and from cultural services being a public consumer good [*13*].

2.2 The social planner’s optimization problem in the private-goods model (GM2)

Suppose now, the cultural services and the stock of cultural goods as input in the production of cultural services are private goods. To explore that case we modify the Hamiltonian (2.16) by substituting the constraints (2.13) and (2.14) for the constraints (2.11) and (2.12). More specifically, in (2.16) we replace $\sum_{j=1}^{n_s} \lambda_{gj} \left(g - g_{sj} \right)$ by $\lambda_g \left(g - \sum_{j=1}^{n_s} g_{sj} \right)$ and $\sum_{i=1}^{n_c} \lambda_{\sigma i} \left(\sum_{j=1}^{n_s} s_j - s_i \right)$ by $\lambda_\sigma \left(\sum_{j=1}^{n_s} s_j - \sum_{i=1}^{n_c} s_i \right)$ to get:

$$H = \sum_{i=1}^{n_c} U^i \left(g_i, k_i, s_i, v_i, y_i \right) + \mu_g \left(\sum_{i=1}^{n_c} v_i - \alpha_g g \right) + \mu_k \left(\sum_{i=1}^{n_c} s_i - \alpha_k k \right) + \lambda_y \left[Y\left(r_y \right) - y \right]$$

$$+ \sum_{j=1}^{n_s} \lambda_{sj} \left[S^j \left(r_{sj}, g_{sj} \right) - s_j \right] + \sum_{i=1}^{n_c} \lambda_{vi} \left[V^i \left(r_{vi}, k_i \right) - v_i \right] + \lambda_r \left(\sum_{i=1}^{n_c} \overline{r_i} - r_y - \sum_{j=1}^{n_s} r_{sj} - \sum_{i=1}^{n_c} r_{vi} \right)$$

$$+ \lambda_c \left(y - \sum_{i=1}^{n_c} y_i \right) + \lambda_g \left(g - \sum_{j=1}^{n_s} g_{sj} \right) + \lambda_\sigma \left(\sum_{j=1}^{n_s} s_j - \sum_{i=1}^{n_c} s_i \right) + \sum_{i=1}^{n_c} \lambda_{gi} \left(g - g_i \right)$$

$$+ \sum_{i=1}^{n_c} \lambda_{ki} \left(k - k_i \right). \qquad (2.41)$$

Solving the Hamiltonian (2.41) yields the marginal conditions (2.17) - (2.29) except that (2.17), (2.20), (2.24) and (2.28) are substituted by, respectively,

$$\frac{\partial H}{\partial s_i} = U_s^i + \mu_k - \lambda_\sigma = 0 \,, \qquad (2.42)$$

$$\frac{\partial H}{\partial s_j} = -\lambda_{sj} + \lambda_\sigma = 0 \qquad j = 1, \ldots, n_s \,, \qquad (2.43)$$

$$\frac{\partial H}{\partial g_{sj}} = \lambda_{sj} S_g^j - \lambda_g = 0 \qquad j = 1, \ldots, n_s \,, \qquad (2.44)$$

$$\dot{\mu}_g = \delta \mu_g - \frac{\partial H}{\partial g} = \delta \mu_g + \alpha_g \mu_g - \lambda_g - \sum_{i=1}^{n_c} \lambda_{gi} \,. \qquad (2.45)$$

After some rearrangement of terms we find that $\mu_g / U_y^* Y_r$ is specified in the same way as in (2.30), whereas (2.35) is replaced by

$$\frac{\mu_k}{U_y^* Y_r} = \frac{1}{S_r^*} - \frac{U_s^i}{U_y^* Y_r} . \tag{2.46}$$

The shadow price of cultural capital (in terms of the resource) in the model GM2 is the difference between the marginal resource cost of production and the consumer-artist's marginal willingness-to-pay for cultural services. Note that owing to the private-good property of cultural services in (2.46) the term $\left(1/n_c\right)$ from (2.35) is missing in (2.46). We will continue the comparison between GM1 and GM2 in section 2.3.

To attain more insight in the properties of the optimal time path in the model GM2 we now focus on the differential equation (2.45), by rearranging equations (2.21) - (2.23), (2.30), (2.43) and (2.44):

$$\frac{\dot{\mu}_g}{\left(\alpha + \alpha_g\right) U_y^* Y_r} = -\left\{ \left[\frac{\sum_{i=1}^{n_c} \frac{U_g^i}{U_y^* Y}}{\left(\alpha + \alpha_g\right)} + \frac{\frac{S_g^*}{S_r^*}}{\left(\alpha + \alpha_g\right)} \right] - \left(\frac{1}{V_r} - \frac{U_v^i}{U_y^* Y_r} \right) \right\} . \tag{2.47}$$

[*4*] = -{([*5*] + [*6a*]) - ([*7*] - [*8*])}.

Our interpretation of (2.47) is similar to that of (2.34), except that the summation sign in [*6a*] disappeared due to the private-good property of cultural services. [*6a*] is now an individual firm's marginal productivity effect of cultural goods in the production of cultural services. (2.47) also gives us more information about the optimal time path of the shadow price of cultural goods

$$\dot{\mu}_g \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad \Leftrightarrow \quad \left([5] + [6a] \right) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \left([7] - [8] \right) \quad \Leftrightarrow \quad \left([5] + [6a] + [8] \right) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} [7] .$$

The direction of change over time of the shadow price of cultural goods depends, on the sign of the difference between marginal benefit ([*5*] + [*6a*]) and marginal cost ([*7*] - [*8*]).

Next we briefly return to the differential equation (2.29).[28] By transforming (2.29) in a similar way as (2.47) yields

$$\frac{\dot{\mu}_k}{(\delta+\alpha_k)U_y^*Y_r} = -\left\{\left[\frac{\sum_{i=1}^{n_c}\frac{U_k^i}{U_y^*Y_r}}{(\delta+\alpha_k)} + \frac{\sum_{j=1}^{n_v}\frac{V_k^j}{V_r^j}}{(\delta+\alpha_k)}\right] - \left(\frac{1}{S_r^*} - \frac{U_s^i}{U_y^*Y_r}\right)\right\} \tag{2.48}$$

$[9]$ = -{($[10]$ + $[11]$) - ($[12]$ - $[13a]$)}.

(2.48) has a similar structure as (2.36). However, (2.48) differs from (2.36) through (i) the absence of the term $(1/n_c)$ and (ii) the summation sign in [*13a*]. The sign of the change in the shadow price of cultural capital is determined as follows:

$$\dot{\mu}_k \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad \Leftrightarrow \quad ([10]+[11]) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} ([12]-[13a]) \quad \Leftrightarrow \quad ([10]+[11]+[13a]) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} [12].$$

As in the public-goods model the sign depends on the sign of the difference between the marginal social benefit ([*10*] + [*11*]) and marginal social cost of cultural capital ([*12*] - [*13a*]).

Consequently, the optimal steady state in the private-goods economy is given by (2.37), (2.38) and

$$[5]+[6a]+[8]=[7] \quad \text{or} \quad \frac{\sum_{i=1}^{n_c}\frac{U_g^i}{U_y^*Y}}{(\alpha+\alpha_g)} + \frac{\frac{S_g^*}{S_r^*}}{(\alpha+\alpha_g)} + \frac{U_v^i}{U_y^*Y_r} = \frac{1}{V_r} \qquad i=1,\dots,n_c, \tag{2.49}$$

$$[10]+[11]+[13a]=[12] \quad \text{or} \quad \frac{\sum_{i=1}^{n_c}\frac{U_k^i}{U_y^*Y_r}}{(\delta+\alpha_k)} + \frac{\sum_{j=1}^{n_v}\frac{V_k^j}{V_r^j}}{(\delta+\alpha_k)} + \frac{U_s^*}{U_y^*Y_r} = \frac{1}{S_r^*}. \tag{2.50}$$

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28 The reader may wonder why we invoke the differential equation (2.29) from the public-goods model again, in the context of tackling the private-goods model. It is readily seen that the marginal condition (2.29) we derived from the solution of the Hamiltonian (2.16) in the public-goods model, has the same "face value" as the marginal condition derived from the solution of the Hamiltonian (2.41) in the private-goods model, although the underlying allocations are different in both models. The same argument applies to (2.18), (2.19), (2.21) - (2.23), and (2.25) - (2.27).

After having extensively discussed the characteristics of the optimal intertemporal allocation in the private-goods model (GM2), we are now interested in exploring the principal differences between the optimal allocation in the public-goods model (GM1) and the private-goods model (GM2).

2.3 Comparing the optimal allocation of the models GM1 and GM2

Clearly, the replacement of [*6*] from (2.39) by [*6a*] in (2.49) results from assuming that the stock of cultural goods is now a private factor in the production of cultural services. Hence the summation condition is no longer warranted. Likewise, [*13*] from (2.40) needs to be substituted by [*13a*] in (2.50) since cultural services cannot be jointly consumed anymore. Therefore the $(1/n_c)th$ part of the aggregate marginal willingness-to-pay for the consumption of cultural-services [*13*] is replaced by an individual consumer's marginal willingness-to-pay [*13a*] which is now the same for all consumers.

Comparing pairwise the steady-state condition (2.39) with (2.49) and the condition (2.40) with (2.50), one may wonder whether it is possible to draw conclusions from that comparison with respect to how the steady-state allocations differ in the economies GM1 and GM2. To fix our ideas we first focus on (2.40) and (2.50). At first glance, it is tempting to argue as follows: n_c([*10*]+[*11*])+[*13*] from (2.40) is greater than [*10*]+[*11*]+[*13a*] from (2.50) because the sum n_c([*10*]+[*11*]) is "naturally" greater than ([*10*]+[*11*]) and [*13a*] is greater than [*13*] anyway due to the summation operation. However, this reasoning is fallacious since the constituent derivatives of both equations belong to different models and are therefore evaluated at different (solution) values of the variables, as we have already pointed out in the context of the differential equation (2.29). As a consequence, the terms that look identical in both equations "at face value" will have different values or magnitudes, in general. We don't know how the terms that look alike in both equations differ from each other, e.g. we don't know the difference between $\left(V_k^j / V_r^j\right)$ from (2.40) and $\left(V_k^j / V_r^j\right)$ from (2.50). For that reason a rigorous comparison between the equations (2.40) and (2.50) is not feasible.

Nevertheless, we believe that some tentative comparison of (2.40) and (2.50) can be made as follows. Recall first that [*13*] from (2.40) is the consumer-artists' aggregate marginal willingness-to-pay for cultural services (in terms of the resources), and this term is the social value of the last unit of cultural services because in the model under consideration (GM1) cultural services are public goods. In contrast, [*13a*] from (2.50) represents the individual marginal willingness-to-pay for cultural services which is also the social value of the last unit of cultural services because in the model considered (GM2) cultural services are private. We take it as plausible that [*13*] is greater than [*13a*] even though this cannot be rigorously proved without further serious restrictions on the model's assumption.[29] Likewise, we consider it is also plausible that n_c([*10*]+[*11*]) from (2.40) is greater than [*10*]+[*11*] from (2.50) in which case n_c([*10*]+[*11*]) + [*13*] from (2.40) would greater, in fact than [*10*]+[*11*]+[*13a*] from (2.50). In other words, plausibility arguments suggest that the sum n_c([*10*]+[*11*])+[*13*] from (2.40) is likely to be greater than the sum [*10*] + [*11*] + [*13a*] from (2.50) even if we account for the fact that in general the terms [*10*] in (2.40) and (2.50) as well as the terms [*11*] in (2.40) and (2.50) differ from each other. As a consequence, S_r^* from (2.40) tends to be smaller than S_r^* from (2.50) which implies, in turn, that the supply of cultural services in the steady state of model GM1 tends to be greater than in GM2.[30] We therefore infer from (2.5) that the steady-state cultural capital tends to be greater in GM1 than in GM2.

We now turn to the comparison of (2.39) and (2.49) applying the same kind of plausibility judgment as in the preceding paragraph. As above, we will mark our conclusions as non-rigorous by qualifiers like "tends to be greater/smaller than". Clearly, [*6*] from (2.39) {[*6a*] from (2.49)} reflects the public-good property {the private-good property} of the stock of cultural goods as an input in the production of cultural services. Due to the summation operation, [*6*] in (2.39) tends to be greater than [*6a*] in (2.49). Along the same line of argument we applied in the comparison of (2.40) and (2.50) above it is therefore plausible that [*5*] + [*6*] + [*8*] is greater than [*5*] + [*6a*] + [*8*]. It follows that V_r^i from (2.39) tends to be smaller than V_r^i from (2.49) and therefore all consumer-artists tend to create a larger amount of new cultural goods in model GM1 than in model GM2. In view of (2.3) we reach the (non-rigorous) conclusion that the steady-state stock of cultural goods will be greater in GM1 than in GM2.

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29 For such restrictions see section 3.

For the purpose of better understanding the difference in the results presented in (2.39), (2.40) and (2.49), (2.50), let us consider temporarily the polar case of a single consumer-artist ($n_c = 1$) who consumes cultural goods and cultural services, and a single cultural-services firm ($n_s = 1$) that demands cultural goods for producing cultural services. By setting $n_c = 1$ and $n_s = 1$ in (2.39) and (2.40), these equations coincide with (2.49) and (2.50), respectively, and consequently the optimal trajectories and steady states of GM1 and GM2 also coincide. For joint consumability to have an impact on the optimal allocation one needs to have a model where the public good is consumed by more than one agent.

Another way to enhance the understanding of the role played by the stock of cultural goods and cultural capital in our model is to assume, for a moment, that all individuals are completely indifferent with respect to both goods. In terms of the formal model we set $U_g^i = U_k^i = V_k^i \equiv 0$ for all $g_i \geq 0$ and $k_i \geq 0$ and all $i = 1,\dots, n_c$, and we refer to such a situation as the absence of cultural externalities.

“Switching off” all cultural externalities in the (otherwise unmodified) model GM2 means that there is no public good left in that economy since by definition of GM2 cultural goods as inputs in the production of cultural services as well as new cultural goods are private goods. In contrast, these two goods are assumed to be public in GM1 and remain public goods in GM1 irrespective of the presence or absence of cultural externalities.

If cultural externalities are absent in the otherwise unmodified public-goods economy GM1, the steady-state optimality conditions (2.39) and (2.40) are turned into

$$[6]+[8]=[7] \qquad \text{or} \qquad \frac{\sum_{j=1}^{n_s} \frac{S_g^j}{S_r^j}}{\left(\delta + \alpha_g\right)} + \frac{U_v^i}{U_y^* Y_r} = \frac{1}{V_r^i} \qquad i = 1,\dots,n_c, \tag{2.51}$$

$$[13]=[12] \qquad \text{or} \qquad \sum_{i=1}^{n_c} \frac{U_s^i}{U_y^* Y_r} = \frac{1}{S_r^*}. \tag{2.52}$$

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[30] This conclusion presupposes that the production functions S^j are strictly concave in r_{sj} and that the cross effects from S_{rg}^j are of second order only.

If cultural externalities are absent in the otherwise unmodified private-goods economy GM2, the steady-state optimality conditions (2.49) and (2.50) become

$$[6a]+[8]=[7] \qquad \text{or} \qquad \frac{\frac{S_g^*}{S_r^*}}{\left(\delta+\alpha_g\right)}+\frac{U_v^i}{U_y^* Y_r}=\frac{1}{V_r^i} \qquad i=1,\ldots,n_c\,. \tag{2.53}$$

$$[13a]=[12] \qquad \text{or} \qquad \frac{U_s^i}{U_y^* Y_r}=\frac{1}{S_r^*} \qquad i=1,\ldots,n_c\,. \tag{2.54}$$

The absence of any summation sign in (2.53) and (2.54) reveals (and confirms) that GM2 without cultural externalities is a society dealing with private goods only. According to (2.54) it is optimal to provide cultural services like any (ordinary) consumer goods: each consumer's marginal willingness-to-pay for those services needs to equal the marginal costs of producing cultural services. The optimal provision of new cultural goods in (2.53) requires to match marginal production costs, $1/V_r^i$, with the sum of the consumer-artist's own marginal utility from creating new cultural goods and the marginal productivity of the stock of cultural goods in each firm that supplies cultural services.

In contrast, as the summation sign in (2.51) and (2.52) indicates, the society GM1 without cultural externalities is still characterized by public goods. Except for the difference between private and public goods, the interpretation of (2.51) and (2.52) is analogous to that of the equations (2.53) and (2.54) and hence need not be repeated here.

The comparison of the steady-state allocations of GM1 and GM2 in the absence of cultural externalities, i.e. the comparison of (2.51) and (2.52) on the one hand and (2.53) and (2.54) on the other hand meets the same difficulties as described at the beginning of the present section. Following the same procedure outlined and founded above the comparison of the steady-state equations (2.51) and (2.52) shows that in GM1 without cultural externalities consumer-artists tend to create more new cultural goods and thus accumulate more cultural capital than in GM2 without cultural externalities. Likewise, the comparison of (2.52) and (2.54) leads us to conclude that in GM1 without cultural externalities more cultural services and hence more cultural capital tend to be provided than in GM2 without cultural externalities. This (tentative) result is qualitatively the same as that of comparing the steady-state allocating of GM1 and GM2 when cultural externalities are present (see above).

It remains to answer the interesting question as to how the allocations differ in any given model, either GM1 or GM2, when the comparison is made between the model with and without cultural externalities. Consider first the equations (2.51) and (2.52) and observe that [*6*] from (2.39) reflect the consumer-artists' aggregate marginal passive-use benefits from cultural goods. Since this term becomes zero when $U_g^i \equiv 0$, the left side of (2.51) tends to be smaller than that of (2.39). As a consequence, V_r^i from (2.51) tends to be smaller than V_r^i from (2.39) implying that the creation of new cultural goods tends to be smaller under the assumption $U_g^i \equiv 0$ than in case of $U_g^i > 0$. With [*8*] from (2.39) vanishing in (2.51), one out of three (partial) marginal benefits from creating new cultural goods is absent. Hence the steady state is likely to be characterized by a positive stock of cultural goods, but it tends to be smaller than in an economy GM1 in which cultural externalities are present.

Consider now the equations (2.52) and (2.40). With setting $U_g^i \equiv 0$ and $V_k^j \equiv 0$, the terms [*10*] and [*11*] from (2.40) become zero and hence are missing in (2.52). The consumer-artists' appreciation of cultural capital [*10*] is absent by assumption as well as their stimulus from cultural capital for creating new cultural goods [*11*]. What is left in (2.52) on the benefit side of cultural services is only the consumer-artists' direct benefit from consuming those services. Since cultural services are public goods in the model under consideration, (2.52) now takes the form of a standard Samuelsonian summation condition. When we apply the analogous reasoning to comparing (2.39) and (2.51) we observe that [*13*] from (2.52) tends to be smaller than n_c ([*10*] +[*11*]) + [*13*] from (2.40) such that the supply of cultural services tends to be smaller in an economy where these externalities are present. Obviously, along with the reduction in cultural services, the steady-state cultural capital will shrink, too, but need not necessarily become zero.

The juxtaposition of GM1 and GM2 with and without cultural externalities showed that as compared with their absence, the presence of cultural externalities tends to make it optimal for societies to step up its cultural activities: the creation of new cultural goods, the stock of cultural goods, the provision of cultural services and the accumulation of cultural capital. We provided reasons and examples supporting the view that cultural externalities are empirically relevant phenomena even though it must be conceded that the order of magnitude of the allocative effects of these externalities is difficult to assess. Anyway, ignoring cultural externalities amounts to ignore culture as an important social phenomenon, in our view, and this is why we will proceed by assuming that cultural externalities are present in both models, GM1 and GM2.

3 Transitional optimal dynamics in simplified models

In the preceding section we investigated the socially optimal intertemporal allocation, and we characterized the nature of the corresponding dynamics. We also demonstrated that depending on the assumption whether cultural services and cultural goods used as productive factors are *public* (GM1) or *private* goods (GM2) the rules guiding the optimal allocation will differ significantly. But due to their general assumptions, the models of section 2 offered only limited information on how the steady state of the economy is characterized, and they did not allow us to characterize the dynamics by means of phase diagrams. To attain more specific results we now impose more restrictive assumptions on the general model. In particular, we will restrict our attention to economies with only one (endogenous) state variable. First, we will set the stock of cultural goods constant and distinguish two scenarios where either both cultural-goods inputs and cultural services are *public* goods ($\mathrm{S\overline{G}1}$) or where both cultural-goods inputs and cultural services are *private* goods ($\mathrm{S\overline{G}2}$). Keeping the stock of cultural goods constant allows us to describe more precisely how the provision of cultural services over time affects the formation of cultural capital.

After having discussed the models $\mathrm{S\overline{G}1}$ and $\mathrm{S\overline{G}2}$ in section 3.1, we will take a different route to simplify the general model by reintroducing the stock of cultural goods as an endogenous state variable while now ignoring the impact of cultural capital on the economy. We will first discuss the case of cultural-goods inputs and cultural services being public goods ($\mathrm{S\overline{K}1}$), and then the scenario where both cultural-goods inputs and cultural services are private goods ($\mathrm{S\overline{K}2}$). With similar procedures as in section 3.1, we will study in section 3.2 the dynamics of a model in which the stock of cultural goods is the only relevant state variable.

3.1 Cultural-capital formation when the stock of cultural goods is constant

We now assume that the initial stock of cultural goods is positive, $g = \overline{g} > 0$, and that $\alpha_g \equiv 0$ in (2.3) and $V^i\left(r_{vi}, k_i\right) = 0$ for all $r_{vi} \geq 0$. As a consequence, $g_t = \overline{g}$ for all $t \geq 0$. In an effort to obtain additional specific information about the optimal time paths of cultural capital and cultural services, we further simplify both the demand and supply side of cultural services by assuming that all consumer-artists and all producers of cultural services are identical: $U^i = U$ for all $i = 1,...,n_c$ and $S^j = S$ for all $j = 1,...,n_s$.

The representative consumer-artist's utility function now reads[31]

$$u = \underset{+\;\;+\;\;+}{U\left(k_c, s_c, y_c\right)}, \tag{3.1}$$

where k_c is her demand for the stock of cultural capital, s_c is her consumption of cultural services and y_c is her consumption of consumer goods. Due to the assumption of identical consumers, (2.5) can be rewritten as:

$$\dot{k} = n_c s_c - \alpha_k k \,. \tag{3.2}$$

Due to the assumption of identical producers of cultural services (2.4) is turned into

$$s_s = S\left(r_s, g_s\right). \tag{3.3}$$

The consumer goods are still produced with the technology (2.6). The constraints (2.7), (2.8) and (2.9) simplify to

$$y_s \geq n_c y_c \,, \tag{3.4}$$

$$n_c \bar{r} \geq n_s r_s + r_y \,, \tag{3.5}$$

$$k \geq k_c \,, \tag{3.6}$$

where $n_c \bar{r}$ is the constant aggregate resource endowment. We know from (2.11) - (2.14) that the supply constraints for cultural-goods inputs and cultural services depend on whether these goods are private or public.

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31 To keep the calculations simple we drop the stock of cultural goods as an argument in the utility function.

3.1.1 A simplified public-goods model with constant stock of cultural goods ($S\bar{G}1$)

In what follows we focus on the case of cultural-goods inputs and cultural services being public goods and hence employ

$$\bar{g} \geq g_s \,, \tag{3.7}$$

$$n_s s_s \geq s_c \,, \tag{3.8}$$

as simplified versions of (2.11) and (2.12), respectively. The model (2.6), (3.1) - (3.8) is referred to as model $S\bar{G}1$.

3.1.1.1 The optimal intertemporal allocation

In the model $S\bar{G}1$ the social planner aims at maximizing the Utilitarian welfare function

$$n_c \int U\left(k_c, s_c, y_c\right) e^{-\delta t} dt \,, \qquad \text{subject to (2.6) and (3.2) - (3.8).} \tag{3.9}$$

The pertaining optimal intertemporal allocation is attained by solving the current-value Hamiltonian:

$$\begin{aligned} H = {} & n_c U\left(k_c, s_c, y_c\right) + \mu_k\left(n_c s_c - \alpha_k k\right) + n_s \lambda_s\left[S\left(r_s, g_s\right) - s_s\right] + \lambda_y\left[Y\left(r_y\right) - y_s\right] \\ & + n_c \lambda_\sigma\left(n_s s_s - s_c\right) + \lambda_c\left(y_s - n_c y_c\right) + \lambda_r\left(n_c \bar{r} - n_s r_s - r_y\right) \\ & + n_s \lambda_g\left(\bar{g} - g_s\right) + n_c \lambda_k\left(k - k_c\right). \end{aligned} \tag{3.10}$$

In case of an interior solution the associated FOCs read:

$$\frac{\partial H}{\partial g_s} = n_s \lambda_s S_g - n_s \lambda_g = 0 \,, \tag{3.11}$$

$$\frac{\partial H}{\partial s_c} = n_c U_s + n_c \mu_k - n_c \lambda_\sigma = 0\,, \tag{3.12}$$

$$\frac{\partial H}{\partial y_c} = n_c U_y - n_c \lambda_c = 0\,, \tag{3.13}$$

$$\frac{\partial H}{\partial r_s} = n_s \lambda_s S_{r_s} - n_s \lambda_r = 0\,, \tag{3.14}$$

$$\frac{\partial H}{\partial r_y} = \lambda_y Y_r - \lambda_r = 0\,, \tag{3.15}$$

$$\frac{\partial H}{\partial s_s} = -n_s \lambda_s + n_c n_s \lambda_\sigma = 0\,, \tag{3.16}$$

$$\frac{\partial H}{\partial y_s} = -\lambda_y + \lambda_c = 0\,, \tag{3.17}$$

$$\frac{\partial H}{\partial k_c} = n_c U_k - n_c \lambda_k = 0\,, \tag{3.18}$$

$$\dot{\mu}_k = \delta \mu_k - \frac{\partial H}{\partial k} = \left(\delta + \alpha_k\right) \mu_k - n_c \lambda_k\,. \tag{3.19}$$

According to (3.12) the choice variable s_c should be selected such that, at each point in time, the marginal benefits are in balance with the value of the marginal contribution to the accumulation of cultural capital. (3.19) indicates that the cultural capital depreciates at the same rate at which it contributes to the output of the cultural external effect. Making use of (3.11) - (3.18) in (3.19) yields, after some rearrangement of terms,

$$\frac{\dot{\mu}_k}{\left(\delta + \alpha_k\right) U_y Y_r} = -\left\{ \frac{n_c U_k}{\left(\delta + \alpha_k\right) U_y Y_r} - \frac{1}{n_c}\left(\frac{1}{S_r} - \frac{n_c U_s}{U_y Y_r} \right) \right\}. \tag{3.20}$$

$$[14] = -\{[15] - \frac{1}{n_c}([16] - [17])\}$$

According to (3.20) the shadow price of cultural capital expressed in terms of the resource [*14*] must equal the difference between the aggregate marginal social benefit [*15*] and the $\left(1/n_c\right)$ *th* part of the marginal social cost of cultural capital ([*16*] - [*17*]). It is easy to see that (3.20) is a special case of (2.36) under the simplifying assumptions of model $S\bar{G}1$ that all consumers are identical, all producers of cultural services are identical and that $v_i = 0$ for all i. Since new cultural goods are not produced anymore, [*11*] is not contained in (3.20), and the remaining summation signs in (2.36) are replaced in (3.20) by multiplying the respective terms with the number of consumer-artists.

3.1.1.2 The optimal time path in a parametric version of model $S\bar{G}1$

Equation (3.20) does not yet provide us with rich information about the optimal time path of cultural services and cultural capital. To obtain more specific results about the transitional dynamics, we introduce further simplifying assumptions:

(i) Leontief technology for producing cultural services:

$$s_s = S\left(r_s, g_s\right) = min\left[a_s r_s, g_s\right], \tag{3.21}$$

where a_s is a positive technological parameter.

(ii) Linear technology for producing consumer goods:

$$y = Y\left(r_y\right) = a_y r_y, \tag{3.22}$$

where a_y is a positive technological parameter.

(iii) The representative individual's utility function (3.1) is additively separable in all its arguments and quadratic with respect to k_c and s_c:

$$u = U\left(k_c, s_c, y_c\right) = U^k\left(k_c\right) + U^s\left(s_c\right) + y_c, \tag{3.23}$$

where $U^k\left(k_c\right) := b_k k_c - \frac{d_k}{2} k_c^{\,2}$, $U^s\left(s_c\right) := b_s s_c - \frac{d_s}{2} s_c^{\,2}$ and where the preference parameters b_k, b_s, d_k and d_s are constant and positive.

With these simplifying assumptions the social planner seeks to maximize

$$n_c \int U\left(k_c, s_c, y_c\right) e^{-\delta t} dt, \qquad \text{subject to (3.2), (3.4) - (3.8), (3.21) and (3.22).} \tag{3.24}$$

The associated current-value Hamiltonian is

$$H = n_c\left(b_k k_c - \frac{d_k}{2}k_c{}^2 + b_s s_c - \frac{d_s}{2}s_c{}^2 + y_c\right) + \mu_k\left(n_c s_c - \alpha_k k\right) + n_s\lambda_{sr}\left(a_s r_s - g_s\right)$$

$$+ n_s\lambda_{sg}\left(g_s - s_s\right) + \lambda_y\left(a_y r_y - y_s\right) + n_c\lambda_\sigma\left(n_s s_s - s_c\right) + \lambda_c\left(y_s - n_c y_c\right)$$

$$+ \lambda_r\left(n_c\bar{r} - n_s r_s - r_y\right) + n_s\lambda_g\left(\bar{g} - g_s\right) + n_c\lambda_k\left(k - k_c\right). \tag{3.25}$$

Observe that the Leontief production function (3.21) is accounted for in (3.25) through the equations $g_s = s_s$ and $a_s r_s = g_s$ implying that inefficient productive plans are excluded from the outset. The FOCs for an interior solution are

$$\lambda_{sg} = \lambda_{sr} + \lambda_g\,, \tag{3.26}$$

$$\lambda_\sigma = b_s - d_s s_c + \mu_k\,, \tag{3.27}$$

$$\lambda_c = \lambda_y = 1\,, \tag{3.28}$$

$$\lambda_r = a_s\lambda_{sr} = a_y\lambda_y\,, \tag{3.29}$$

$$\lambda_{sg} = n_c\lambda_\sigma\,, \tag{3.30}$$

$$\lambda_k = b_k - d_k k_c\,, \tag{3.31}$$

$$\dot{\mu}_k = \left(\delta + \alpha_k\right)\mu_k - n_c\lambda_k\,. \tag{3.32}$$

Note first that (3.28) and (3.29) imply that $\lambda_c, \lambda_r, \lambda_{sr}$ and λ_y are positive. Consequently all corresponding Lagrange constraints hold as equalities. However, we still need to distinguish two cases depending on whether or not the constraint $g_s \le \bar{g}$ is strictly binding. Due to the Kuhn-Tucker condition $\lambda_g\left(\bar{g} - g_s\right) = 0$, the optimal allocation exhibits either "$\lambda_g > 0$ and $g_s = \bar{g}$" or "$\lambda_g = 0$ and $g_s < \bar{g}$" (suppressing the knife-edge case where $\lambda_g = 0$ and $g_s = \bar{g}$). Since λ_g is readily interpreted as the shadow price of cultural goods, we interpret the case "$\lambda_g > 0$ and $g_s = \bar{g}$" as an optimal allocation in which cultural goods used as inputs for cultural-services firms are scarce. Correspondingly, "$\lambda_g = 0$ and $g_s < \bar{g}$" portrays an optimal allocation in which cultural goods used as inputs are abundant.[32] Both cases will be investigated in more depth in the next subsection, in which we will “visualize the evolution” of the model $S\bar{G}1$ with the help of the phase diagram technique.

3.1.1.3 The phase diagram

a) Case $g_s = \bar{g}$ and $\lambda_g > 0$

If $\lambda_g > 0$ and $g_s = \bar{g}$ for all *t*, inspection of (3.5), (3.8), (3.21) and (3.22) readily yields, after some rearrangement of terms,

$$r_s^{\bar{G}1a} := \frac{\bar{g}}{a_s},\ s_s^{\bar{G}1a} := \bar{g},\ r_y^{\bar{G}1a} := n_c\bar{r} - n_s\frac{\bar{g}}{a_s},\ y^{\bar{G}1a} := a_y r_y^{\bar{G}1a} \text{ and } s_c^{\bar{G}1a} := n_s\bar{g}. \tag{3.33}$$

Hence (3.2) is modified to read

$$\dot{k} = n_c n_s \bar{g} - \alpha_k k. \tag{3.34}$$

For $\dot{k} = 0$, equation (3.34) determines the optimal steady-state value of cultural capital as

$$k^{\bar{G}1a} := n_c n_s\left(\frac{\bar{g}}{\alpha_k}\right), \tag{3.34'}$$

and (3.34) also specifies the optimal path of accumulation or decrease of cultural capital along the path towards the steady state. The graph of the $\dot{k} = 0$ locus of (3.34) is depicted in Figure 3.1. To determine the direction of motion of *k* over time to the right and left of this locus, we consider an arbitrary point on the $\dot{k} = 0$ locus, e.g. the point *R* in Figure 3.1 whose coordinates are (k_0, s_0). A deviation by $\Delta k \neq 0$ from point *R* gives

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32 It is interesting to note that the necessity of distinguishing between scarce and abundant cultural goods is due to the lack of smooth differentiability of the Leontief production function (3.21). To see that reconsider the model (2.6), (3.1) - (3.8). As the FOCs (3.11) and (3.14) show, the production function (3.3) has implicitly been assumed to be continuously differentiable (as in the entire section 2). One can easily check that in case of an interior solution the FOCs (3.11) - (3.19) unambiguously imply $\lambda_g > 0$ and hence $g_s = \bar{g}$. Therefore it is the specific functional form of the production function (3.21) that introduces the possibility of abundant cultural goods along the optimal time path.

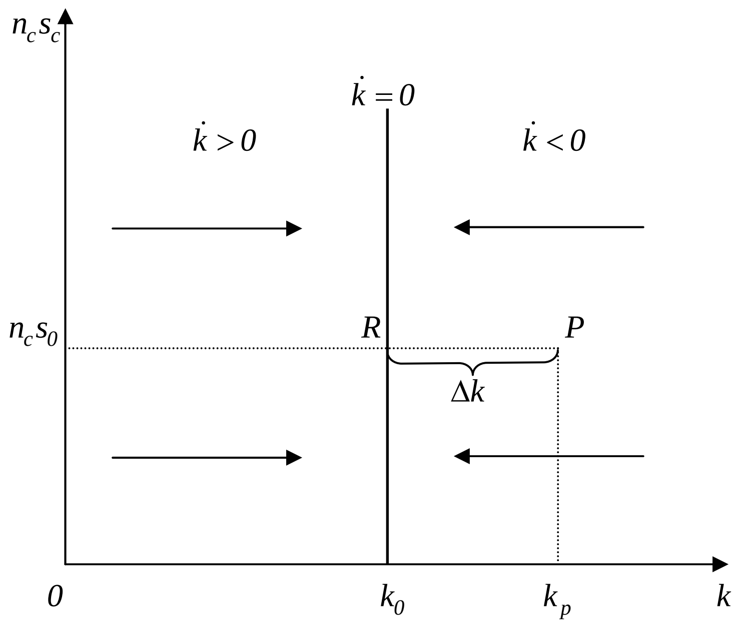
$$\dot{k} = n_c n_s \bar{g} - \alpha_k \left(k_0 + \Delta k \right),$$

and it clearly follows that

$$\dot{k} = n_c n_s \bar{g} - \alpha_k \left(k_0 + \Delta k \right) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0, \qquad \text{if and only if} \qquad \Delta k \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 .$$

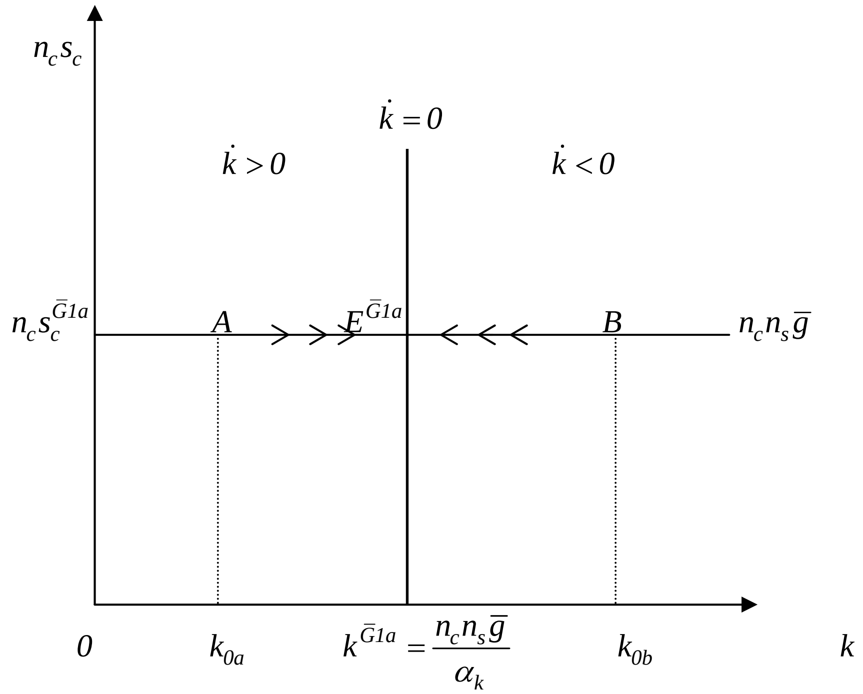
Consequently, for all points (*k*, *s*) to the left of the $\dot{k} = 0$ locus, $\dot{k}$ is positive, as indicated by the arrows pointing east in Figure 3.1. The opposite holds for all points (*k*, *s*) to the right of the $\dot{k} = 0$ locus. At those points, $\dot{k}$ is negative, and hence the arrows point westward. Having determined the direction of motion of cultural capital *k*, we proceed to further characterize the optimal time path of cultural capital.

Figure 3.1 Direction of motion of cultural capital



As depicted in Figure 3.2, the adjustment path depends on the initial stock of cultural capital, k_0. If $k_0 < k^{\bar{G}1a}$, such as k_{0a} in Figure 3.2, we start at point *A* and move along the horizontal straight line to point $E^{\bar{G}1a}$. Conversely, if $k_0 > k^{\bar{G}1a}$, such as k_{0b} in Figure 3.2, we start at point *B* and move along the horizontal line to point $E^{\bar{G}1a}$. The point $E^{\bar{G}1a}$ represents the optimal steady state.

Figure 3.2 The optimal time path of cultural capital in the parametric version of model $S\bar{G}1$ **when cultural goods are scarce** $\left(\lambda_g > 0\right)$



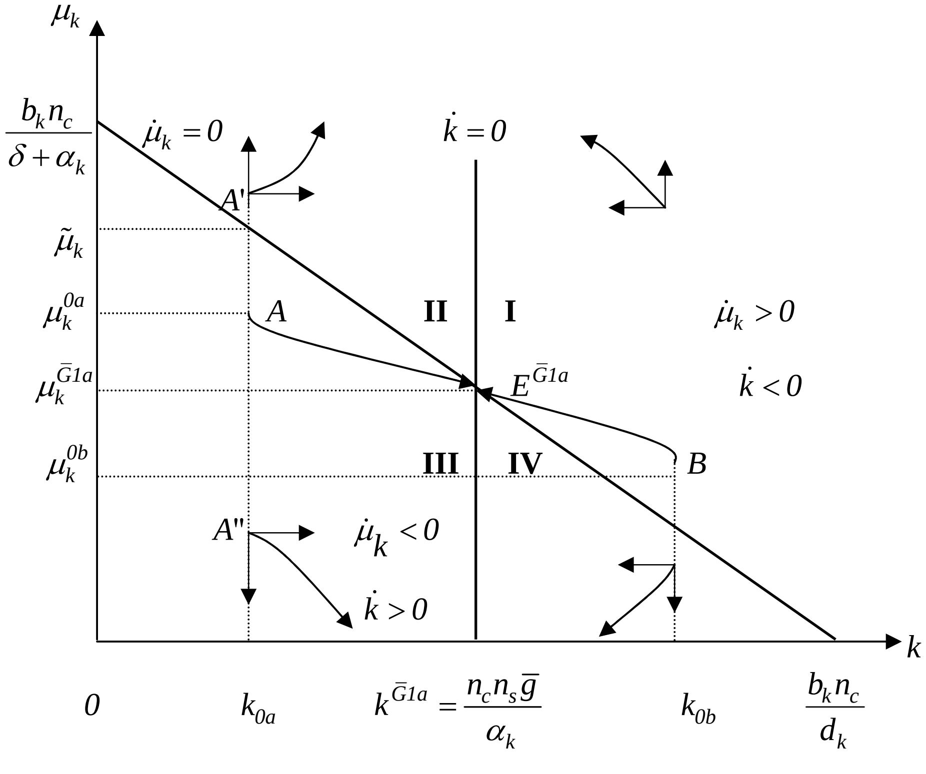
It remains to characterize the change in the shadow price μ_k of cultural capital over time as specified in (3.32). In the optimal steady state $\dot{\mu}_k$ is zero and hence we obtain from (3.32) the $\dot{\mu}_k = 0$ isocline

$$\mu_k^{\bar{G}1a} = \frac{n_c\left(b_k - d_k k^{\bar{G}1a}\right)}{\delta + \alpha_k}. \tag{3.35}$$

This isocline is plotted in Figure 3.3 along with the $\dot{k} = 0$ isocline that has already been employed in Figure 3.2. The $\dot{\mu}_k = 0$ and $\dot{k} = 0$ isoclines in Figure 3.3 partition the space into four regions, denoted by I, II, III and IV, respectively. The point of intersection of both isoclines, $E^{\bar{G}1a}$, is the unique interior steady state.

In region **I** the direction of motion is northwest; there exists only one path starting e.g. from *B* in this region that leads to the steady state $E^{\bar{G}1a}$. If a starting point is chosen above or below the point *B*, the system will never reach the steady state $E^{\bar{G}1a}$. In region **II** the arrows point northeast, implying that all trajectories starting in this region will fail to reach the steady-state point $E^{\bar{G}1a}$. The properties of time paths starting in region **III** are analogous to those starting in region **I**. The arrows point southeast and therefore there exists an optimal time path starting e. g. from *A* which leads to the steady state $E^{\bar{G}1a}$. If the starting position is above or below *A*, the economy will not reach the steady state. In region **IV** the arrows point southwest so that all trajectories starting here will not reach the steady state either.

Figure 3.3 The optimal time path of the shadow price of cultural capital in the parametric version of model $S\bar{G}1$ when cultural goods are scarce $\left(\lambda_g > 0\right)$



As shown in Figure 3.3, the adjustment path of the shadow price of cultural capital depends on the initial stock of cultural capital, k_0. If $k_0 < k^{\bar{G}1a}$ [$k_0 > k^{\bar{G}1a}$], such as k_{0a} [k_{0b}] in Figure 3.3, we start at some point *A* below the $\dot{\mu}_k = 0$ isocline but above $\mu_k^{\bar{G}1a}$ [at some point *B* above the $\dot{\mu}_k = 0$ isocline but below $\mu_k^{\bar{G}1a}$] and μ_k continuously decreases [increases] until it attains its steady-state value $\mu_k^{\bar{G}1a}$ for $k = k^{\bar{G}1a}$.

To prove that the time path of μ_k is as described above consider the case $k_0 = k_{0a}$ in Figure 3.3 and suppose the initial value μ_k^0 of μ_k is set such that $\mu_k^0 = \overline{k_{0a}A'} > \tilde{\mu}_k$. Since $\dot{\mu}_k > 0$ at the point A', μ_k would continue to increase with $\dot{k} > 0$ implying that μ_k could never converge to $\mu_k^{\bar{G}1a}$. Similarly, if $\mu_k^0 < \mu_k^{\bar{G}1a}$ is set initially, for example $\mu_k^0 = \overline{k_{0a}A''}$ in Figure 3.3, then $\dot{\mu}_k < 0$ and μ_k would continuously decline with $\dot{k} > 0$. Hence μ_k would not converge to $\mu_k^{\bar{G}1a}$. We conclude, therefore, that there is some $\mu_k^{0a} \in \left]\mu_k^{\bar{G}1a}, \tilde{\mu}_k\right[$ such that at some point *A* the crucial trajectory is hit which "takes" the stock of cultural capital from k_{0a} to its steady-state value $k^{\bar{G}1a}$. Analogous arguments hold for the case $k_0 > k^{\bar{G}1a}$.

b) Case $g_s < \bar{g}$ and $\lambda_g = 0$

Consider now the case that cultural-goods inputs are abundant, i.e. that $g_s < \bar{g}$ and hence $\lambda_g = 0$. From (3.26) and (3.28) - (3.30) we infer

$$\lambda_{sg} = n_c \lambda_\sigma = \lambda_{sr} = \frac{a_y \lambda_y}{a_s} = \frac{a_y}{a_s}.$$

Combining this information with (3.27) we obtain

$$\mu_k = \frac{a_y}{a_s n_c} - b_s + d_s s_c, \qquad \text{or, equivalently,} \qquad s_c = \frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c} + \frac{\mu_k}{d_s}. \tag{3.36}$$

We differentiate (3.36) with respect to time:

$$\dot{\mu}_k = d_s \dot{s}_c. \tag{3.36'}$$

The next step is to insert (3.31), (3.36) and (3.36') into (3.32) to get

$$d_s \dot{s}_c = (\delta + \alpha_k)\left(\frac{a_y}{a_s n_c} - b_s + d_s s_c\right) - n_c b_k + n_c d_k k_c, \tag{3.37}$$

The equation (3.37) yields, after some rearrangement of terms,

$$\dot{s}_c = -M_1 + M_2 n_c s_c + M_3 k_c , \tag{3.38}$$

where $M_1 := \frac{(\delta+\alpha_k)(a_s b_s n_c - a_y) + a_s b_k n_c^2}{a_s d_s n_c}$, $M_2 := \frac{(\delta+\alpha_k)}{n_c}$ and $M_3 := \frac{d_k n_c}{d_s}$.

The terms M_2 and M_3 are positive. The rather formidably looking term M_1 can be shown to be positive, if and only if

$$n_c > n_{c0} := \frac{-a_s b_s(\delta+\alpha_k) + \sqrt{(a_s b_s)^2(\delta+\alpha_k)^2 + 4 a_s a_y b_k (\delta+\alpha_k)}}{2 a_s b_k}.$$

This inequality which we assume to hold in the following does not seem to be severely restrictive, since our focus is on economies with a large number of consumer-artists. Since $n_c < 1$ doesn't make economic sense, we restrict our analysis to

$$n_c \geq max[1, n_{c0}]. \tag{3.39}$$

For $\dot{s}_c = 0$ equation (3.38) yields

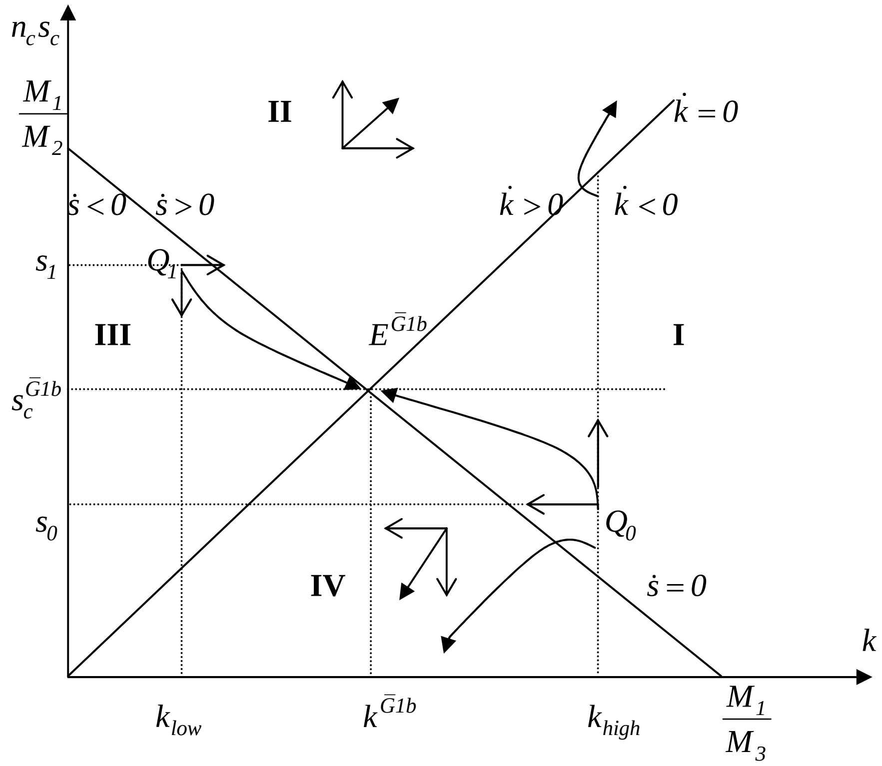
$$n_c s_c = \frac{M_1}{M_2} - \frac{M_3}{M_2} k_c .$$

We now combine the isoclines associated to the differential equations (3.2) and (3.38) to obtain Figure 3.4. A steady state of the parametric version of the economy S$\bar{\text{G}}$1 is defined by $\dot{s}_c = \dot{k} = 0$ and hence by

$$\left.\begin{aligned} &-M_1 + M_2 n_c s_c + M_3 k_c = 0, \\ &n_c s_c - \alpha_k k = 0. \end{aligned}\right\} \tag{3.40}$$

In view of (3.40) the $\dot{s}_c = 0$ and $\dot{k} = 0$ isoclines in Figure 3.4 partition the space into four regions, denoted by I, II, III and IV. The point of intersection of both isoclines, $E^{\bar{G}1b}$, is the unique interior steady state.

Figure 3.4 Phase diagram for the parametric version of model $S\bar{G}1$ when cultural goods are abundant ($\lambda_g = 0$)



In region **I** the direction of motion is northwest; there exists only one path passing e.g. through Q_0 that leads to the steady state $E^{\bar{G}1b}$. If the system starts below the point Q_0, cultural services are eventually driven down to zero; if it starts above Q_0, the consumption of cultural services will increase but cultural capital will always stay above its steady-state value $k^{\bar{G}1b}$; the system will never reach the steady state. In region **II** the time path of the variables (s_c, k) moves northeast. No trajectory starting from this region will ever reach the steady state. Time paths starting in region **III** [region **IV**] are characterized in an analogous way as time paths starting in region **I** [region **II**].

Two situations can be straightforwardly distinguished in Figure 3.4:

First, if the initial stock of cultural capital of a society is lower than its steady-state level, like e.g. k_{low} in Figure 3.4, then the optimal trajectory towards the steady state $E^{\bar{G}1b}$ is characterized by:

$$\dot{s}_t < 0, \dot{y}_t > 0, \dot{k} > 0 \text{ and } \dot{\mu}_k < 0 \text{ for all } t \geq 0 .$$

Starting at k_{low}, cultural capital will be accumulated through the consumption of cultural services. With the relatively low initial stock of cultural capital k_{low}, the socially optimal policy is to set the initial level of cultural services at $s_1 > 0$, well above its steady-state level $s^{\bar{G}1b}$. Starting at point Q_1 whose coordinates are (k_{low}, s_1) the economy is safely driven to the steady state $E^{\bar{G}1b}$ along the optimal trajectory. Before the steady state $E^{\bar{G}1b}$ is reached, more cultural services are provided than in the steady state. Moreover, the total derivative of equation (3.5) with respect to time yields $\dot{r}_s = -\dot{r}_y$, implying $\dot{s}_c = -\dot{y}_c$. Hence in case of a low initial stock of cultural capital all individuals pay for raising that stock by reducing their consumption of the private consumer good.

Conversely, if the initial stock of cultural capital is higher than its steady-state level, e.g. k_{high} in Figure 3.4, then the optimal trajectory towards the steady state is characterized by:

$$\dot{s}_t > 0, \dot{y}_t < 0, \dot{k} < 0 \text{ and } \dot{\mu}_k > 0 \text{ for all } t \geq 0 .$$

Since the stock of cultural capital is already high, the socially optimal policy is to start at a low level of cultural services. More specifically, if k_{high} in Figure 3.4 is the initial endowment of cultural capital, the social planner needs to set the initial value $s_0 < s^{\bar{G}1b}$ of cultural services. Along the trajectory passing through Q_0 in Figure 3.4 the steady state $E^{\bar{G}1b}$ is eventually reached. From the equation of motion $\dot{s}_c = -\dot{y}_c > 0$ we infer that the provision of cultural services rises over time at the opportunity cost of reduced private-good consumption.

For later reference, we calculate the value of cultural capital in the optimal steady state by solving $n_c s_c = \alpha_k k$ (obtained from setting $\dot{k} = 0$ in (3.2)) and (3.39):

$$k^{\bar{G}1b} := K^{\bar{G}1b}(n_c) = \frac{M_1}{\alpha_k M_2 + M_3} = \frac{a_s b_k n_c^2 + a_s b_s (\delta + \alpha_k) n_c - a_y (\delta + \alpha_k)}{a_s d_k n_c^2 + a_s d_s \alpha_k (\delta + \alpha_k)}, \qquad (3.41)$$

$$n_c s_c^{\bar{G}1b} = \frac{\alpha_k M_1}{\alpha_k M_2 + M_3} = \alpha_k \left[\frac{a_s b_k n_c^2 + a_s b_s (\delta + \alpha_k) n_c - a_y (\delta + \alpha_k)}{a_s d_k n_c^2 + a_s d_s \alpha_k (\delta + \alpha_k)} \right]. \qquad (3.42)$$

Invoking (3.8) and (3.21) one gets $s_c^{\bar{G}1b} = n_s s_s^{\bar{G}1b} = n_s g_s^{\bar{G}1b}$ such that

$$g_s^{\bar{G}1b} = \frac{\alpha_k M_1}{\left(\alpha_k M_2 + M_3\right) n_c n_s} = \frac{\alpha_k \left[a_s b_k n_c^2 + a_s b_s \left(\delta + \alpha_k\right) n_c - a_y \left(\delta + \alpha_k\right)\right]}{a_s d_k n_c^3 n_s + a_s d_s \alpha_k \left(\delta + \alpha_k\right) n_c n_s} . \tag{3.43}$$

$k^{\bar{G}1b}$, $s_c^{\bar{G}1b}$ and $g_s^{\bar{G}1b}$ are positive due to (3.39). Recall that $g_s^{\bar{G}1b}$ from (3.43) is the steady state of the solution to (3.25) in case of $\lambda_g = 0$ and $\bar{g} \geq g_s^{\bar{G}1b}$. Neglecting the limiting case of the constraint $\bar{g} \geq g_s^{\bar{G}1b}$ being weakly binding the steady-state solution (3.43) implicitly presupposes that all parameters are such that

$$\bar{g} > \frac{\alpha_k M_1}{\left(\alpha_k M_2 + M_3\right) n_c n_s} .$$

From this observation follows that if the parameters belong to the subset of parameters satisfying $\alpha_k M_1 / \left[\left(\alpha_k M_2 + M_3\right) n_c n_s\right] < \bar{g}$ then the optimal solution to (3.25) exhibits $\lambda_g > 0$.

The interaction of parameters in the terms on the RHS of the equations (3.41), (3.42) and (3.43) is quite complex. One can show that $k^{\bar{G}1b}$ as well as $n_c s_c^{\bar{G}1b}$ and $g_s^{\bar{G}1b}$ are strictly increasing in a_s, b_k and b_s and strictly decreasing in a_y, d_k and d_s. However, the signs of the first derivative of $k^{\bar{G}1b}$ with respect to α_k, δ and n_c are ambiguous[33].

As far as the first derivates of $k^{\bar{G}1b}$, $n_c s_c^{\bar{G}1b}$ and $g_s^{\bar{G}1b}$ with respect to the parameters are unambiguous in sign they allow for interesting interpretations of what determines the size of the steady-state values and which political implication. These interpretations are straightforward and will therefore be left to the reader. We will proceed instead, by focusing on the relationship between $k^{\bar{G}1b}$ and n_c which has been shown to be indeterminate in sign. This ambiguity

[33] The derivative of $k^{\bar{G}1b}$ with respect to α_k and δ yields

$$\frac{dk^{\bar{G}1b}}{d\alpha_k} = \frac{-a_y + a_s b_s n_c}{a_s \left[d_k n_c^2 + d_s \alpha_k \left(\delta + \alpha_k\right)\right]} - \frac{\left[d_s \alpha_k + d_s \left(\delta + \alpha_k\right)\right]\left[a_s b_k n_c^2 + a_s b_s \left(\delta + \alpha_k\right) n_c - a_y \left(\delta + \alpha_k\right)\right]}{a_s \left[d_k n_c^2 + d_s \alpha_k \left(\delta + \alpha_k\right)\right]^2} \overset{?}{\gtreqless} 0$$

and $$\frac{dk^{\bar{G}1b}}{d\delta} = \frac{a_s b_s d_k n_c^3 - \left(a_s b_k d_s \alpha_k + a_y d_k\right) n_c^2}{a_s \left[d_k n_c^2 + d_s \alpha_k \left(\delta + \alpha_k\right)\right]^2} \overset{?}{\gtreqless} 0 .$$

is particularly intriguing because the plausible conjecture is clearly that $k^{\bar{G}1b}$ is strictly increasing in n_c. After all, cultural capital is a public good and due to the Samuelsonian summation condition for its optimal allocation the marginal provision cost should be equal to the consumer-artists' aggregate willingness-to-pay for cultural capital. Since consumer-artists are assumed to be identical, their *aggregate* marginal willingness-to-pay can be expected to be strictly increasing in the numbers of consumer-artists, n_c. Hence good economic intuition suggests that $dk^{\bar{G}1b}/dn_c$ should be positive.

However this conjecture is easily shown to be wrong by observing that

$$\frac{dk^{\bar{G}1b}}{dn_c} = \frac{(\delta+\alpha_k)\left\{2a_y d_k n_c + a_s\left[2b_k d_s \alpha_k n_c + b_s\left(-d_k n_c^2 + d_s\alpha_k + d_s\alpha_k\delta\right)\right]\right\}}{a_s\left[d_k n_c^2 + d_s\alpha_k(\delta+\alpha_k)\right]^2}, \tag{3.44}$$

and therefore

$$\frac{dk^{\bar{G}1b}}{dn_c} \gtreqless 0 \;\Leftrightarrow\; n_c \lesseqgtr n_M := \frac{a_y d_k + a_s b_k d_s \alpha_k + \sqrt{\left(a_y d_k + a_s b_k d_s \alpha_k\right)^2 + a_s^2 b_s^2 d_k d_s(\delta+\alpha_k)}}{a_s b_s d_k}.$$

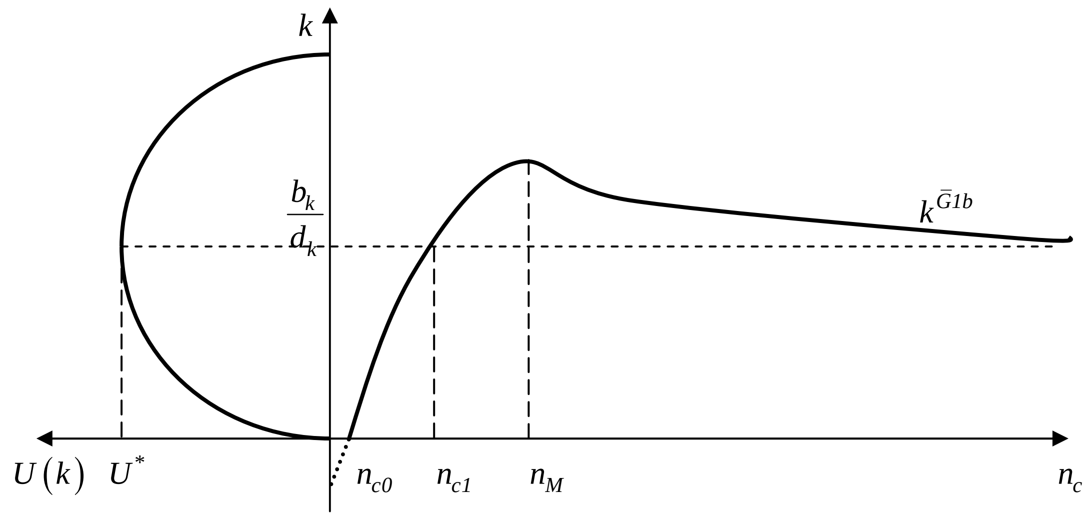
Moreover, it is also straightforward that

$$\lim_{n_c\to\infty} k^{\bar{G}1b} = \frac{b_k}{d_k}.$$

The reason why our intuition failed is simply that our preceding argument has been based on the implicit assumption that the marginal utility of cultural capital is positive for all positive stocks of cultural capital, whereas the replacement of the function *U* from (3.1) by the parametric utility function (3.23) implies that $U_k \gtreqless 0$ if and only if $k \lesseqgtr (b_k/d_k)$. Increasing *k* beyond (b_k/d_k) would be utility reducing. One is tempted to conclude from this observation that optimality requires *k* not to exceed (b_k/d_k).

Note, however, that $dk^{\bar{G}1b}/dn_c < 0$ for all $n_c > n_M$ and $\lim_{n_c\to\infty} k^{\bar{G}1b} = b_k/d_k$ imply that $k^{\bar{G}1b} > b_k/d_k$ for all $n_c > n_M$. Hence it is optimal to raise *k* beyond the value b_k/d_k that maximizes instantaneous utility. In other words, *k* turns out to overshoot (b_k/d_k) for sufficiently high values of n_c as illustrated in Figure 3.5. This surprising result calls for an explanation.

Figure 3.5 Cultural capital $k^{\bar{G}1b}$, number of consumer-artists n_c and marginal utility of cultural capital[34]



To better understand why the optimal stock of cultural capital, $k^{\bar{G}1b}$, depends on n_c as drawn in Figure 3.5, we set $\dot{\mu}_k = 0$ in (3.32) and rearrange (3.26) - (3.32) to obtain

$$\left[n_c \left(b_s - d_s s_c \right) + n_c \frac{n_c \left(b_k - d_k k_c \right)}{\delta + \alpha_k} \right] = \frac{a_y}{a_s}. \tag{3.45}$$

In (3.45) the term $n_c \left(b_s - d_s s_c \right)$ is the consumer-artists' instantaneous aggregate marginal willingness-to-pay for cultural services, and the term $n_c \left(b_k - d_k k_c \right) / \left(\delta + \alpha_k \right)$ is their instantaneous aggregate marginal willingness-to-pay for cultural capital. The RHS of (3.45) represents the marginal rate of transforming the consumer good into cultural services. If we substitute $s_c = \alpha_k k_c / n_c$ in (3.45) the resultant equation uniquely determines the optimal steady-state value of *k*. Our discussion above has shown that, except for small n_c $\left(n_c < n_{c1} \right)$, solving (3.45) for *k* (after consideration of $s_c = \alpha_k k_c / n_c$) implies

[34] Figure 3.5 is a free-hand graph, and some subsequent figures are also free-hand graphed.

$$\frac{U_k}{U_y} = b_k - d_k k_c < 0 \quad \text{and} \quad \frac{U_s}{U_y} = b_s - d_s s_c = b_s - \frac{\alpha_k d_s k_c}{n_c} > 0 . \tag{3.46}$$

Taking (3.46) into account, (3.45) can be rewritten in terms of (3.20):

$$\left[\underbrace{n_c \frac{U_s}{U_y}}_{+} + \underbrace{n_c \frac{n_c U_k}{(\delta + \alpha_k) U_y}}_{-} \right] = \underbrace{\frac{Y_r}{S_s}}_{\text{const.}} . \tag{3.47}$$

Since the marginal willingness-to-pay for cultural services is greater than that for cultural capital, it is optimal to increase s_c to a point where the marginal willingness-to-pay for cultural capital has turned negative.

In the following subsection we further explore the impact of n_c on $k^{\bar{G}1b}$ and $s_c^{\bar{G}1b}$ by means of numerical examples.

3.1.1.4 Numerical examples for the dependence of $k^{\bar{G}1b}$ and $s_c^{\bar{G}1b}$ on n_c

Our subsequent numerical calculations are based on the parameters values

$$a_s = 2,\ b_k = 3,\ d_k = 2,\ \alpha_k = 0{,}5,\ a_y = 1,\ b_s = 3,\ d_s = 2 \text{ and } \delta = 0{,}5 .$$

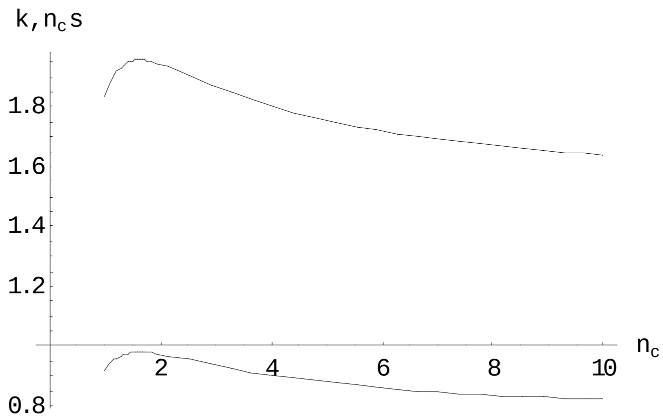
These numerical specifications (and those in later numerical calculations) are not based on realistic empirical data but are chosen for convenience of exposition only. Table 3.1 lists the results.

Table 3.1 Dependence of $k^{\bar{G}1b}$ and $s_c^{\bar{G}1b}$ on n_c

n_c	*10*	*100*	10^6
$k^{\bar{G}1b}$	*1.64*	*1.51*	*1.50*
$n_c\ s_c^{\bar{G}1b}$	*0.82*	*0.76*	*0.75*

In Table 3.1 $k^{\bar{G}1b}$ and $n_c s_c^{\bar{G}1b}$ from (3.41) and (3.42), respectively, are calculated[35] for three different values of n_c. Confirming our preceding conclusion $k^{\bar{G}1b}$ converges to $k = b_k / d_k = 1.5$ for very large numbers of consumer-artists. As already discussed in the previous subsection, it is surprising that the value $k = b_k / d_k$ yielding maximum instantaneous utility is exceeded for $n_c = 10$ and $n_c = 100$. Further insights are provided by Figure 3.6 that plots the steady-state values of $k^{\bar{G}1b}$ and $n_c s_c^{\bar{G}1b}$ for all $n_c \in [1, 10]$. Confirming the graph in Figure 3.5 both steady-state values are first increasing in n_c, attain their unique maximum at about $n_c = 1.72$ and are then monotone decreasing in n_c tending toward $k^{\bar{G}1b} = 1.5$ and $n_c s_c^{\bar{G}1b} = 0.75$, respectively.

Figure 3.6 Numerical example for the dependence of $k^{\bar{G}1b}$ and $s_c^{\bar{G}1b}$ on n_c



3.1.2 The simplified private-goods model with constant stock of cultural goods ($S\bar{G}2$)

With cultural-goods inputs and cultural services being private, the resource constraints (3.7) and (3.8) of the model $S\bar{G}1$ are replaced by

[35] These numerical calculations and all following ones are calculated with the computer program *Mathematica.*

$$\bar{g} \geq n_s g_s \,, \tag{3.48}$$

$$n_s s_s \geq n_c s_c \,. \tag{3.49}$$

3.1.2.1 The optimal intertemporal allocation

The social planner aims at maximizing the welfare function

$$n_c \int U\left(k_c, s_c, y_c\right) e^{-\delta t} dt \,, \qquad \text{subject to (2.6), (3.2) - (3.6) and (3.48) - (3.49).} \tag{3.50}$$

The Hamiltonian (3.10) is now modified to read:

$$H = n_c U\left(k_c, s_c, y_c\right) + \mu_k\left(n_c s_c - \alpha_k k\right) + n_s \lambda_s\left[S\left(r_s, g_s\right) - s_s\right] + \lambda_y\left[Y\left(r_y\right) - y_s\right]$$
$$+ \lambda_\sigma\left(n_s s_s - n_c s_c\right) + \lambda_c\left(y_s - n_c y_c\right) + \lambda_r\left(n_c \bar{r} - n_s r_s - r_y\right) + \lambda_g\left(\bar{g} - n_s g_s\right)$$
$$+ n_c \lambda_k\left(k - k_c\right). \tag{3.51}$$

The FOCs (3.11) - (3.19) carry over, except that the equation (3.16) is replaced by

$$\frac{\partial H}{\partial s_s} = -n_s \lambda_s + n_s \lambda_\sigma = 0 \,. \tag{3.52}$$

As a consequence, (3.20) is turned into

$$\frac{\dot{\mu}_k}{\left(\delta + \alpha_k\right) U_y Y_r} = -\left\{\frac{n_c U_k}{\left(\delta + \alpha_k\right) U_y Y_r} - \left(\frac{1}{S_r} - \frac{U_s}{U_y Y_r}\right)\right\}. \tag{3.53}$$

[*14*] = -{[*15*] - ([*16*] - [*17a*])}

Our interpretation of (3.20) also applies to (3.53). Comparing (3.20) and (3.53) in the respective steady states ($\dot{\mu}_k = 0$) yields

i) for the public-goods model $\mathrm{S\bar{G}1}$:

$$n_c\left(\frac{n_c U_k}{(\delta+\alpha_k)U_y Y_r}+\frac{U_s}{U_y Y_r}\right)=\frac{1}{S_r},$$

ii) for the private-goods model $\mathrm{S\bar{G}2}$:

$$\left(\frac{n_c U_k}{(\delta+\alpha_k)U_y Y_r}+\frac{U_s}{U_y Y_r}\right)=\frac{1}{S_r}.$$

Invoking plausibility arguments similar to those used in section 2.3, we infer that the supply of cultural services tends to be greater in $\mathrm{S\bar{G}1}$ than in $\mathrm{S\bar{G}2}$.

3.1.2.2 The optimal time path in a parametric version of model $\mathrm{S\bar{G}2}$

To obtain more specific results consider now the parametric functional forms (3.21), (3.22) and (3.23). The associated current-value Hamiltonian turns out to be

$$\begin{aligned}H = {} & n_c\left(b_k k_c-\frac{d_k}{2}k_c^{\,2}+b_s s_c-\frac{d_s}{2}s_c^{\,2}+y_c\right)+\mu_k\left(n_c s_c-\alpha_k k\right)+n_s\lambda_{sr}\left(a_s r_s-g_s\right)\\ & +n_s\lambda_{sg}\left(g_s-s_s\right)+\lambda_y\left(a_y r_y-y_s\right)+\lambda_\sigma\left(n_s s_s-n_c s_c\right)+\lambda_c\left(y_s-n_c y_c\right)\\ & +\lambda_r\left(n_c\bar{r}-n_s r_s-r_y\right)+\lambda_g\left(\bar{g}-n_s g_s\right)+n_c\lambda_k\left(k-k_c\right).\end{aligned} \tag{3.54}$$

The FOCs (3.26) - (3.32) remain unchanged, except that the equation (3.30) is replaced by:

$$\lambda_{sg}=\lambda_\sigma. \tag{3.55}$$

As in case of the parametric version of the public-goods economy $\mathrm{S\bar{G}1}$, we have to distinguish solutions with $\lambda_g>0$ and $\lambda_g=0$.

a) Case $g_s=\bar{g}$ and $\lambda_g>0$

Suppose first that cultural-goods inputs are scarce, $\lambda_g > 0$. Using the procedure applied above, we consider (3.2), (3.21), (3.48) and (3.49) and find that:

$$\bar{g} = n_s g_s^{\bar{G}2a} = n_s s_s^{\bar{G}2a} = n_c s_c^{\bar{G}2a} = \alpha_k k^{\bar{G}2a}, \tag{3.56}$$

implying that $k^{\bar{G}2a} = \bar{g} / \alpha_k$ is now the value of cultural capital in the optimal steady state. In addition, the pertaining shadow price of cultural capital can be determined by setting $\dot{\mu}_k = 0$ in (3.32):

$$\mu_k^{\bar{G}2a} = \frac{n_c \left(b_k - d_k k^{\bar{G}2a} \right)}{\delta + \alpha_k} = \frac{n_c b_k}{\delta + \alpha_k} - \frac{d_k \bar{g}}{\alpha_k \left(\delta + \alpha_k \right)}. \tag{3.57}$$

b) Case $g_s < \bar{g}$ and $\lambda_g = 0$

Consider now the parametric private-goods model S$\bar{\text{G}}$2 in which the solution exhibits $\lambda_g = 0$. Repeating the calculations carried out above with the appropriate modifications we find that (3.35) and (3.38) are now replaced by, respectively,

$$\mu_k = \frac{a_y}{a_s} - b_s + d_s s_c, \qquad \text{or, equivalently,} \qquad s_c = \frac{b_s}{d_s} - \frac{a_y}{a_s d_s} + \frac{\mu_k}{d_s}, \tag{3.58}$$

$$\dot{s}_c = -M_4 + M_2 n_c s_c + M_3 k_c, \tag{3.59}$$

where $M_4 := \dfrac{\left(\delta + \alpha_k\right)\left(a_s b_s - a_y\right) + a_s b_k n_c}{a_s d_s}$. The sign of M_4 is positive, if and only if[36]

$$n_c > n_{c0} := \frac{\left(a_y - a_s b_s\right)\left(\delta + \alpha_k\right)}{a_s b_k}.$$

[36] The value of n_{c0} in (3.39) is apparently different from n_{c0} as defined here. However, with a slight abuse of notation we use the same symbol n_{c0} throughout the subsequent analysis to avoid the clutter.

Since we are interested in economies with sufficiently large numbers of consumer-artists, n_c, we assume that this inequality holds. Moreover, since the condition $n_c \geq 1$ must also be satisfied, we impose the restriction

$$n_c \geq max\left[1, n_{c0}\right].$$

The optimal dynamics driven by (3.2) and (3.59) can be characterized by means of a phase diagram that is the same, in qualitative terms, as that shown in Figure 3.4. It suffices, therefore, to calculate the pertaining steady-state values

$$k^{\bar{G}2b} := K^{\bar{G}2b}(n_c) = \frac{M_4}{\alpha_k M_2 + M_3} = \frac{a_s b_k n_c^2 + (a_s b_s - a_y)(\delta + \alpha_k) n_c}{a_s d_k n_c^2 + a_s d_s \alpha_k (\delta + \alpha_k)}, \tag{3.60}$$

$$n_c s_c^{\bar{G}2b} = \frac{\alpha_k M_4}{\alpha_k M_2 + M_3} = \alpha_k \left[\frac{a_s b_k n_c^2 + (a_s b_s - a_y)(\delta + \alpha_k) n_c}{a_s d_k n_c^2 + a_s d_s \alpha_k (\delta + \alpha_k)}\right]. \tag{3.61}$$

From $n_c s_c^{\bar{G}2b} = n_s s_s^{\bar{G}2b} = n_s g_s^{\bar{G}2b}$, (3.41) and (3.42) follows

$$g_s^{\bar{G}2b} = \frac{\alpha_k M_4}{(\alpha_k M_2 + M_3) n_c n_s} = \frac{\alpha_k \left[a_s b_k n_c^2 + (a_s b_s - a_y)(\delta + \alpha_k) n_c\right]}{a_s d_k n_c^3 n_s + a_s d_s \alpha_k (\delta + \alpha_k) n_c n_s}. \tag{3.62}$$

$k^{\bar{G}2b}, s_c^{\bar{G}2b}$ and $g_s^{\bar{G}2b}$ are positive, since $n_c \geq max\left[1, n_{c0}\right]$. Observe that the only difference between (3.38) and (3.59) is that M_1 is substituted by M_4. $k^{\bar{G}2b}$ as well as $n_c s_c^{\bar{G}2b}$ are still strictly increasing in a_s, b_k and b_s, strictly decreasing in a_y, d_k and d_s, and the signs of the first derivatives of $k^{\bar{G}2b}$ with respect to α_k, δ and n_c are still ambiguous. Similar as in the previous model, we restrict our discussion to the relationship between $k^{\bar{G}2b}$ and n_c. The derivative of $k^{\bar{G}2b}$ with respect to n_c yields

$$\frac{dk^{\bar{G}2b}}{dn_c} = \frac{(a_y - a_s b_s) d_k (\delta + \alpha_k) n_c^2 + 2 a_s b_k d_s \alpha_k (\delta + \alpha_k) n_c - (a_y - a_s b_s) d_s \alpha_k (\delta + \alpha_k)^2}{a_s \left[d_k n_c^2 + d_s \alpha_k (\delta + \alpha_k)\right]^2}. \tag{3.63}$$

To determine the sign of $dk^{\bar{G}2b} / dn_c$, it is convenient to distinguish three cases depending on the sign of the term $\left(a_y - a_s b_s\right)$.

Case 1 $\left(a_y - a_s b_s\right) < 0$:

Under that condition (3.63) implies

$$\frac{dk^{\bar{G}2b}}{dn_c} \gtreqless 0 \;\Leftrightarrow\; n_c \lesseqgtr n_{M1} := \frac{-a_s b_k d_s \alpha_k - \sqrt{\left[a_s{}^2 b_k{}^2 d_s{}^2 \alpha_k{}^2 + \left(a_y - a_s b_s\right)^2 d_k d_s \alpha_k \left(\delta + \alpha_k\right)\right]}}{\left(a_y - a_s b_s\right) d_k}.$$

$k^{\bar{G}2b}$ is first strictly increasing in n_c, up to the threshold value $n_c = n_{M1}$, and is then strictly decreasing in n_c for all $n_c > n_{M1}$.

Case 2 $\left(a_y - a_s b_s\right) = 0$:

In this case (3.63) becomes

$$\frac{dk^{\bar{G}2b}}{dn_c} = \frac{2 b_k d_s \alpha_k \left(\delta + \alpha_k\right) n_c}{\left[d_k n_c{}^2 + d_s \alpha_k \left(\delta + \alpha_k\right)\right]^2} > 0 .$$

Case 3 $\left(a_y - a_s b_s\right) > 0$:

(3.63) now implies

$$\frac{dk^{\bar{G}2b}}{dn_c} \lesseqgtr 0 \;\Leftrightarrow\; n_c \lesseqgtr n_{M3} := \frac{-a_s b_k d_s \alpha_k + \sqrt{\left[a_s{}^2 b_k{}^2 d_s{}^2 \alpha_k{}^2 + \left(a_y - a_s b_s\right)^2 d_k d_s \alpha_k \left(\delta + \alpha_k\right)\right]}}{\left(a_y - a_s b_s\right) d_k}.$$

$k^{\bar{G}2b}$ is first strictly decreasing in n_c, up to the threshold value $n_c = n_{M3}$, and is then strictly increasing in n_c for all $n_c > n_{M3}$. To understand that curvature, note that from $k^{\bar{G}2b} = 0$ for $n_c = 0$ and $dk^{\bar{G}2b} / dn_c < 0$ for $n_c > n_{M3}$ we conclude that $k^{\bar{G}2b} < 0$ for all

$n_c \in]0, n_M]$. Consequently $n_M < n_{c0}$ since by definition of n_{c0} ($n_{c0} := (a_y - a_s b_s)(\delta + \alpha_k) / a_s b_k$) it is true that $k^{\bar{G}2b} > 0$ for all $n_c > n_{c0}$. Therefore the relevant domain of $dk^{\bar{G}2b} / dn_c > 0$ is $[max[1, n_{c0}], \infty[$.

Observe also that

$$\lim_{n_c \to \infty} k^{\bar{G}2b} = \frac{b_k}{d_k},$$

i.e. with very large numbers of consumer-artists, irrespective of which case is considered, the steady-state value of cultural capital $k^{\bar{G}2b}$ converges to the positive value b_k / d_k.

We illustrate those different cases in Figure 3.7. To explain the different shapes of the curves illustrated in Figure 3.7, we now apply the argument used in the previous subsection of setting $\dot{\mu}_k = 0$ in (3.32) and rearrange (3.26) - (3.29), (3.31) and (3.55) to get

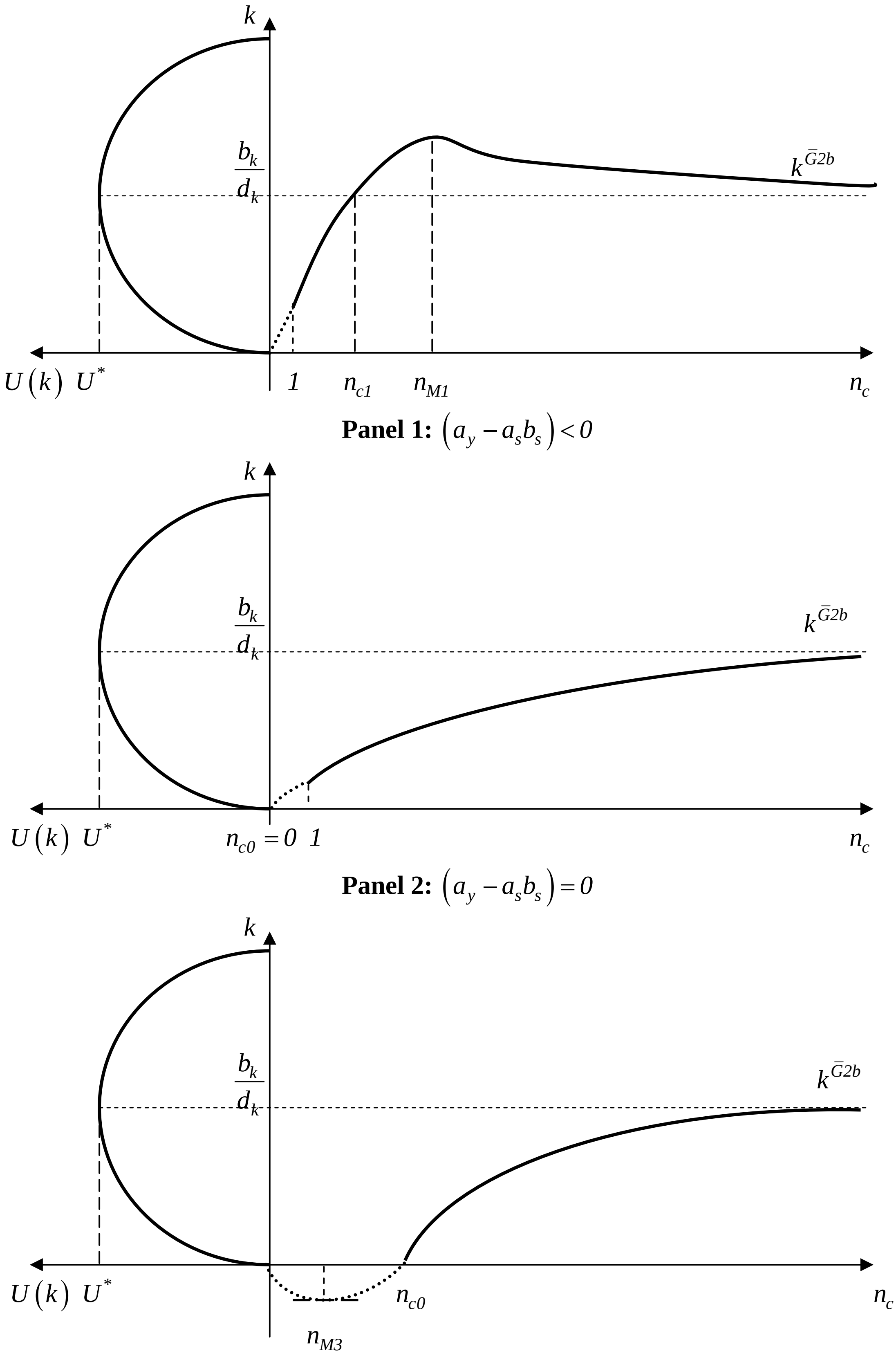
$$\left[(b_s - d_s s_c) + \frac{n_c (b_k - d_k k_c)}{\delta + \alpha_k}\right] = \frac{a_y}{a_s}. \tag{3.64}$$

On the LHS of (3.64) the term $(b_s - d_s s_c)$ is the consumer-artist's instantaneous marginal willingness-to-pay for private-goods cultural services, where b_s is the consumer-artist's maximum marginal willingness-to-pay for cultural services at $U_s|_{s=0}$, and the term $n_c (b_k - d_k k_c) / (\delta + \alpha_k)$ is the instantaneous aggregate marginal willingness-to-pay for cultural capital. The RHS of (3.64) represents the marginal rate of transforming the consumer good into cultural services. We now rearrange (3.64) to get:

$$a_y - a_s b_s = -a_s d_s s_c + \frac{a_s n_c (b_k - d_k k_c)}{\delta + \alpha_k}. \tag{3.65}$$

By considering $s_c = \alpha_k k_c / n_c$, (3.65) implies that in the case 2 and case 3 (defined by $(a_y - a_s b_s) = 0$ and $(a_y - a_s b_s) > 0$, respectively) $(b_k - d_k k_c) > 0$ or $k_c < b_k / d_k$ for all $n_c > 1$ which reconfirms our results derived above. If $(a_y - a_s b_s) < 0$ (case 1), $k_c > b_k / d_k$ is not ruled out by (3.65).

Figure 3.7 Different shape of the function $k^{\bar{G}2b} = K^{\bar{G}2b}(n_c)$ depending on different parameter-values sets a_s, a_y and b_s



Panel 3: $(a_y - a_s b_s) > 0$ **and** $n_{c0} > 1$

In fact, given the similarity between (3.64) and (3.45) the rationale of case 1 is the same as that of the steady-state value cultural capital in the parametric version of model $S\bar{G}1$ in (3.41).

Next we use some numerical simulations to illustrate the impact of n_c on $k^{\bar{G}2b}$ by taking the sign of the term $\left(a_s b_s - a_y\right)$ into account.

3.1.2.3 Numerical examples

We provide two numerical examples that are based on the parameter values

Example I: $a_s = 2,\ b_k = 3,\ d_k = 2,\ \alpha_k = 0{,}5,\ a_y = 1,\ b_s = 3,\ d_s = 2$ and $\delta = 0{,}5$.

Example II: $a_s = 2,\ b_k = 3,\ d_k = 2,\ \alpha_k = 0{,}5,\ a_y = 7,\ b_s = 3,\ d_s = 2$ and $\delta = 0{,}5$.

Example I corresponds to case 1, example II corresponds to case 3 in Figure 3.7. The following Table 3.2 lists the calculation results.

Table 3.2 Dependence of $k^{\bar{G}2b}$ and $s_c^{\bar{G}2b}$ on n_c

Example I. $\left(a_y - a_s b_s\right) < 0$

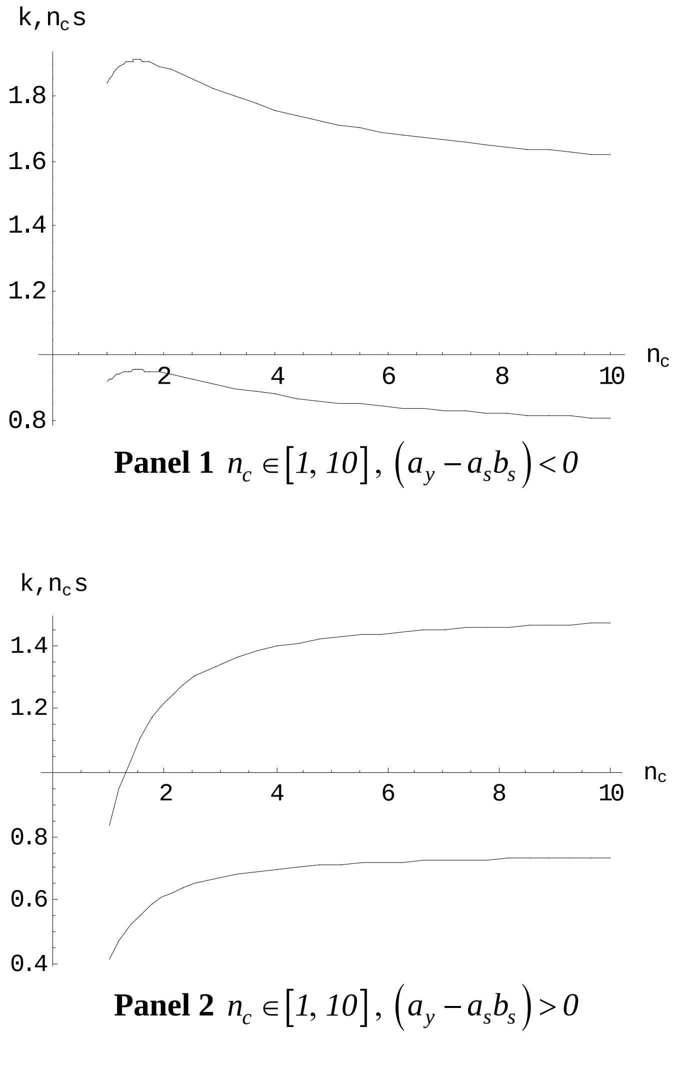
n_c	10	100	10^6
$k^{\bar{G}2b}$	1.62	1.51	1.50
$n_c s_c^{\bar{G}2b}$	0.81	0.755	0.75

Example II. $\left(a_y - a_s b_s\right) > 0$

n_c	10	100	10^6
$k^{\bar{G}2b}$	1.46	1.49	1.50
$n_c s_c^{\bar{G}2b}$	0.73	0.745	0.75

In Table 3.2 $k^{\bar{G}2b}$ and $n_c s_c^{\bar{G}2b}$ from (3.60) and (3.61), respectively, are calculated for three different values of n_c for parameters that satisfy either $a_y < a_s b_s$ or $a_y > a_s b_s$. In example I the cultural capital $k^{\bar{G}2b}$ exceeds the level of cultural capital that yields the maximum instantaneous utility for $n_c = 10$ and $n_c = 100$, and (almost) reaches its limit value $b_k / d_k = 1.50$ at $n_c = 10^6$. In contrast, in example II $k^{\bar{G}2b}$ increases when n_c is raised from *10* to *100* and 10^6 where it (almost) reaches its limit value. Panel 1 in Figure 3.8 contains the values of $k^{\bar{G}2b}$ and $n_c s_c^{\bar{G}2b}$ for all $n_c \in [1, 10]$ when $a_y < a_s b_s$. Both variables are first increasing in n_c, attain their unique maximum at about $n_c = 1.53$ and are then monotone decreasing in n_c tending toward $k^{\bar{G}2b} = 1.5$ and $n_c s_c^{\bar{G}2b} = 0.75$, respectively. Panel 2 shows for $a_y > a_s b_s$ that from $n_c = 1$ onward $k^{\bar{G}2b}$ and $n_c s_c^{\bar{G}2b}$ are monotone increasing in n_c and approach $k^{\bar{G}2b} = 1.5$ and $n_c s_c^{\bar{G}2b} = 0.75$, respectively, for very large numbers of consumer-artists.

Figure 3.8 Numerical examples for the dependence of $k^{\bar{G}2b}$ and $s_c^{\bar{G}2b}$ on n_c



Panel 1 $n_c \in [1, 10]$, $(a_y - a_s b_s) < 0$

Panel 2 $n_c \in [1, 10]$, $(a_y - a_s b_s) > 0$

Now we are in the position to compare the optimal trajectories and steady states of the models $S\bar{G}1$ and $S\bar{G}2$.

3.1.3 Comparing the optimal steady states of the models $S\bar{G}1$ and $S\bar{G}2$

We first consider the case of cultural-goods inputs being scarce.

a) Case $g_s = \bar{g}$ and $\lambda_g > 0$

In the optimal steady state, the difference between the values of cultural capital in the models $S\bar{G}1$ (cf. (3.34')) and $S\bar{G}2$ (cf. (3.56)) is easily calculated as

$$k^{\bar{G}1a} - k^{\bar{G}2a} = \left(n_c n_s - 1\right)\frac{\bar{g}}{\alpha_k}\begin{cases} > 0 & \text{for} \quad n_c n_s > 1, \\ = 0 & \text{for} \quad n_c n_s = 1. \end{cases} \tag{3.66}$$

The steady-state shadow price μ_k in (3.35) differs from that in (3.57) as follows:

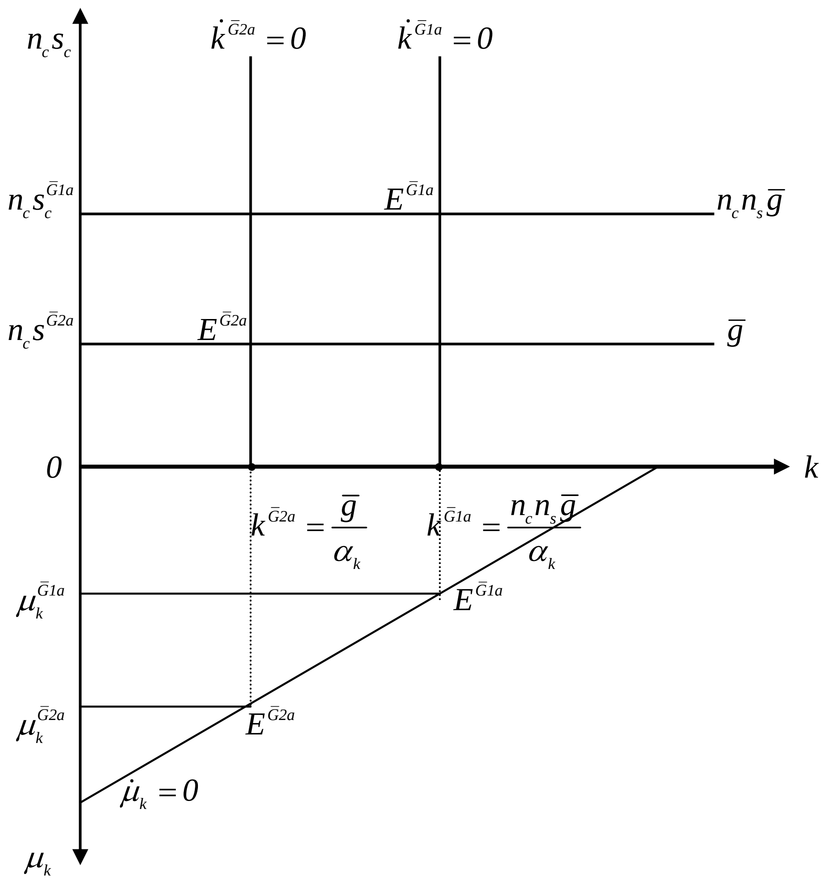
$$\mu_k^{\bar{G}1a} - \mu_k^{\bar{G}2a} = \frac{\left(1 - n_c n_s\right) d_k \bar{g}}{\alpha_k\left(\delta + \alpha_k\right)}\begin{cases} < 0 & \text{for} \quad n_c n_s > 1, \\ = 0 & \text{for} \quad n_c n_s = 1. \end{cases} \tag{3.67}$$

Commenting on these findings, we observe that in economies with only one consumer-artist and one cultural-services firm $\left(n_c = n_s = 1\right)$, both steady-state values coincide (and so do the values along the entire optimal time paths, too). This result is not surprising since the difference between public and private goods disappear for $n_c = n_s = 1$; if there is one agent only, jointly consumable goods are not jointly consumed.

Since $n_c = n_s = 1$ is an irrelevant polar case we conclude that (for $n_c n_s > 1$) in the public-goods model the optimal level of cultural capital is higher, and its shadow price lower, than in the private-goods model.

The results of the comparison of cultural capital and its shadow price for the case $g_s = \bar{g}$ and $\lambda_g > 0$ between models $S\bar{G}1$ and $S\bar{G}2$ are illustrated in Figure 3.9.

Figure 3.9 Comparing the optimal steady states in the parameterized models $S\bar{G}1$ **and** $S\bar{G}2$ **when the stock of cultural goods is scarce** $\left(\lambda_g > 0\right)$



b) Case $g_s < \bar{g}$ **and** $\lambda_g = 0$

In view of (3.41) - (3.43) and (3.60) - (3.62), the comparison between the optimal steady-state values of k, $n_c s_c$ and g_s in the parameterized models $S\bar{G}1$ and $S\bar{G}2$ with an abundant stock of cultural goods is straightforward:

$$k^{\bar{G}1b} - k^{\bar{G}2b} = \frac{M_1 - M_4}{\alpha_k M_2 + M_3} = \frac{\left(\delta + \alpha_k\right) a_y \left(n_c - 1\right)}{a_s d_k n_c^2 + a_s d_s \alpha_k \left(\delta + \alpha_k\right)} > 0, \tag{3.68}$$

$$n_c s_c^{\bar{G}1b} - n_c s_c^{\bar{G}2b} = \frac{\alpha_k \left(M_1 - M_4\right)}{\alpha_k M_2 + M_3} = \frac{\alpha_k \left(\delta + \alpha_k\right) a_y \left(n_c - 1\right)}{a_s d_k n_c^2 + a_s d_s \alpha_k \left(\delta + \alpha_k\right)} > 0, \tag{3.69}$$

and

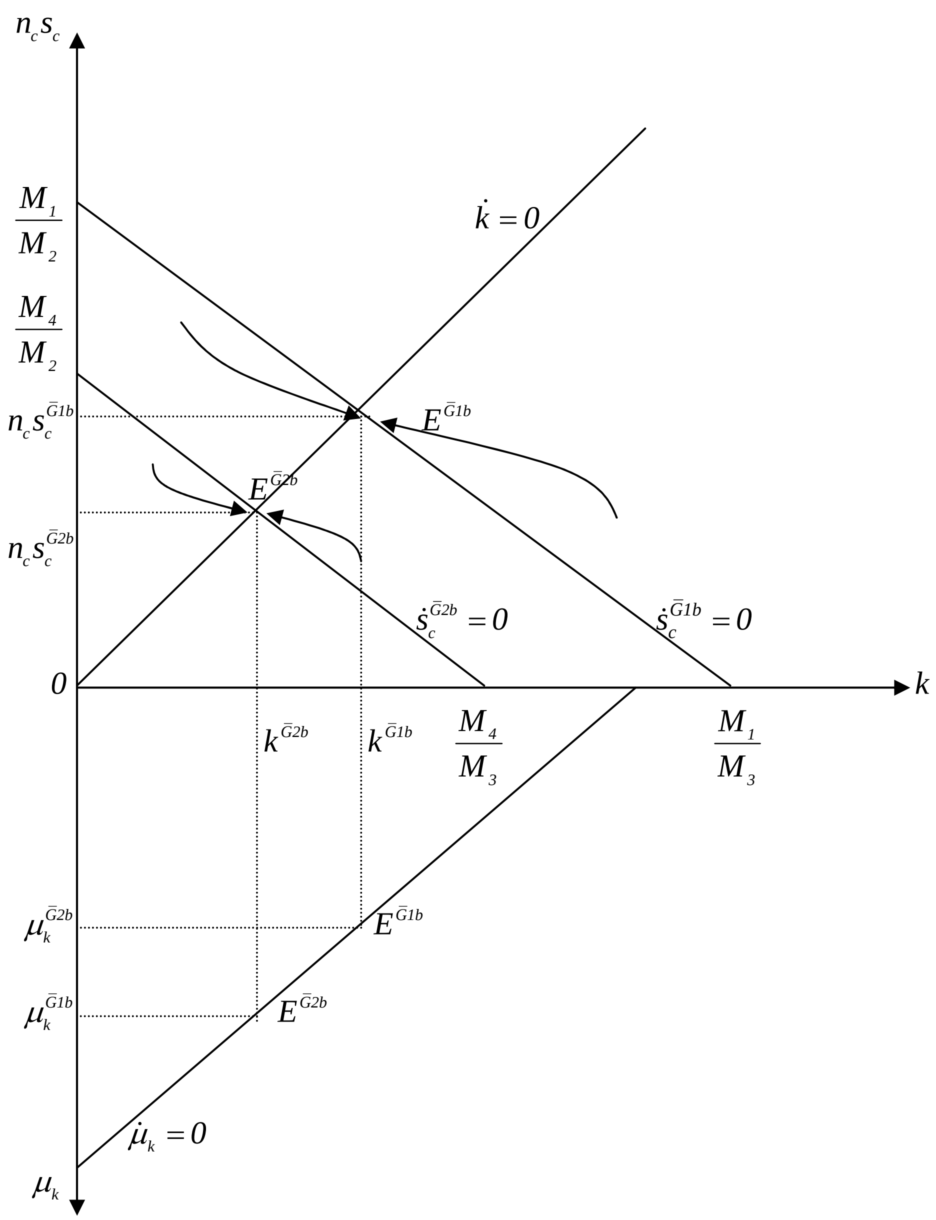
$$g_s^{\bar{G}1b} - g_s^{\bar{G}2b} = \frac{M_1 - M_4}{\left(\alpha_k M_2 + M_3\right) n_c n_s} = \frac{a_y \left(\delta + \alpha_k\right)\left(n_c - 1\right)}{a_s d_k n_c^{\,2} + a_s d_s \alpha_k \left(\delta + \alpha_k\right)} > 0 . \tag{3.70}$$

Observe first that in economies with one consumer-artist only $\left(n_c = 1\right)$, both steady states coincide. As noted before, in this case the jointly consumable goods cannot be jointly consumed, the difference between public and private goods hence vanishes. For the economies with more than one consumer-artist ($n_c > 1$), (3.68), (3.69) and (3.70) unambiguously yield $s_c^{\bar{G}1b} - s_c^{\bar{G}2b} > 0$, $k^{\bar{G}1b} - k^{\bar{G}2b} > 0$ and $g_s^{\bar{G}1b} - g_s^{\bar{G}2b} > 0$. We conclude that if cultural-goods input and cultural services are public, the optimal steady-state levels are higher than in case of cultural-goods input and cultural services being private goods.

The intuition of our comparison between models $\mathrm{S\bar{G}1}$ and $\mathrm{S\bar{G}2}$ suggests, that the differences (3.68), (3.69) and (3.70) might be increasing in the number of consumer-artists. Yet closer inspection of (3.68) - (3.70) reveals that these differences are not monotone increasing in n_c. Roughly speaking, since both $k^{\bar{G}1b}$ and $k^{\bar{G}2b}$ are not monotone increasing in n_c, the difference $k^{\bar{G}1b} - k^{\bar{G}2b}$ is not likely to be monotone increasing in n_c either. In the next subsection we will support that conjecture by offering some numerical examples.

For the case $g_s < \bar{g}$ and $\lambda_g = 0$, the models $\mathrm{S\bar{G}1}$ and $\mathrm{S\bar{G}2}$ are compared in Figure 3.10 that illustrates the differential equations (3.2), (3.38) and (3.59).

Figure 3.10 Comparing the optimal time paths in the parameterized models $S\bar{G}1$ and $S\bar{G}2$ when cultural-goods inputs are abundant



Numerical examples of the comparison between $S\bar{G}1$ and $S\bar{G}2$ ($g_s < \bar{g}$)

The comparison between the models $S\bar{G}1$ and $S\bar{G}2$ is now continued by presenting some examples with the numerical specification of parameters that have already been used in section 3.1.1.3 and 3.1.2.3:

Example I: $a_s = 2, b_k = 3, d_k = 2, \alpha_k = 0{,}5,\ a_y = 1, b_s = 3, d_s = 2$ and $\delta = 0{,}5$,

Example II: $a_s = 2, b_k = 3, d_k = 2, \alpha_k = 0{,}5,\ a_y = 7, b_s = 3, d_s = 2$ and $\delta = 0{,}5$.

When those parameter values are plugged into (3.68) and (3.69), we get for the case $\left(a_y - a_s b_s\right) < 0$:

$$k^{\bar{G}1b} - k^{\bar{G}2b} = \frac{n_c - 1}{4n_c^{\,2} + 2},\ n_c s_c^{\bar{G}1b} - n_c s_c^{\bar{G}2b} = \frac{n_c - 1}{8n_c^{\,2} + 4}\ ;$$

and for the case $\left(a_y - a_s b_s\right) > 0$:

$$k^{\bar{G}1b} - k^{\bar{G}2b} = \frac{5n_c - 7}{4n_c^{\,2} + 2},\ n_c s_c^{\bar{G}1b} - n_c s_c^{\bar{G}2b} = \frac{5n_c - 7}{8n_c^{\,2} + 4}.$$

The results are listed in Table 3.3.

Table 3.3 Dependence of $k^{\bar{G}1b} - k^{\bar{G}2b}$ and $s_c^{\bar{G}1b} - s_c^{\bar{G}2b}$ on n_c

Example I. $\left(a_y - a_s b_s\right) < 0$

n_c	10	100	10^6
$k^{\bar{G}1b} - k^{\bar{G}2b}$	0.022	0.002	0.00
$n_c s_c^{\bar{G}1b} - n_c s_c^{\bar{G}2b}$	0.011	0.001	0.00

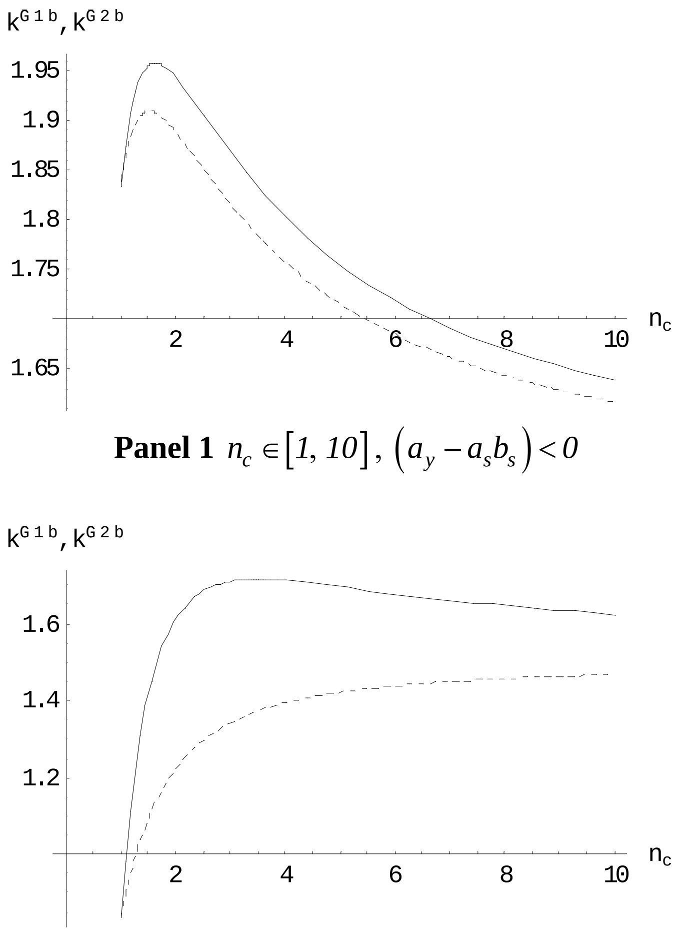
Example II. $\left(a_y - a_s b_s\right) > 0$

n_c	10	100	10^6
$k^{\bar{G}1b} - k^{\bar{G}2b}$	0.107	0.012	0.00
$n_c s_c^{\bar{G}1b} - n_c s_c^{\bar{G}2b}$	0.054	0.006	0.00

In Table 3.3 the differences $k^{\bar{G}1b} - k^{\bar{G}2b}$ and $n_c s_c^{\bar{G}1b} - n_c s_c^{\bar{G}2b}$ from (3.68) and (3.69), respectively, are calculated for three different values of n_c with parameters satisfying either $\left(a_y - a_s b_s\right) > 0$ or $\left(a_y - a_s b_s\right) < 0$. Table 3.3 suggests that the differences $k^{\bar{G}1b} - k^{\bar{G}2b}$, $s_c^{\bar{G}1b} - s_c^{\bar{G}2b}$ and $g_s^{\bar{G}1b} - g_s^{\bar{G}2b}$ are strictly declining in n_c and satisfy, moreover, $\lim_{n_c \to \infty} \left(s_c^{\bar{G}1b} - s_c^{\bar{G}2b}\right) = 0$, $\lim_{n_c \to \infty} \left(k^{\bar{G}1b} - k^{\bar{G}2b}\right) = 0$ and $\lim_{n_c \to \infty} \left(g_s^{\bar{G}1b} - g_s^{\bar{G}2b}\right) = 0$. We plot the curves from Figure 3.6 and Figure 3.8 together into Figure 3.11 that portrays the comparison. Panel 1 in Figure 3.11 shows the example I that contains the values of $k^{\bar{G}1b} - k^{\bar{G}2b}$ and $n_c s_c^{\bar{G}1b} - n_c s_c^{\bar{G}2b}$ for all $n_c \in [1, 10]$, Panel 2 represents example II.

Figure 3.11 Numerical examples of the comparison between $k^{\bar{G}1b}$ and $k^{\bar{G}2b}$

(The solid lines stand for $k^{\bar{G}1b}$, the dashed lines represent $k^{\bar{G}2b}$)



Panel 1 $n_c \in [1, 10]$, $\left(a_y - a_s b_s\right) < 0$

Panel 2 $n_c \in [1, 10]$, $\left(a_y - a_s b_s\right) > 0$

Hence, the preceding numerical examples reconfirm our conjecture from section 2.3 that the steady-state values of the cultural capital and cultural services tend to be greater in GM1 than in GM2. Recall that in the general analysis of section 2.3 the only way to give substance to this conjecture has been plausibility arguments. For the more restrictive parametric models $S\bar{G}1$ and $S\bar{G}2$ we have now proved this conjecture to be valid.

3.2 Accumulation of cultural goods when the stock of cultural capital has no impact

In the general model of section 2 we assumed that an individual's felicity is positively affected by the state variables cultural goods, g, and cultural capital, k, and by the control variables cultural services, s, newly created cultural goods, v, and consumer goods, y. In the previous section 3.1 we simplified the model by keeping the stock of cultural goods constant to obtain more informative results about the dynamics of the provision of cultural services and the formation of cultural capital. In the present section, we allow cultural goods to accumulate again, as in the general model, but this time we disregard the process of cultural-capital formation. In terms of the formal model, there are two different ways to exclude the formation of cultural capital from the analysis. One way is to set $\alpha_k \equiv 0$ and $s_j \equiv 0$ for all $j = 1,\ldots, n_s$ which implies $\dot{k} \equiv 0$ and hence $k_t = \bar{k}$ for all t. But this procedure would prevent individuals from enjoying cultural goods via consuming cultural services. We therefore take the other route of setting $U_k^i = V_k^i = 0$ for all $i = 1,\ldots, n_c$. In this case, the differential equation (2.5) is still in operation. But since nobody cares about the values cultural capital takes on, (2.5) becomes irrelevant for the formal model and can therefore be dropped altogether. To keep the notation simple we also drop the variable k as an argument of U^i and V^i. As in the previous section we further simplify the exposition by assuming that all consumer-artists are identical.

With this setup it is necessary again to treat separately the cases of cultural-goods inputs and cultural services being public or private for the consumer-artists. We denote by $S\bar{K}1$ and $S\bar{K}2$ the submodels where these goods are public and private, respectively, and start our analysis with $S\bar{K}1$.

3.2.1 A simplified public-goods model with zero impact of cultural capital ($\mathrm{S\overline{K}1}$)

In model $\mathrm{S\overline{K}1}$, the representative individual has the utility function

$$u = U\left(g_c, s_c, v_c, y_c\right), \tag{3.71}$$

where g_c is her demand for the stock of cultural goods, s_c is her consumption of cultural services, v_c are her newly created cultural goods and y_c is her consumption of consumer goods. Since all consumers are identical, equation (2.3) is now modified to read

$$\dot{g} = n_c v_c - \alpha_g g . \tag{3.72}$$

Since cultural-goods inputs are public, the pertinent supply constraints are:

$$g \geq g_s , \tag{3.73}$$

$$g \geq g_c . \tag{3.74}$$

Suppressing cultural capital as a variable in V^i from (2.2), the production of cultural goods is now given by:

$$v_c = V\left(r_v\right). \tag{3.75}$$

Cultural services are produced with the technology (3.3). The consumer goods are produced with the technology (2.6) and the associated supply constraint is the same as in (3.4). The model $\mathrm{S\overline{K}1}$ is completed by adding the resource constraint:

$$n_c \bar{r} \geq n_c r_v + n_s r_s + r_y . \tag{3.76}$$

3.2.1.1 The optimal intertemporal allocation

The social planner aims at maximizing the Utilitarian welfare function

$$n_c \int U\left(g_c, s_c, v_c, y_c\right) e^{-\delta t} dt,$$

subject to (2.6), (3.3), (3.4), (3.8) and (3.72) to (3.76). (3.77)

This optimization problem is solved by means of the current-value Hamiltonian:

$$H = n_c U\left(g_c, s_c, v_c, y_c\right) + \mu_g\left(n_c v_c - \alpha_g g\right) + n_c \lambda_v\left[V\left(r_v\right) - v_c\right] + \lambda_y\left[Y\left(r_y\right) - y_s\right]$$
$$+ \lambda_c\left(y_s - n_c y_c\right) + \lambda_r\left(n_c \bar{r} - n_c r_v - n_s r_s - r_y\right) + n_s \lambda_s\left[S\left(r_s, g_s\right) - s_s\right]$$
$$+ n_c \lambda_{gc}\left(g - g_c\right) + n_s \lambda_{gs}\left(g - g_s\right) + n_c \lambda_\sigma\left(n_s s_s - s_c\right). \tag{3.78}$$

In case of an interior solution the FOCs read:

$$\frac{\partial H}{\partial g_c} = n_c U_g - n_c \lambda_{gc} = 0, \tag{3.79}$$

$$\frac{\partial H}{\partial s_c} = n_c U_s - n_c \lambda_\sigma = 0, \tag{3.80}$$

$$\frac{\partial H}{\partial v_c} = n_c U_v + n_c \mu_g - n_c \lambda_v = 0, \tag{3.81}$$

$$\frac{\partial H}{\partial y_c} = n_c U_y - n_c \lambda_c = 0, \tag{3.82}$$

$$\frac{\partial H}{\partial s_s} = n_c n_s \lambda_\sigma - n_s \lambda_s = 0, \tag{3.83}$$

$$\frac{\partial H}{\partial g_s} = -n_s \lambda_{gs} + n_s \lambda_s S_g = 0, \tag{3.84}$$

$$\frac{\partial H}{\partial y_s} = -\lambda_y + \lambda_c = 0, \tag{3.85}$$

$$\frac{\partial H}{\partial r_s} = -n_s \lambda_r + n_s \lambda_s S_r = 0, \tag{3.86}$$

$$\frac{\partial H}{\partial r_v} = n_c \lambda_v V_r - n_c \lambda_r = 0, \tag{3.87}$$

$$\frac{\partial H}{\partial r_y} = \lambda_y Y_r - \lambda_r = 0\,, \tag{3.88}$$

$$\dot{\mu}_g = \left(\delta + \alpha_g\right)\mu_g - n_c \lambda_{gc} - n_s \lambda_{gs}\,. \tag{3.89}$$

We combine (3.79) and (3.82) - (3.88), rearrange terms and turn (3.81) into:

$$\frac{\mu_g}{U_y Y_r} = \frac{1}{V_r} - \frac{U_v}{U_y Y_r}\,. \tag{3.90}$$

The interpretation of (3.90) is analogous to that of (2.30): The LHS gives us the shadow price expressed in terms of the resource and the RHS presents the difference between the individual's marginal cost to produce new cultural goods and her marginal willingness-to-pay for new cultural goods created by herself.

Consider now the differential equation (3.89). We use (3.79) and (3.81) through (3.88) to transform (3.90) into:

$$\frac{\dot{\mu}_g}{\left(\delta + \alpha_g\right) U_y Y_r} = -\left\{ \left(\frac{n_c \dfrac{U_g}{U_y Y_r}}{\delta + \alpha_g} + \frac{n_s \dfrac{S_g}{S_r}}{\delta + \alpha_g} \right) - \left(\frac{1}{V_r} - \frac{U_v}{U_y Y_r} \right) \right\}. \tag{3.91}$$

[*18*] = -{([*19*] + [*20*]) - ([*21*] - [22])}

According to (3.91) the change in the shadow price of cultural goods in terms of the resource, [*18*], must equal the difference between the aggregate marginal benefit of cultural goods ([*19*] + [*20*]), and the marginal production cost of cultural goods, ([*21*] - [*22*]). (3.91) is easily identified as a special case of (2.34) that determines the motion in time of the costate variable μ_g in the general model with heterogeneous consumer-artists, while (3.91) deals with identical consumer-artists. Our comments on equation (3.20) also apply to equation (3.91).

3.2.1.2 The optimal time path in a parametric version of model $S\bar{K}1$

As in section 3.1.1.2, to further characterize the dynamic process of the accumulation of cultural goods we introduce some additional simplifications:

(i) New cultural goods are produced with the linear production technology:

$$v_c = V(r_v) = a_v r_v, \tag{3.92}$$

where a_v is a constant and positive production coefficient.

(ii) As in section 3.1.1.2, the cultural services are produced with the Leontief technology (3.21):

$$s_s = S(r_s, g_s) = min[a_s r_s, g_s].$$

(iii) The consumer goods are produced with the linear technology (3.22):

$$y = Y(r_y) = a_y r_y.$$

(iv) The representative consumer-artist's utility function is parametric and additive separable:

$$U(g_c, s_c, v_c, y_c) = U^g(g_c) + U^s(s_c) + U^v(v_c) + y_c, \tag{3.93}$$

where $U^g(g_c) := b_g g_c - \frac{d_g}{2} g_c^{\,2}$, $U^s(s_c) := b_s s_c - \frac{d_s}{2} s_c^{\,2}$, $U^v(v_c) := b_v v_c - \frac{d_v}{2} v_c^{\,2}$ and where b_g, d_g, b_v and d_v are constant, positive parameters.

The Hamiltonian associated to that parametric model reads

$$\begin{aligned} H = {} & n_c b_g g_c - n_c \frac{d_g}{2} g_c^{\,2} + n_c b_s s_c - n_c \frac{d_s}{2} s_c^{\,2} + n_c b_v v_c - n_c \frac{d_v}{2} v_c^{\,2} + n_c y_c + \mu_g (n_c v_c - \alpha_g g) \\ & + n_c \lambda_v [a_v r_v - v_c] + \lambda_y (a_y r_y - y_s) + \lambda_c (y_s - n_c y_c) + \lambda_r (n_c \bar{r} - n_c r_v - n_s r_s - r_y) \\ & + n_s \lambda_{s1} (a_s r_s - s_s) + n_s \lambda_{s2} (g_s - s_s) + n_c \lambda_{gc} (g - g_c) + n_s \lambda_{gs} (g - g_s) \\ & + n_c \lambda_\sigma (n_s s_s - s_c). \end{aligned} \tag{3.94}$$

The FOCs for an interior solution are given by (3.28) and

$$\lambda_{gc} = b_g - d_g g_c , \tag{3.95}$$

$$\lambda_{\sigma} = b_s - d_s s_c , \tag{3.96}$$

$$\mu_g = d_v v_c - b_v + \lambda_v . \tag{3.97}$$

$$\lambda_r = a_v \lambda_v = a_y \lambda_y , \tag{3.98}$$

$$\lambda_{s1} = \frac{\lambda_r}{a_s} , \tag{3.99}$$

$$\lambda_{s2} = \lambda_{gs} , \tag{3.100}$$

$$n_c \lambda_{\sigma} = \lambda_{s1} + \lambda_{s2} , \tag{3.101}$$

$$\dot{\mu}_g = \left(\delta + \alpha_g\right) \mu_g - n_c \lambda_{gc} - n_s \lambda_{gs} . \tag{3.102}$$

We account for (3.28) and (3.98) to turn (3.97) into:

$$\mu_g = \frac{a_y}{a_v} - b_v + d_v v_c . \tag{3.103}$$

In order to get more information about the laws of motion in this economy, we consider (3.95) through (3.101) in (3.103) to obtain:

$$\dot{\mu}_g = \left[\left(\delta + \alpha_g\right) \left(\frac{a_y}{a_v} - b_v \right) - b_g n_c - b_s n_c n_s + \frac{a_y}{a_s} n_s \right] + \left(\delta + \alpha_g\right) d_v v_c + d_g n_c g_c + d_s n_c n_s s_c . \tag{3.104}$$

Next we plug $\dot{\mu}_g = d_v \dot{v}_c$ from (3.103) into (3.104), then take the conditions (3.8), (3.73) and (3.74) into account to write, after some rearrangement of terms,

$$\dot{v}_c = \frac{1}{d_v} \left[\left(\delta + \alpha_g\right) \left(\frac{a_y}{a_v} - b_v \right) - b_g n_c - b_s n_c n_s + \frac{a_y}{a_s} n_s \right] + \left(\delta + \alpha_g\right) v_c + \frac{n_c}{d_v} \left(d_g + d_s n_s^2 \right) g . \tag{3.105}$$

or:

$$\dot{v}_c = -M_5 + M_6 n_c v_c + M_7 g \,, \tag{3.106}$$

where

$$M_5 := \frac{\left(\delta + \alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) + a_s a_v b_g n_c + a_s a_v b_s n_c n_s - a_v a_y n_s}{a_s a_v d_v} \,,$$

$$M_6 := \frac{\left(\delta + \alpha_g\right)}{n_c} > 0 \text{ and } M_7 := \frac{n_c}{d_v}\left(d_g + d_s n_s^{\,2}\right) > 0 \,.$$

The sign of M_5 is positive, if and only if

$$n_c > n_{c0} := \frac{\left(\delta + \alpha_g\right)\left(a_s a_y - a_s a_v b_v\right) + a_v a_y n_s}{a_s a_v b_g + a_s a_v b_s n_s} \,.$$

Since we are interested in economies in which the number of consumer-artists, n_c, is sufficiently large, we assume this inequality to hold. Moreover, the condition $n_c \geq 1$ needs to be satisfied, hence $n_c > max\left[1, n_{c0}\right]$.

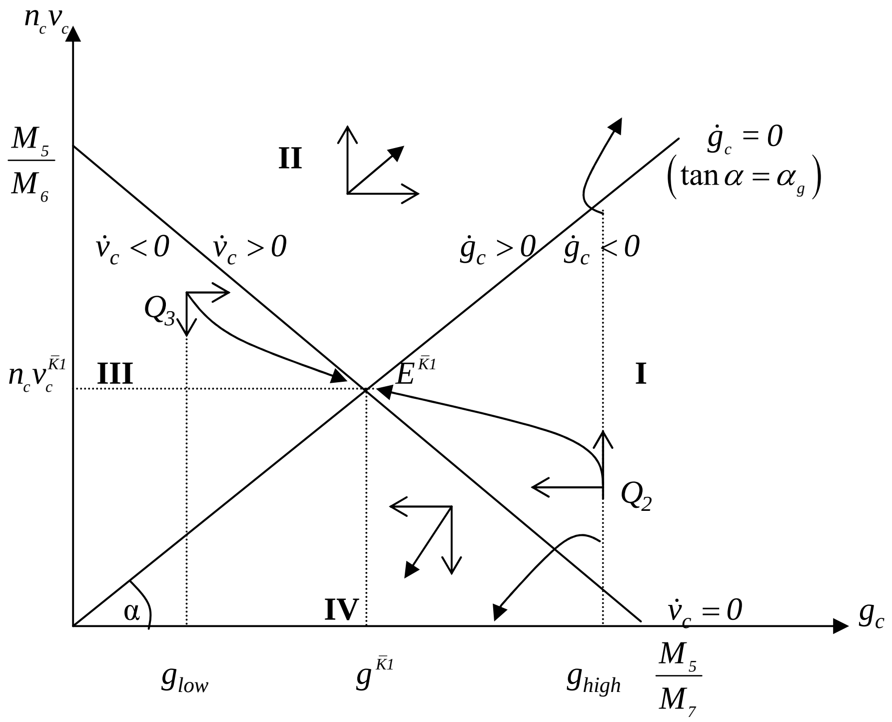
3.2.1.3 The phase diagram

(3.72) and (3.106) represent a system of two differential equations which yields the steady-state conditions:

$$\left.\begin{aligned} -M_5 + M_6 n_c v_c + M_7 g &= 0, \\ n_c v_c - \alpha_g g &= 0. \end{aligned}\right\} \tag{3.107}$$

The construction of the phase diagram associated to (3.107) is analogous to our procedure in the previous section based on (3.40). The $\dot{v}_c = 0$ and $\dot{g} = 0$ isoclines in Figure 3.12 partition the space into four regions. The point of intersection $E^{\bar{K}1}$ is the unique interior steady state.

Figure 3.12 Phase diagram for the parametric version of model $S\bar{K}1$



In region **I** the direction of motion is northwest. There exists one trajectory only starting e.g. from Q_2 that leads to the steady state $E^{\bar{K}1}$. If the system starts below the point Q_2, the investment in cultural goods would eventually be driven to zero; a starting point above the point Q_2 would imply that the creation of new cultural goods will be strongly stimulated inducing the stock of cultural goods to accumulate so fast that the system will never reach the steady state. Ever increasing investment in new cultural goods eventually uses up all available resources for creating cultural goods which clearly is not an optimal trajectory. In region **II** the economy moves northeast. No trajectory starting from this region will ever reach the steady state. In region **III** the economy behaves as in region **I**, and in region **IV** it behaves as in region **II**.

We distinguish two alternative initial situations in Figure 3.12: Suppose first, the initial stock of cultural goods is smaller than its steady-state level, e.g. g_{low} in Figure 3.12. In that case putting the economy on the optimal trajectory towards the steady state requires to choose an initial investment in cultural goods that is higher than its steady-state level, $n_c v^{\bar{K}1}$. The high (but not too high) investment in cultural goods induces the stock of cultural goods to grow until its steady-state level $g^{\bar{K}1}$ is reached. Second, suppose the initial stock of cultural goods is greater than its steady-state level, e.g. g_{high} in Figure 3.12. In that case the optimal trajectory towards the steady state is such that the initial investment in cultural goods must be set below its steady-state level, $n_c v^{\bar{K}1}$. Again the equation of motion $\dot{v}_c = -\dot{y}_c > 0$ implies that the optimal investment in the stock of cultural goods is increasing over time to the effect that private consumption shrinks.

Solving (3.107) give us the following steady-state values

$$g^{\bar{K}1} := G^{\bar{K}1}(n_c, n_s) = \frac{M_5}{\alpha_g M_6 + M_7}$$

$$= \frac{\left(a_s a_v b_g + a_s a_v b_s n_s\right) n_c^2 - \left[a_v a_y n_s - \left(a_s a_v b_v - a_s a_y\right)\left(\delta + \alpha_g\right)\right] n_c}{a_s a_v d_g n_c^2 + a_s a_v d_s n_c^2 n_s^2 + a_s a_v d_v \alpha_g \left(\delta + \alpha_g\right)}, \tag{3.108}$$

$$= \frac{\left[a_v \left(a_s b_s n_c - a_y\right) n_c\right] n_s + a_s \left[a_v b_g n_c + \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right)\right] n_c}{a_s a_v d_g n_c^2 + a_s a_v d_s n_c^2 n_s^2 + a_s a_v d_v \alpha_g \left(\delta + \alpha_g\right)}, \tag{3.108'}$$

$$n_c v^{\bar{K}1} = \frac{\alpha_g M_5}{\alpha_g M_6 + M_7}$$

$$= \alpha_g \left\{ \frac{\left(a_s a_v b_g + a_s a_v b_s n_s\right) n_c^2 - \left[a_v a_y n_s - \left(a_s a_v b_v - a_s a_y\right)\left(\delta + \alpha_g\right)\right] n_c}{a_s a_v d_g n_c^2 + a_s a_v d_s n_c^2 n_s^2 + a_s a_v d_v \alpha_g \left(\delta + \alpha_g\right)} \right\}. \tag{3.109}$$

$g^{\bar{K}1}$ and $v^{\bar{K}1}$ are positive, since $n_c > max[1, n_{c0}]$. The impact of exogenous changes in the parameters, n_c, n_s, and $a_s, a_v, a_y, b_g, b_s, b_v, d_g, d_s, d_v, \delta$ and α_g on the formation of cultural-goods stock and creating new cultural goods in (3.108) and (3.109) is remarkably more complex than in the model $S\bar{G}1$ from (3.41) - (3.43). We therefore list those impacts and interactions in Table 3.4 wgich shows that $g^{\bar{K}1}$ (and $v^{\bar{K}1}$) is strictly increasing in a_s, a_v, b_g, b_s and b_v, and is strictly decreasing in a_y, d_g, d_s and d_v. The signs of the first derivative of $g^{\bar{K}1}$ with respect to α_g, δ and n_c, n_s are ambiguous.

Table 3.4 The impact of parameter changes on the steady-state values of the stock of cultural goods and newly created cultural goods

		Impact on $g^{\bar{K}1}$ and $v^{\bar{K}1}$
Production technology Parameters	a_s	+
	a_v	+
	a_y	-
Individual preference Parameters	b_g	+
	b_s	+
	b_v	+
	d_g	-
	d_s	-
	d_v	-
Depreciation factor	α_g	?
	δ	?
Number of consumer-artists	n_c	?
Number of cultural-services firms	n_s	?

We now focus our attention on the link between the variables n_c, n_s and $g^{\bar{K}1}$. The derivative of $g^{\bar{K}1}$ with respect to n_c is

$$\frac{dg^{\bar{K}1}}{dn_c} = \frac{B\left(d_g + d_s n_s^{2}\right)n_c^{2} + 2a_s a_v d_v\left(b_g + b_s n_s\right)\alpha_g\left(\delta + \alpha_g\right)n_c - Bd_v\alpha_g\left(\delta + \alpha_g\right)}{a_s a_v\left[\left(d_g + d_s n_s^{2}\right)n_c^{2} + d_v\alpha_g\left(\delta + \alpha_g\right)\right]^{2}}, \quad (3.110)$$

where $B := a_v a_y n_s - a_s\left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right)$. The derivative of $g^{\bar{K}1}$ with respect to n_s reads

$$\frac{dg^{\bar{K}1}}{dn_s} = \frac{\left[a_s d_s\left(a_y - a_s b_s n_c\right)n_c^{3}\right]n_s^{2} - 2a_s d_s n_c^{3} C n_s}{a_s a_v\left[d_g n_c^{2} + d_s n_c^{2} n_s^{2} + d_v\alpha_g\left(\delta + \alpha_g\right)\right]^{2}} - \frac{a_v\left(a_y - a_s b_s n_c\right)\left[d_g n_c^{2} + d_v\alpha_g\left(\delta + \alpha_g\right)\right]n_c}{a_s a_v\left[d_g n_c^{2} + d_s n_c^{2} n_s^{2} + d_v\alpha_g\left(\delta + \alpha_g\right)\right]^{2}}, \quad (3.111)$$

where $C := \left[a_v b_g n_c - \left(a_y - a_v b_v \right)\left(\delta + \alpha_g \right) \right]$. Apparently, due to the complexity of (3.110) and (3.111), it is a formidable task to determine the interdependence between $g^{\bar{K}1}$ and n_c, n_s. Since the signs of the terms B, C and $\left(a_y - a_s b_s n_c \right)$ are ambiguous, the signs of $dg^{\bar{K}1} / dn_c$, $dg^{\bar{K}1} / dn_s$ and $\delta g^{\bar{K}1} / \delta n_c \delta n_s$ are ambiguous, either. In order to get some useful results, we disregard $\delta g^{\bar{K}1} / \delta n_c \delta n_s$ altogether and confine our discussion to the signs of $dg^{\bar{K}1} / dn_c$ and $dg^{\bar{K}1} / dn_s$. With regard to $dg^{\bar{K}1} / dn_c$ we need to distinguish three cases depending on the sign of the term B:

$$B \lesseqgtr 0 \quad \Leftrightarrow \quad n_s \lesseqgtr n_{s0} := \frac{a_s \left(a_v b_v - a_y \right)\left(\delta + \alpha_g \right)}{a_v a_y}.$$

Case 1.1 $n_{s0} \geq 1$ and $n_s \in \left[1, n_{s0} \right[$ (implying $B < 0$)

Under this condition (3.110) implies

$$\frac{dg^{\bar{K}1}}{dn_c} \gtreqless 0 \Leftrightarrow n_c \lesseqgtr n_{M1},$$

where

$$n_{M1} := \frac{-a_s a_v d_v \left(b_g + b_s n_s \right) \alpha_g \left(\delta + \alpha_g \right) - \left\{ \left[a_s a_v d_v \left(b_g + b_s n_s \right) \alpha_g \left(\delta + \alpha_g \right) \right]^2 \right.}{B \left(d_g + d_s n_s^2 \right)}$$

$$+ \frac{\left. B^2 \left(d_g + d_s n_s^2 \right) d_v \alpha_g \left(\delta + \alpha_g \right) \right\}^{\frac{1}{2}}}{B \left(d_g + d_s n_s^2 \right)}.$$

$g^{\bar{K}1}$ is strictly growing in n_c for $n_c \leq n_{M1}$, reaches its maximum at the threshold value $n_c = n_{M1}$, and then declines in n_c for $n_c > n_{M1}$.

Case 1.2 $n_{s0} \geq 1$ and $n_s = n_{s0}$ (implying $B = 0$)

(3.110) now becomes

$$\frac{dg^{\bar{K}1}}{dn_c} = \frac{2a_s a_v d_v \left(b_g + b_s n_s\right)\alpha_g \left(\delta + \alpha_g\right) n_c}{a_s a_v \left[\left(d_g + d_s n_s^2\right) n_c^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]^2} > 0 .$$

Case 1.3 $n_s \geq max\left[1, n_{s0}\right]$ (implying $B > 0$)

In this case (3.110) implies

$$\frac{dg^{\bar{K}1}}{dn_c} \lesseqgtr 0 \Leftrightarrow n_c \lesseqgtr n_{M3} ,$$

where

$$n_{M3} := \frac{-a_s a_v d_v \left(b_g + b_s n_s\right)\alpha_g \left(\delta + \alpha_g\right) + \left\{\left[a_s a_v d_v \left(b_g + b_s n_s\right)\alpha_g \left(\delta + \alpha_g\right)\right]^2\right.}{B\left(d_g + d_s n_s^2\right)}$$

$$\frac{\left. + B^2 \left(d_g + d_s n_s^2\right) d_v \alpha_g \left(\delta + \alpha_g\right)\right\}^{\frac{1}{2}}}{B\left(d_g + d_s n_s^2\right)} .$$

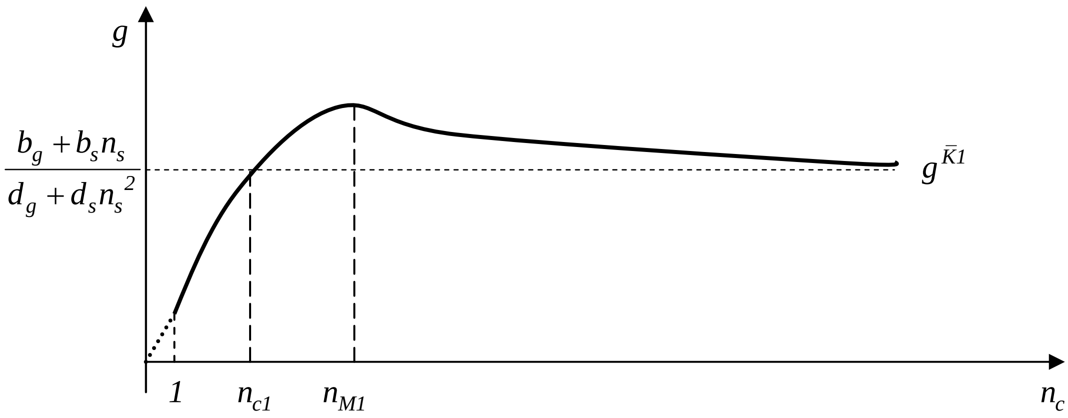
$g^{\bar{K}1}$ is first strictly declining in n_c up to the threshold value $n_c = n_{M3}$ and then strictly increases in n_c for all $n_c > n_{M3}$. Since $g^{\bar{K}1} = 0$ for $n_c = 0$ and $dg^{\bar{K}1} / dn_c < 0$ for $n_c < n_{M3}$, we conclude that $g^{\bar{K}1} < 0$ for all $n_c \in \left]0, n_{c0}\right[$, where $n_{c0} := B / \left(a_s a_v b_g + a_s a_v b_s n_s\right)$ and $g^{\bar{K}1} > 0$ for all $n_c > n_{c0}$. Therefore the relevant domain of $dg^{\bar{K}1} / dn_c > 0$ is $\left[max\left[1, n_{c0}\right], \infty\right[$.

In addition, we find that

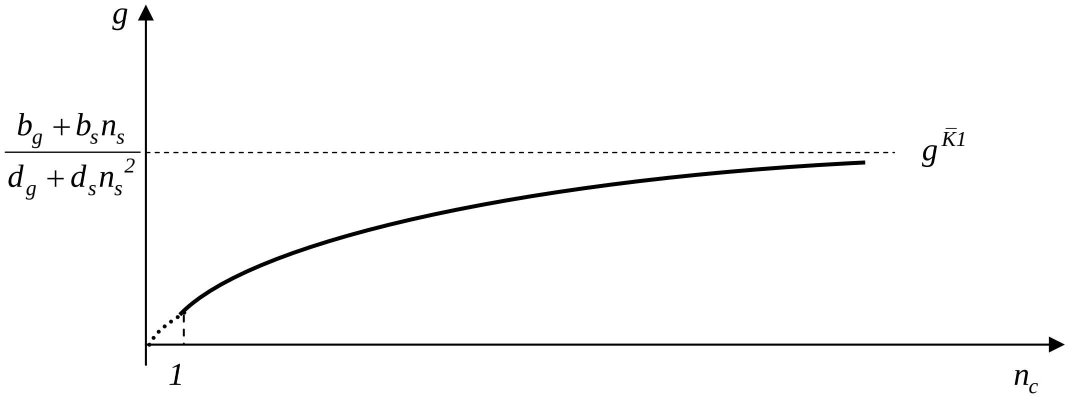
$$\lim_{n_c \to \infty} g^{\bar{K}1} = \frac{b_g + b_s n_s}{d_g + d_s n_s^2} ,$$

i.e. that with very large numbers of consumer-artists, the steady-state value of cultural capital $g^{\bar{K}1}$ converges to the positive value $\left(b_g + b_s n_s\right) / \left(d_g + d_s n_s^2\right)$. We now illustrate those cases in Figure 3.13.

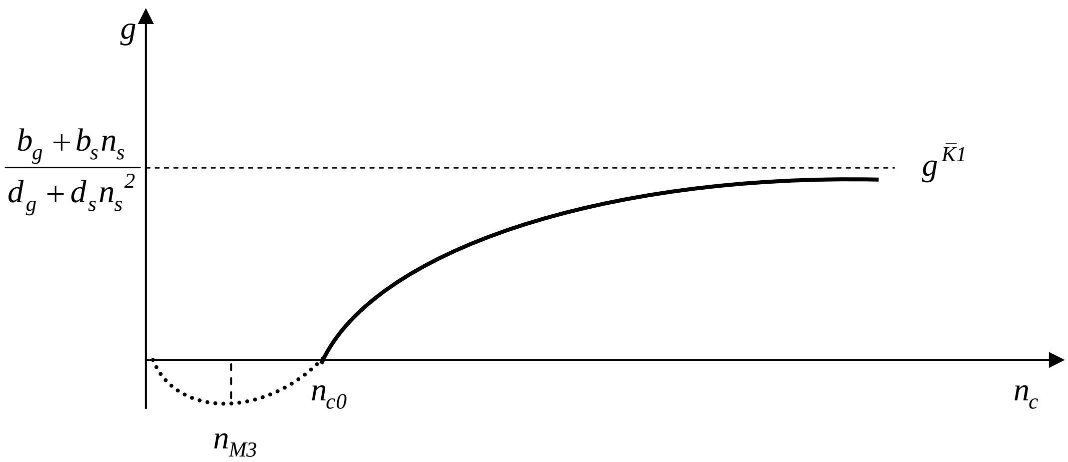
Figure 3.13 Different shapes of the function $g^{\bar{K}1} = G^{\bar{K}1}(n_c, \bar{n}_s)$



Panel 1: The case of $B < 0$



Panel 2: The case of $B = 0$



Panel 3: The case of $B > 0$

We now turn to the discussion of the sign of $dg^{\bar{K}1} / dn_s$. Obviously, we need to distinguish three different cases depending on the sign of the terms $(a_y - a_s b_s n_c)$ and C. One has

$$\left(a_y - a_s b_s n_c\right) \lesseqgtr 0 \qquad \Leftrightarrow \qquad n_c \gtreqless n_{c1} := \frac{a_y}{a_s b_s},$$

and

$$C \gtreqless 0 \qquad \Leftrightarrow \qquad n_c \lesseqgtr n_{c2} := \frac{\left(a_y - a_v b_v\right)\left(\delta + \alpha_g\right)}{a_v b_g}.$$

To keep the analysis simple, we suppose[37] that $\left(a_y - a_v b_v\right) > 0$ and hence $n_{c2} > 0$.

Case 2.1 $n_c > max\left[1, n_{c1}\right]$ (implying $\left(a_y - a_s b_s n_c\right) < 0$)

The equation (3.111) implies:

(i) If $C > 0$, then $\frac{dg^{\bar{K}1}}{dn_s} \gtreqless 0 \Leftrightarrow n_s \lesseqgtr n_{sM1}$, where

$$n_{sM1} := \frac{a_s d_s n_c^{\,2} C - \left\{\left(a_s d_s n_c^{\,2} C\right)^2 + a_s d_s n_c a_v \left[d_g n_c^{\,2} + d_v \alpha_g \left(\delta + \alpha_g\right)\right]\left(a_y - a_s b_s n_c\right)\right\}^{\frac{1}{2}}}{\left(a_y - a_s b_s n_c\right) a_s d_s n_c^{\,2}}.$$

$g^{\bar{K}1} > 0$ for $n_s = 0$. $g^{\bar{K}1}$ is first strictly growing in n_s, until it reaches the threshold value $n_s = n_{sM1}$, and is then strictly declining in n_s for $n_s > n_{sM1}$.

(ii) If $C = 0$, then $\frac{dg^{\bar{K}1}}{dn_s} \gtreqless 0 \Leftrightarrow n_s \lesseqgtr n_{sr}$, where

$$n_{sr} := \left\{\frac{a_v \left[d_g n_c^{\,2} + d_v \alpha_g \left(\delta + \alpha_g\right)\right]}{a_s d_s n_c^{\,2}}\right\}^{\frac{1}{2}}.$$

$g^{\bar{K}1} > 0$ for $n_s = 0$. It is first strictly increasing in n_s, reaches its maximum at the threshold value $n_s = n_{sr}$, and is then strictly declining in n_s for $n_s > n_{sr}$.

(iii) If $C < 0$, then $\frac{dg^{\bar{K}1}}{dn_s} \gtreqless 0 \Leftrightarrow n_s \lesseqgtr n_{sM3}$, where

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37 The only reason for this restriction is to avoid the distinction of additional subcases. More specially, $C > 0$ would be meaningless in case of $a_y < a_s b_s n_c$.

$$n_{sM3} := \frac{a_s d_s n_c^2 C + \left\{ \left(a_s d_s n_c^2 C \right)^2 + a_s d_s n_c a_v \left[d_g n_c^2 + d_v \alpha_g \left(\delta + \alpha_g \right) \right] \left(a_y - a_s b_s n_c \right) \right\}^{\frac{1}{2}}}{\left(a_y - a_s b_s n_c \right) a_s d_s n_c^2} .$$

$g^{\bar{K}1}$ is first strictly increasing in n_s for $n_s < n_{sM3}$, reaches its maximum at the threshold value $n_s = n_{sM3}$, and is then declining in n_s for $n_s > n_{sM1}$. Furthermore,

$$g^{\bar{K}1} = \frac{\left[a_v \left(a_s b_s n_c - a_y \right) n_c \right] n_s + a_s C n_c}{a_s a_v d_g n_c^2 + a_s a_v d_s n_c^2 n_s^2 + a_s a_v d_v \alpha_g \left(\delta + \alpha_g \right)} < 0$$

for all $n_s \in \left[0, n_{s0} \right[$, where $n_{s0} := \left[-a_s C / a_v \left(a_s b_s n_c - a_y \right) n_c \right]$. Hence it is true that $g^{\bar{K}1} > 0$ for $n_s \in \left[max\left[1, n_{s0} \right], \infty \right[$.

Case 2.2 $n_c = n_{c1}$ and $n_{c1} \geq 1$ (implying $\left(a_y - a_s b_s n_c \right) = 0$)

Under this condition, the sign of $dg^{\bar{K}1} / dn_s$ is ambiguous. We distinguish three subcases depending on the sign of *C*:

$$\frac{dg^{\bar{K}1}}{dn_s} = \frac{-2 a_s d_s n_c^3 C n_s}{a_s a_v \left[d_g n_c^2 + d_s n_c^2 n_s^2 + d_v \alpha_g \left(\delta + \alpha_g \right) \right]^2} \gtreqless 0 \text{ for } C \lesseqgtr 0 . \tag{3.112}$$

However, in view of $n_c = n_{c1}$ and (3.108) we conclude that

$$C \gtreqless 0 \iff g^{\bar{K}1} = \left. \frac{C \bar{n}_c}{a_v d_g \bar{n}_c^2 + a_v d_s \bar{n}_c^2 n_s^2 + a_v d_v \alpha_g \left(\delta + \alpha_g \right)} \right|_{n_s = 0} \gtreqless 0 .$$

We therefore rule out $C \leq 0$ such that in case 2.2 an interior solution ($g^{\bar{K}1} > 0$) only exists if $C > 0$, and under that condition (3.112) yields $dg^{\bar{K}1} / dn_s < 0$ for $n_s \in \left[1, \infty \right[$. $g^{\bar{K}1}$ is then strictly decreasing in n_s.

Case 2.3 $n_c \in \left[1, n_{c1} \right[$ and $n_{c1} \geq 1$ (implying $\left(a_y - a_s b_s n_c \right) > 0$)

The equation (3.111) implies

$$\frac{dg^{\bar{K}1}}{dn_s} \gtreqless 0 \Leftrightarrow n_s \gtreqless n_{sM3},$$

where

$$n_{sM3} := \frac{a_s d_s n_c^{\,2} C + \left\{ \left(a_s d_s n_c^{\,2} C \right)^2 + a_s d_s n_c a_v \left[d_g n_c^{\,2} + d_v \alpha_g \left(\delta + \alpha_g \right) \right] \left(a_y - a_s b_s n_c \right) \right\}^{\frac{1}{2}}}{\left(a_y - a_s b_s n_c \right) a_s d_s n_c^{\,2}}.$$

$g^{\bar{K}1}$ is first strictly decreasing in n_s, and reaches its minimum at $n_s = n_{sM3}$, $g^{\bar{K}1}$ is then increasing in n_s for $n_s > n_{sM3}$. From (3.108') we conclude that $g^{\bar{K}1} \gtreqless 0$, if and only if $n_s \lesseqgtr n_{s03}$, where $n_{s03} := \left[-a_s C / a_v \left(a_s b_s n_c - a_y \right) n_c \right]$. An interior solution ($g^{\bar{K}1} > 0$) only exists for $n_s \in \left[1, n_{s03} \right]$. Since[38]

$$\left. \frac{dg^{\bar{K}1}}{dn_s} \right|_{n_s = n_{s03}} < 0 \qquad \text{and} \qquad \left. \frac{dg^{\bar{K}1}}{dn_s} \right|_{n_s = n_{sM3}} = 0,$$

it is true, that $n_{s03} < n_{sM3}$.

In addition, it is straightforward that

$$\lim_{n_s \to \infty} g^{\bar{K}1} = 0,$$

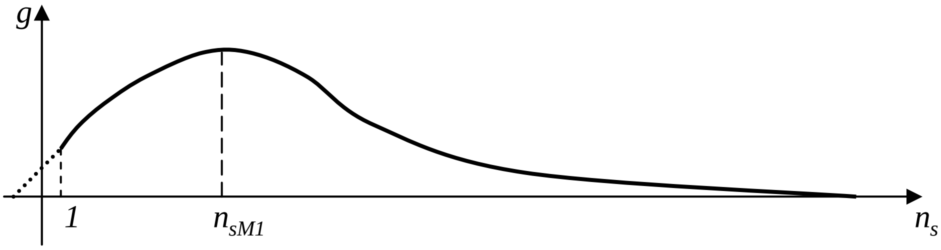
i.e. with very large numbers of cultural-services firms, $g^{\bar{K}1}$ converges to zero.

We now depict all cases discussed above in Figure 3.14.

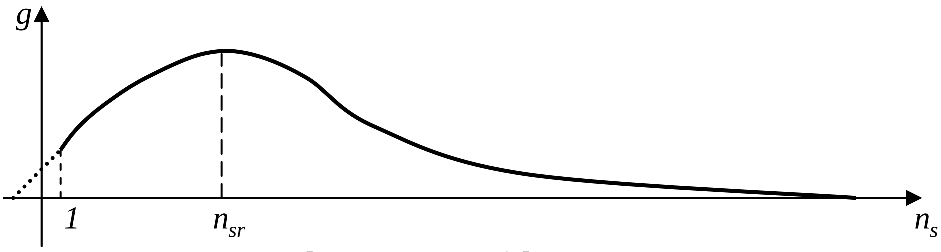
38 Note, that $\left. \frac{dg^{\bar{K}1}}{dn_s} \right|_{n_s = n_{s03}} = \frac{\left(a_s b_s n_c - a_y \right) n_c}{\left[a_s d_g n_c^{\,2} + a_s d_s n_c^{\,2} n_s^{\,2} + a_s d_v \alpha_g \left(\delta + \alpha_g \right) \right]} < 0$.

Figure 3.14 Different shapes of the function $g^{\bar{K}1} = G^{\bar{K}1}\left(\bar{n}_c, n_s\right)$

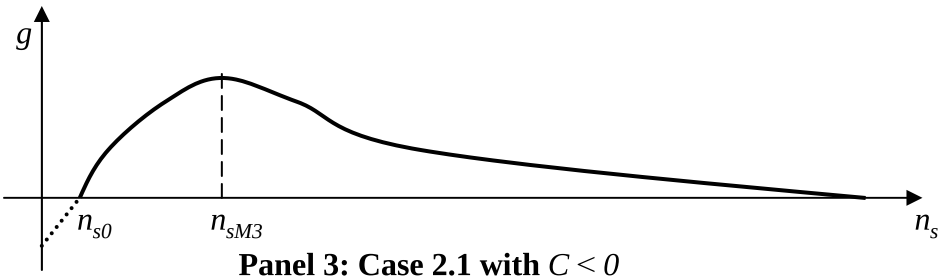
1) Case 2.1



Panel 1: Case 2.1 with $C > 0$

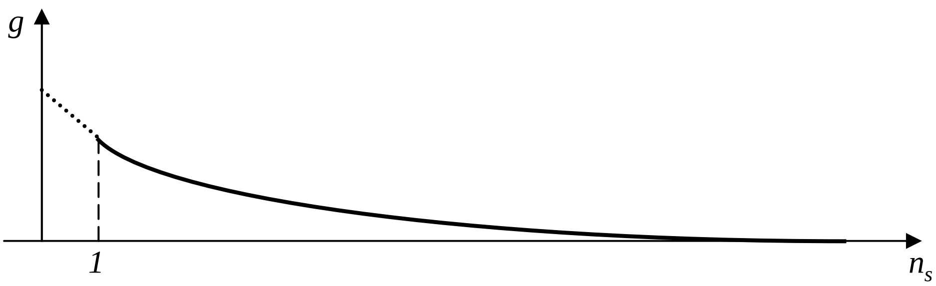


Panel 2: Case 2.1 with $C = 0$



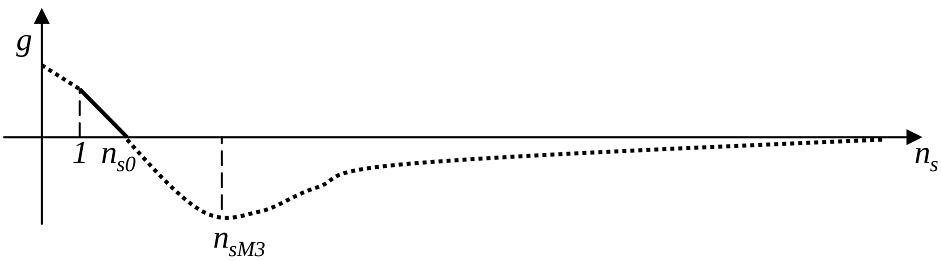
Panel 3: Case 2.1 with $C < 0$

2) Case 2.2



Panel 4: Case 2.2 with $C > 0$

3) Case 2.3



Panel 5: Case 2.3 with $C < 0$

To interpret the shapes of the curves drawn in Figures 3.13 and 3.14, we set $\dot{\mu}_g = 0$ in (3.102) to obtain after some algebraic manipulation of (3.95) - (3.101)

$$\left[\left(b_v - d_v v_c\right) + \frac{n_c\left(b_g - d_g g_c\right)}{\delta + \alpha_g} + \frac{n_c n_s\left(b_s - d_s s_c\right)}{\delta + \alpha_g} \right] = \frac{a_y}{a_v} + \frac{n_s}{\delta + \alpha_g}\frac{a_y}{a_s}. \tag{3.113}$$

According to (3.113), the LHS stands for the aggregate marginal social benefits, the RHS captures the aggregate marginal production costs. The term $\left(b_v - d_v v_c\right)$ on the LHS is the consumer-artist's instantaneous marginal willingness-to-pay for newly created cultural goods, where b_v is the consumer-artist's maximum marginal willingness-to-pay for newly created cultural goods at $U_v\big|_{v=0}$, and the term $n_c\left(b_g - d_g g_c\right)/\left(\delta + \alpha_g\right)$ is the instantaneous aggregate marginal willingness-to-pay for the stock of cultural goods, and the term $n_c n_s\left(b_s - d_s s_c\right)/\left(\delta + \alpha_g\right)$ is the instantaneous aggregate marginal willingness-to-pay for cultural services. $\left(a_y / a_v\right)$ on the RHS represents the marginal rate of transforming the consumer good into newly created cultural goods, $\left(a_y / a_s\right)$ is the marginal rate of transforming the consumer goods into cultural services.

We now discuss the curvature of the function $g^{\bar{K}1} = G^{\bar{K}1}\left(n_c, \bar{n}_s\right)$ in Figure 3.13. We rewrite (3.113) as

$$\left[-d_v v_c + \frac{n_c\left(b_g - d_g g_c\right)}{\delta + \alpha_g} + \frac{n_c n_s\left(b_s - d_s s_c\right)}{\delta + \alpha_g} \right] = \frac{a_y}{a_v} + \frac{n_s}{\delta + \alpha_g}\frac{a_y}{a_s} - b_v. \tag{3.114}$$

We then substitute $v_c = \alpha_g g_c / n_c$ and $s_c = n_s g_c$ (from the equilibrium conditions (3.21), (3.72) and (3.74)) on the LHS in (3.114) and solve for g_c to obtain the implication:

$$B \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \quad \Leftrightarrow \quad g_c \begin{cases} \begin{cases} > g_M \left(> g_N\right) & \text{for } n_c > n_{c1} \\ \leq g_N & \text{for } n_c \in \left[max\left[1, n_{c1}\right], n_{c1} \right[\end{cases} \\ = g_N \left(< g_M\right) & \text{for } n_c \in \left[1, \infty\right[\\ < g_N \left(< g_M\right) & \end{cases} \tag{3.115}$$

where

$$g_N := \frac{b_g n_c^{\,2} + b_s n_c^{\,2} n_s - A_1 n_c}{d_g n_c^{\,2} + d_s n_c^{\,2} n_s^{\,2} + d_v \alpha_g \left(\delta + \alpha_g\right)} \quad \text{and} \quad g_M := \frac{b_g + b_s n_s}{d_g + d_s n_s^{\,2}},$$

and $A_1 := \left[a_s \left(a_y - a_v b_v \right) \left(\delta + \alpha_g \right) + a_v a_y n_s \right] / a_s a_v$.

In addition, we find $\lim_{n_c \to \infty} g_N := \left(b_g + b_s n_s \right) / \left(d_g + d_s n_s^{\,2} \right) = g_M$. (3.115) hence provides the explanation of the different shapes of the curves in Figure 3.13, since that $U_g \gtreqless 0$ if and only if $g \lesseqgtr \left(b_g + b_s n_s \right) / \left(d_g + d_s n_s^{\,2} \right)$. Increasing g beyond $\left(b_g + b_s n_s \right) / \left(d_g + d_s n_s^{\,2} \right)$ would be utility reducing.

Next we turn to the case $g^{\bar{K}1} = G^{\bar{K}1}\left(\bar{n}_c , n_s \right)$ by using the same procedure. We readily get the implication

$$a_s b_s n_c - a_y \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad \Leftrightarrow \quad g_c \begin{cases} \lesseqgtr g_T & \text{for } C \gtreqless 0, \\ < g_T & \text{for } C > 0, \\ > g_N & \text{for } C < 0, \end{cases} \tag{3.116}$$

where

$$g_T := \frac{n_c n_s \left(a_s b_s n_c - a_y \right)}{a_s d_s n_c^{\,2} + a_s d_s n_c^{\,2} n_s^{\,2} + a_s d_v \alpha_g \left(\delta + \alpha_g \right)}.$$

We also have, $\lim_{n_s \to \infty} g_T := 0$. The conditions in (3.116) thus offer the explanation of the different shapes of the curves in Figure 3.14.

To attain more specific information about the interaction between n_c, n_s and $g^{\bar{K}1}$, we resort to some numerical examples in the following sub-section.

3.2.1.4 Numerical examples for the dependence of $g^{\bar{K}1}$ and $v_c^{\bar{K}1}$ on n_c and n_s

The subsequent numerical calculations are based on the parameters values

$a_s = 2, a_v = 3, a_y = 1, b_g = 3, b_s = 3, b_v = 3, d_g = 2, d_s = 2, d_v = 2, \delta = 0{,}5$ and $\alpha_k = 0{,}5$,

with two alternative values for n_s:

Example I: $n_s = 1$; Example II: $n_s = 10$.

Example I corresponds to the panel 1, example II corresponds to the panel 3 of Figure 3.13. Table 3.5 lists the results.

Table 3.5 Dependence of $g^{\bar{K}1}$ and $v_c^{\bar{K}1}$ on n_c and n_s

Example I ($n_s = 1$): $B := a_v a_y n_s - a_s\left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) < 0$

n_c	10	100	10^6
$g^{\bar{K}1}$	1.55	1.51	1.50
$n_c v_c^{\bar{K}1}$	0.78	0.75	0.75

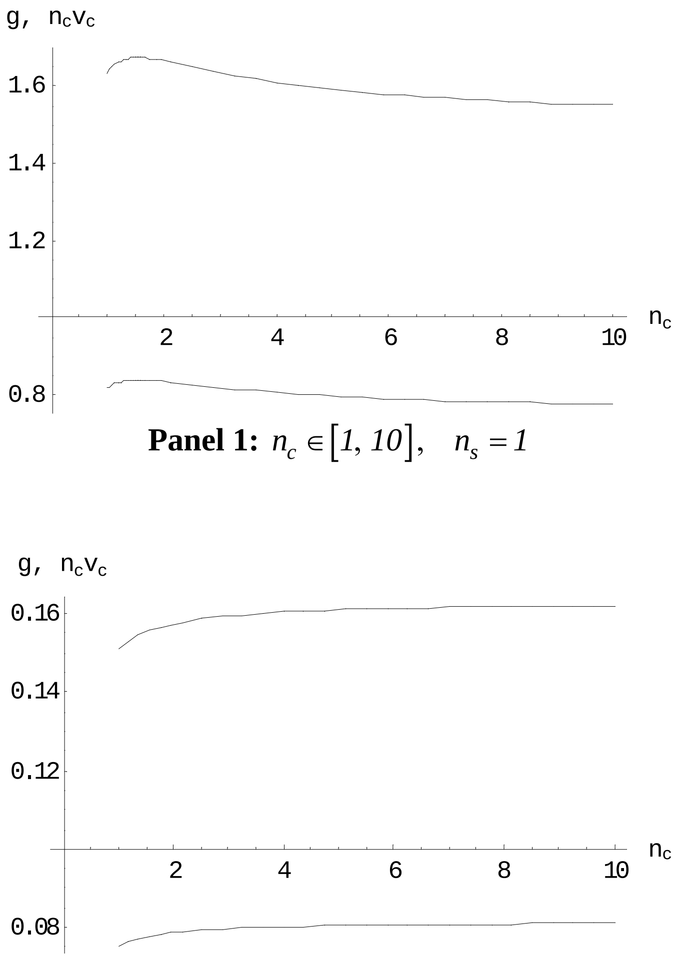
Example II ($n_s = 10$): $B := a_v a_y n_s - a_s\left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) > 0$

n_c	10	100	10^6
$g^{\bar{K}1}$	0.1630	0.1633	0.1634
$n_c v_c^{\bar{K}1}$	0.0815	0.0816	0.0817

In Table 3.5 $g^{\bar{K}1}$ and $n_c v_c^{\bar{K}1}$ from (3.108) and (3.109), respectively, are calculated for three different values of n_c with alternative specifications of parameters as described above. In example I, for $n_c = 10$ and $n_c = 100$ $g^{\bar{K}1}$ exceeds the level of the stock of cultural goods that yields the maximum instantaneous utility, and it attains the value $g = b_g / d_g \left(= 1.5\right)$ at $n_c = 10^6$. In example II $g^{\bar{K}1}$ increases when n_c is raised from 10 to 100 and 10^6 where it (almost) attains the value at which it converges for arbitrarily large n_c. Panel 1 in Figure 3.15

contains the values of $g^{\bar{K}1}$ and $n_c v_c^{\bar{K}1}$ for all $n_c \in [1, 10]$, for $n_s = 1$ and $B < 0$. Both graphs are first increasing in n_c, then they attain their unique maximum and finally decrease monotonically in n_c tending toward $g^{\bar{K}1} = 1.5$ and $n_c v_c^{\bar{K}1} = 0.75$, respectively, for n_c tending to infinity. Panel 2 shows that the values of $g^{\bar{K}1}$ and $n_c v_c^{\bar{K}1}$ are monotone increasing in n_c and approach the values $g^{\bar{K}1} = 0.1634$ and $n_c v_c^{\bar{K}1} = 0.0816$, respectively, from below when the number of consumer-artists becomes arbitrarily large.

Figure 3.15 Numerical examples of the dependence of $g^{\bar{K}1}$ and $v_c^{\bar{K}1}$ on n_c

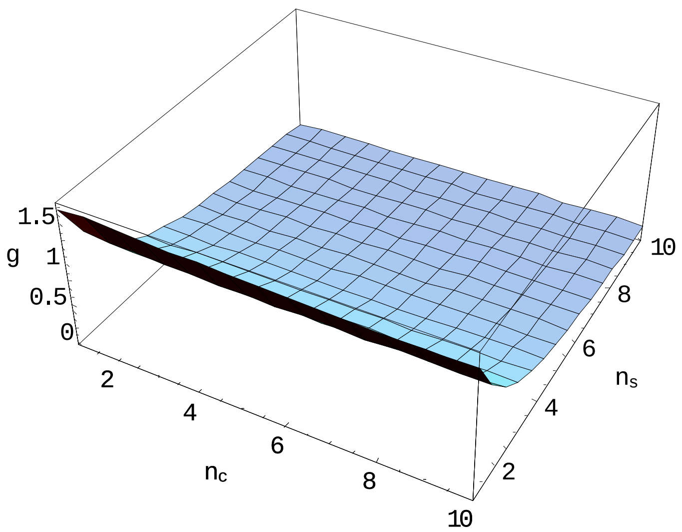


Panel 1: $n_c \in [1, 10], \quad n_s = 1$

Panel 2: $n_c \in [1, 10], \quad n_s = 10$

We now illustrate the graph of $g^{\bar{K}1} = G^{\bar{K}1}(n_c, n_s)$ in the three-dimensional diagram of Figure 3.16 where the values of n_c and n_s are varied, while the remaining parameters values are specified as before.

Figure 3.16 The graph of $g^{\bar{K}1} = G^{\bar{K}1}(n_c, n_s)$ with numerical specification of parameters



$n_c \in [1, 10], \quad n_s \in [1, 10]$

For the numerical specification of parameters, introduced at the beginning of the present sub-section Figure 3.15 provides a complete description of how the socially optimal steady-state value of $g^{\bar{K}1}$ depends on the parameters of n_c and n_s.

3.2.2 A simplified private-goods model with zero impact of cultural capital ($S\bar{K}2$)

If cultural goods are private inputs for the cultural-services firms and cultural services are private goods for consumer-artists, the associated resource constraints turn out to be (3.49) and:

$$g \geq n_s g_s \,. \tag{3.117}$$

3.2.2.1 The optimal intertemporal allocation

The Hamiltonian (3.78) is now modified to read

$$\begin{aligned} H = {} & n_c U\left(g_c, s_c, v_c, y_c\right) + \mu_g\left(n_c v_c - \alpha_g g\right) + n_c \lambda_v\left[V\left(r_v\right) - v_c\right] + \lambda_y\left[Y\left(r_y\right) - y_s\right] \\ & + \lambda_c\left(y_s - n_c y_c\right) + \lambda_r\left(n_c \bar{r} - n_c r_v - n_s r_s - r_y\right) + n_s \lambda_s\left[S\left(r_s, g_s\right) - s_s\right] \\ & + n_c \lambda_{gc}\left(g - g_c\right) + \lambda_{gs}\left(g - n_s g_s\right) + \lambda_\sigma\left(n_s s_s - n_c s_c\right). \end{aligned} \tag{3.118}$$

Observe that the Hamiltonians (3.78) and (3.118) differ only with respect to their last two terms accounting for cultural-goods input and cultural services being either public or private. The FOCs (3.79) - (3.89) carry over, but the equations (3.83) and (3.89) are replaced by, respectively,

$$\frac{\partial H}{\partial s_s} = n_s \lambda_\sigma - n_s \lambda_s = 0 \,, \tag{3.119}$$

$$\dot{\mu}_g = \left(\delta + \alpha_g\right)\mu_g - n_c \lambda_{gc} - \lambda_{gs} \,. \tag{3.120}$$

We reorganize the equations (3.80) - (3.88), (3.119) and (3.120) to obtain

$$\frac{\dot{\mu}_g}{\left(\delta + \alpha_g\right) U_y Y_r} = -\left\{\left(\frac{n_c \dfrac{U_g}{U_y Y_r}}{\delta + \alpha_g} + \frac{\dfrac{S_g}{S_r}}{\delta + \alpha_g}\right) - \left(\frac{1}{V_r} - \frac{U_v}{U_y Y_r}\right)\right\}. \tag{3.121}$$

[*18*] = -{([*19*] + [*20a*]) - ([*21*] - [*22*])}

In view of (3.121), the change in the shadow price of cultural goods in terms of the resource, [*18*], must equal the difference between the aggregate marginal benefit of cultural goods ([*19*]

+ [*20a*]) and the marginal production cost of cultural goods, ([*21*] - [*22*]). Our comments on equation (3.91) also apply to equation (3.121).

To compare the conditions characterizing the steady states of the models $S\bar{K}1$ and $S\bar{K}2$ we set $\dot{\mu}_g = 0$ in (3.91) and (3.121) to obtain, respectively,

$$\left[\left(\frac{n_c \dfrac{U_g}{U_y Y_r}}{\delta+\alpha_g} + \frac{n_s \dfrac{S_g}{S_r}}{\delta+\alpha_g}\right) + \frac{U_v}{U_y Y_r}\right] = \frac{1}{V_r},$$

for the model $S\bar{K}1$ and

$$\left[\left(\frac{n_c \dfrac{U_g}{U_y Y_r}}{\delta+\alpha_g} + \frac{\dfrac{S_g}{S_r}}{\delta+\alpha_g}\right) + \frac{U_v}{U_y Y_r}\right] = \frac{1}{V_r},$$

for the model $S\bar{K}2$. Hence the supply of newly created cultural goods tends to be greater in the former than in the latter.

3.2.2.2 The optimal time path in a parametric version of model $S\bar{K}2$

To make further progress we proceed as in section 3.2.1.2 by resorting to a parametric version of the model $S\bar{K}2$. The associated Hamiltonian is obtained by replacing the last two items in (3.94), $n_s\lambda_{gs}\left(g-g_s\right)+n_c\lambda_\sigma\left(n_s s_s - s_c\right)$, by $\lambda_{gs}\left(g-n_s g_s\right)+\lambda_\sigma\left(n_s s_s - n_c s_c\right)$. The FOCs (3.95) - (3.101) carry over unchanged, and (3.102) is substituted by (3.120). Similar calculations as those which led to (3.106) now yield

$$\dot{v}_c = -M_8 + M_6 n_c v_c + M_9 g, \qquad (3.122)$$

where

$$M_8 := \frac{\left(\delta+\alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) + a_s a_v b_g n_c + a_s a_v b_s - a_v a_y}{a_s a_v d_v}$$

and

$$M_9 := \frac{n_c}{d_v}\left(d_g + \frac{d_s}{n_c^2}\right) > 0\,.$$

A necessary and sufficient condition for $M_8 > 0$ is

$$n_c > n_{c0} := \frac{\left(\delta+\alpha_g\right)\left(a_s a_y - a_s a_v b_v\right) - a_s a_v b_s + a_v a_y}{a_s a_v b_g}\,.$$

As argued in section 3.2.1.2, we are primarily interested in economies with a large number of consumer-artists and therefore assume that M_8 is positive, in addition, the condition $n_c \geq max\left[1, n_{c0}\right]$ needs to be satisfied.

The combination of (3.72) and (3.122) represents a system of two differential equations whose steady state is determined by

$$\left.\begin{array}{l} -M_8 + M_6 n_c v_c + M_9 g = 0, \\ n_c v_c - \alpha_g g = 0. \end{array}\right\} \tag{3.123}$$

The phase diagram associated to (3.123) has an analogous structure as that of the model $\mathrm{S\bar{K}1}$. We therefore refrain from repeating the phase diagram and proceed to calculating the solution of (3.123):

$$\begin{aligned} g^{\bar{K}2} := G^{\bar{K}2}\left(n_c\right) &= \frac{M_8}{\alpha_g M_6 + M_9} \\ &= \frac{a_s a_v b_g n_c^2 + \left[a_s a_v b_s - a_v a_y + \left(a_s a_v b_v - a_s a_y\right)\left(\delta+\alpha_g\right)\right] n_c}{a_s a_v d_g n_c^2 + a_s a_v d_s + a_s a_v d_v \alpha_g\left(\delta+\alpha_g\right)}\,, \end{aligned} \tag{3.124}$$

$$\begin{aligned} n_c v^{\bar{K}2} &= \frac{\alpha_g M_8}{\alpha_g M_6 + M_9} \\ &= \alpha_g \left\{\frac{a_s a_v b_g n_c^2 + \left[a_s a_v b_s - a_v a_y + \left(a_s a_v b_v - a_s a_y\right)\left(\delta+\alpha_g\right)\right] n_c}{a_s a_v d_g n_c^2 + a_s a_v d_s + a_s a_v d_v \alpha_g\left(\delta+\alpha_g\right)}\right\}. \end{aligned} \tag{3.125}$$

The impact of the exogenous parameters n_c, $a_s, a_v, a_y, b_g, b_s, b_v, d_g, d_s, d_v, \delta$ and α_g on the formation of the stock of cultural goods and newly created cultural goods in (3.124) and (3.125) in the private-goods model is similar to the public-goods model $S\bar{K}1$, except that in the present model the results do not depend on the number of cultural-services firms. We pay our attention solely to the interdependence between $g^{\bar{K}2}$ and n_c. The derivative of $g^{\bar{K}2}$ with respect to n_c yields

$$\frac{dg^{\bar{K}2}}{dn_c} = \frac{-d_g D n_c^{\,2} + 2\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]a_s a_v b_g n_c + \left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]D}{a_s a_v\left[d_g n_c^{\,2} + d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]^2}, \quad (3.126)$$

where $D := a_v\left(a_s b_s - a_y\right) + a_s\left(a_v b_v - a_y\right)\left(\delta+\alpha_g\right)$. Obviously, to determine the sign of $dg^{\bar{K}2}/dn_c$ we have to distinguish three cases differing with respect to whether $D > 0$, $D = 0$ or $D < 0$.

Case 1: $D > 0$

For $D > 0$ (3.126) implies

$$\frac{dg^{\bar{K}2}}{dn_c} \gtreqless 0 \Leftrightarrow n_c \lesseqgtr n_{M1},$$

where

$$n_{M1} := \frac{-\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]a_s a_v b_g}{-d_g D} - \frac{\left\{\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]^2\left(a_s a_v b_g\right)^2 + d_g\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]D^2\right\}^{\frac{1}{2}}}{-d_g D}.$$

In this case, $g^{\bar{K}2}$ is first increasing in n_c, up to the threshold value $n_c = n_{M1}$, and then decreases in n_c for $n_c > n_{M1}$.

Case 2: $D = 0$

For $D = 0$ (3.126) implies

$$\frac{dg^{\bar{K}2}}{dn_c} = \frac{2\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]b_g n_c}{\left[d_g n_c^{\,2} + d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]^2} > 0 .$$

Case 3: $D < 0$

For $D < 0$ (3.126) implies

$$\frac{dg^{\bar{K}2}}{dn_c} \lesseqgtr 0 \;\Leftrightarrow\; n_c \lesseqgtr n_{M3},$$

where

$$n_{M3} := \frac{-\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]a_s a_v b_g}{-d_g D}$$

$$+\frac{\left\{\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]^2\left(a_s a_v b_g\right)^2 + d_g\left[d_s + d_v\alpha_g\left(\delta+\alpha_g\right)\right]D^2\right\}^{\frac{1}{2}}}{-d_g D}.$$

In this case, $g^{\bar{K}2}$ is first declining in n_c, up to the threshold value $n_c = n_{M3}$, and is then increasing in n_c for $n_c > n_{M1}$. Moreover, since $g^{\bar{K}2} = 0$ for $n_c = 0$ and $dg^{\bar{K}2} / dn_c < 0$ for $n_c < n_{M3}$, we conclude that $g^{\bar{K}2} < 0$ for all $n_c \in \left]0, n_{c0}\right]$, where $n_{c0} := -D / a_s a_v b_g$. Thus the relevant domain for $g^{\bar{K}2} > 0$ is $dg^{\bar{K}2} / dn_c > 0$ for $n_c \in \left[max\left[1, n_{c0}\right], \infty\right[$.

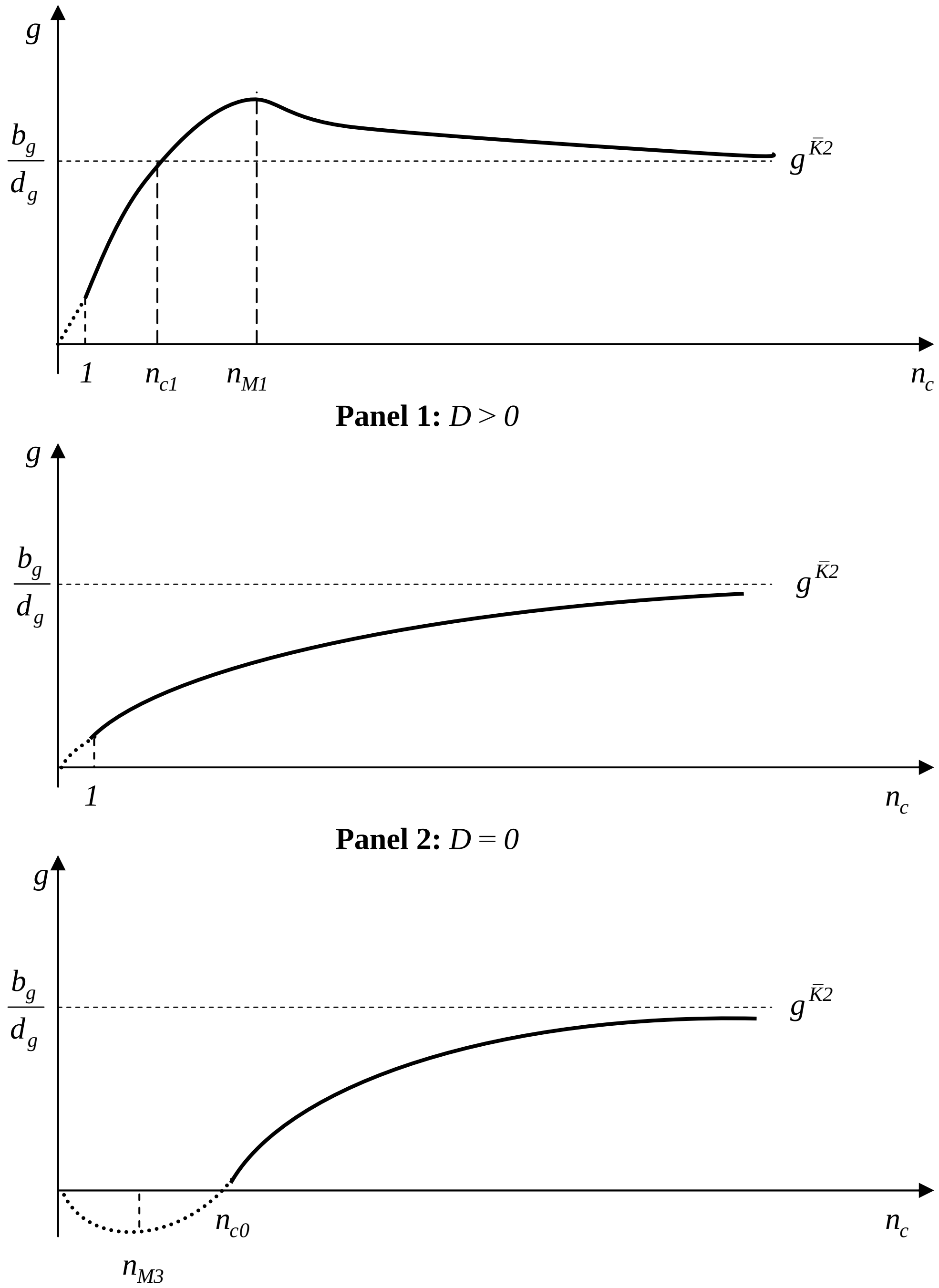
Furthermore, it is straightforward that

$$\lim_{n_c \to \infty} g^{\bar{K}2} = \frac{b_g}{d_g},$$

that is, with very large numbers of consumer-artists, the steady-state value of the stock of cultural goods, $g^{\bar{K}2}$, converges to the positive value b_g / d_g .

We now depict those cases in Figure 3.17.

Figure 3.17 Different shapes of the function $g^{\bar{K}2} = G^{\bar{K}2}(n_c)$



Panel 3: *D < 0*

The interpretation of the different shapes of the curves drawn in Figure 3.17 is very similar to the previous models and hence need no further comment. We set $\dot{\mu}_g = 0$ in (3.120) and get, after some algebraic manipulation of (3.95) - (3.101) and (3.119),

$$\left[\left(b_v - d_v v_c\right) + \frac{n_c\left(b_g - d_g g_c\right)}{\delta + \alpha_g} + \frac{\left(b_s - d_s s_c\right)}{\delta + \alpha_g}\right] = \frac{a_y}{a_v} + \frac{a_y}{a_s\left(\delta + \alpha_g\right)}. \tag{3.127}$$

Observe that (3.113) and (3.127) differ slightly in the far right term on the LHS, and the far right term on the RHS, since the cultural-goods inputs and the cultural services are private goods in the present model. Now the term $\left(b_s - d_s s_c\right)/\left(\delta + \alpha_g\right)$ is a single consumer-artist's instantaneous marginal willingness-to-pay for cultural services and the term $\left(a_y / a_s\right)$ is the marginal rate of transforming the private consumer goods into private cultural services. We apply the same procedure as before and rearrange (3.127) to get the implication:

$$D \gtreqless 0 \Leftrightarrow \left[-d_v v_c + \frac{n_c\left(b_g - d_g\right)g_c}{\delta + \alpha_g} - \frac{d_s s_c}{\delta + \alpha_g}\right] = \frac{a_y}{a_v} + \frac{a_y}{a_s\left(\delta + \alpha_g\right)} - \frac{b_s}{\delta + \alpha_g} - b_v \lesseqgtr 0 . \tag{3.128}$$

We then substitute $v_c = \alpha_g g_c / n_c$ and $s_c = \left(n_s / n_c\right) g_c$ (from the equilibrium conditions (3.21) and (3.117)) on the LHS in (3.128) and solve for g_c and obtain the implication:

$$D \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \quad \Leftrightarrow \quad g_c \begin{cases} \begin{cases} > g_T & \text{for } n_c > n_{c1} \\ \le g_T & \text{for } n_c \in \left[\max\left[1, n_{c1}\right], n_{c1}\right[\end{cases} \\ = g_T\left(< g_Q\right) & \text{for } n_c \in \left[1, \infty\right[\\ < g_T\left(< g_Q\right) & \text{for } n_c \in \left[\max\left[1, n_{c0}\right], \infty\right[\end{cases} \tag{3.129}$$

where

$$g_T := \frac{b_g n_c^2}{d_g n_c^2 + d_s n_s + d_v \alpha_g\left(\delta + \alpha_g\right)} \quad \text{and} \quad g_Q := \frac{b_g}{d_g}.$$

Moreover, $\lim_{n_c \to \infty} g_T := \left(b_g / d_g\right) = g_Q$. (3.129) therefore provides the rationale of the different shapes of the curves in Figure 3.17.

We now present some numerical examples which illustrate these different shapes of the function $g^{\bar{K}2} = G^{\bar{K}2}(n_c)$.

3.2.2.3 Numerical examples

The subsequent numerical calculations are based on the parameters values

$$a_s = 2,\ a_v = 3,\ b_g = 3,\ b_s = 3,\ b_v = 3,\ d_g = 2,\ d_s = 2,\ d_v = 2,\ \delta = 0{,}5 \text{ and } \alpha_k = 0{,}5,$$

with two alternative values of the parameter a_y:

Example I: $a_y = 1$; Example II: $a_y = 10$.

Example I corresponds to panel 1, example II corresponds to panel 3 in Figure 3.17. Table 3.6 lists the calculation results.

Table 3.6 Dependence of $g^{\bar{K}2}$ (and $v_c^{\bar{K}2}$) on n_c with alternative parameters

Example I. $D := a_v\left(a_s b_s - a_y\right) + a_s\left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) > 0$ (for $a_y = 1$)

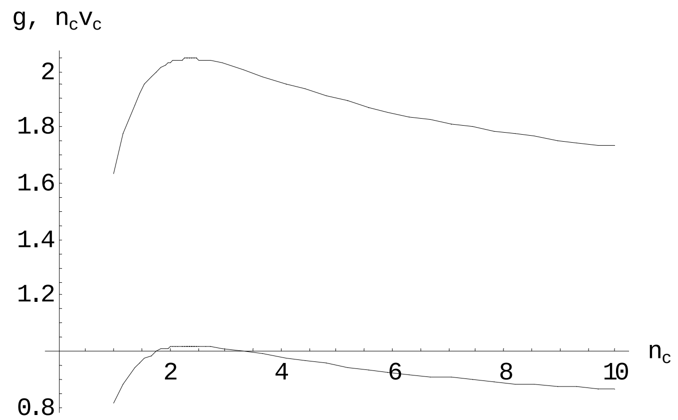
n_c	10	100	10^6
$g^{\bar{K}2}$	1.73	1.53	1.50
$n_c v_c^{\bar{K}2}$	0.86	0.76	0.75

Example II. $D := a_v\left(a_s b_s - a_y\right) + a_s\left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) < 0$ (for $a_y = 10$)

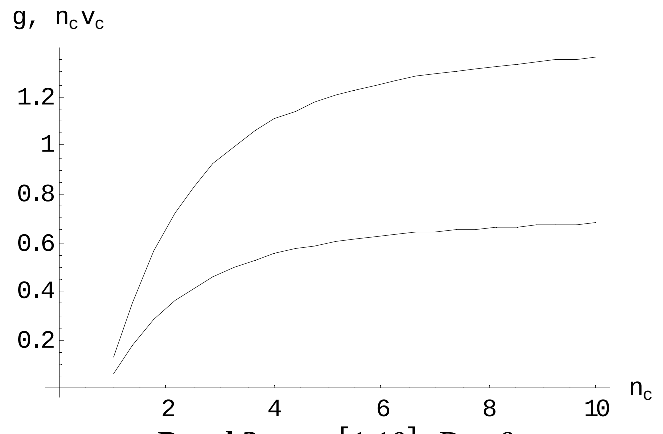
n_c	10	100	10^6
$g^{\bar{K}2}$	1.36	1.48	1.5
$n_c v_c^{\bar{K}2}$	0.68	0.74	0.75

In Table 3.6 we calculate $g^{\bar{K}2}$ and $n_c v_c^{\bar{K}2}$ from (3.124) and (3.125), respectively, for three different values of n_c with alternative values of the parameter a_y. In example I the stock of cultural goods $g^{\bar{K}2}$ exceeds the level which yields the maximum instantaneous utility for $n_c = 10$ and $n_c = 100$ and reaches its limit value $g = b_g / d_g \left(=1.5\right)$ at $n_c = 10^6$. In example II the stock of cultural goods increases when n_c is raised from *10* to *100* and 10^6 where it reaches its limit value. Those numerical examples are illustrated in Figure 3.18. Panel 1 in Figure 3.18 contains the values of $g^{\bar{K}2}$ and $n_c v_c^{\bar{K}2}$ for all $n_c \in [1,10]$ for $D > 0$. Both variables are first increasing in n_c, then attain their unique maximum and finally decrease monotonically in n_c tending toward $g^{\bar{K}2} = 1.5$ and $n_c v_c^{\bar{K}2} = 0.75$, respectively. Panel 2 shows for $D < 0$ that $g^{\bar{K}2}$ and $n_c v_c^{\bar{K}2}$ are strictly increasing in n_c and approach the upper bound $g^{\bar{K}2} = 1.5$ and $n_c v_c^{\bar{K}2} = 0.75$, respectively, from below with the increasing numbers of consumer-artists.

Figure 3.18 Numerical examples for the dependence of $g^{\bar{K}2}$ and $v_c^{\bar{K}2}$ on n_c



Panel 1: $n_c \in [1,10], D > 0$



Panel 2: $n_c \in [1,10], D < 0$

Having characterized the optimal provision of both the stock of cultural goods and newly created cultural goods in the steady state of the public-goods model $S\bar{K}1$ and the private-goods model $S\bar{K}2$, we are interested to explore the differences between the optimal allocation of the models $S\bar{K}1$ and $S\bar{K}2$.

3.2.3 Comparing the optimal allocations in the parametric versions of the models $S\bar{K}1$ and $S\bar{K}2$

From (3.107) and (3.123) we know that depending on whether cultural-goods input and cultural services are public ($S\bar{K}1$) or private ($S\bar{K}2$) the steady-state values differ in newly created cultural goods and the stock of cultural goods. The comparison of (3.108) and (3.124) yields

$$D_g := g^{\bar{K}1} - g^{\bar{K}2} = \frac{M_5}{\alpha_g M_6 + M_7} - \frac{M_8}{\alpha_g M_6 + M_9} = \frac{Q\left(\alpha_g M_6 + M_9\right) - M_8 T}{\left(\alpha_g M_6 + M_7\right)\left(\alpha_g M_6 + M_9\right)}, \tag{3.130}$$

and the comparison of (3.109) and (3.125) yields

$$D_v := n_c v_c^{\bar{K}1} - n_c v_c^{\bar{K}2} = \frac{\alpha_g M_5}{\alpha_g M_6 + M_7} - \frac{\alpha_g M_8}{\alpha_g M_6 + M_9} = \frac{\alpha_g \left[Q\left(\alpha_g M_6 + M_9\right) - M_8 T\right]}{\left(\alpha_g M_6 + M_7\right)\left(\alpha_g M_6 + M_9\right)}, \tag{3.131}$$

where

$$Q = M_5 - M_8 = \frac{a_s a_v b_s \left(n_c n_s - 1\right) - a_v a_y \left(n_s - 1\right)}{a_s a_v d_v} \geq 0$$

and

$$T = M_7 - M_9 = \frac{d_s}{d_v}\left(n_c n_s^{\,2} - \frac{1}{n_c}\right) \geq 0 .$$

It follows from (3.130) and (3.131) that in economies with only one cultural-services producer ($n_s = 1$) and one single consumer-artist ($n_c = 1$), both steady states coincide. Since on the demand side the “joint consumption” doesn’t take place any more, it makes no difference whether the respective goods are public or private.

In the sequel, we restrict our attention to (3.130) and (3.131) for economies with more than one cultural-services producer ($n_s > 1$) and consumer-artist ($n_c > 1$). In this case the signs of D_g and D_v are still ambiguous. To obtain further information, we expand the numerator on the RHS in (3.130) and rewrite D_g as $D_g = \left[A / \left(\alpha_g M_6 + M_7 \right)\left(\alpha_g M_6 + M_9 \right) \right]$, where

$$A := A_1 n_c^{4} + A_2 n_c^{3} + A_3 n_c^{2} + A_4 n_c \,, \tag{3.132}$$

and where

$$A_1 := \left(a_s a_v b_s d_g n_s - a_s a_v b_g d_s n_s^{2} \right),$$

$$A_2 := \left[a_v a_y d_s - a_s a_v b_s d_s + \left(a_s a_y d_s - a_s a_v b_v d_s \right) \right] n_s^{2} - a_v a_y d_g n_s + a_v a_y d_g - a_s a_v b_s d_g \,,$$

$$A_3 := a_s a_v b_s \left[d_s + d_v \alpha_g \left(\delta + \alpha_g \right) \right] n_s + a_s a_v b_g \,,$$

$$\begin{aligned} A_4 := & -\left[a_s a_y d_v \alpha_g \left(\delta + \alpha_g \right) + a_v a_y d_s \right] n_s \\ & + \left(a_v a_y d_v \alpha_g + a_s a_v b_v d_s - a_s a_v b_s d_v \alpha_g - a_s a_y d_s \right)\left(\delta + \alpha_g \right). \end{aligned}$$

Since M_6, M_7 and M_9 are positive (see(3.106) and (3.122) we have sign D_g = sign *A*. Unfortunately, the term *A* is also ambiguous in sign. But we can determine, at least,

$$\begin{aligned} \text{sign} \left(\lim_{n_c \to \infty} A \right) &= \text{sign} \lim_{n_c \to \infty} \left(A_1 n_c^{4} + A_2 n_c^{3} + A_3 n_c^{2} + A_4 n_c \right) \\ &= \text{sign} \lim_{n_c \to \infty} \left[n_c^{4} \left(A_1 + \frac{A_2}{n_c} + \frac{A_3}{n_c^{2}} + \frac{A_4}{n_c^{3}} \right) \right] = \text{sign } A_1 . \end{aligned}$$

Obviously it is true that

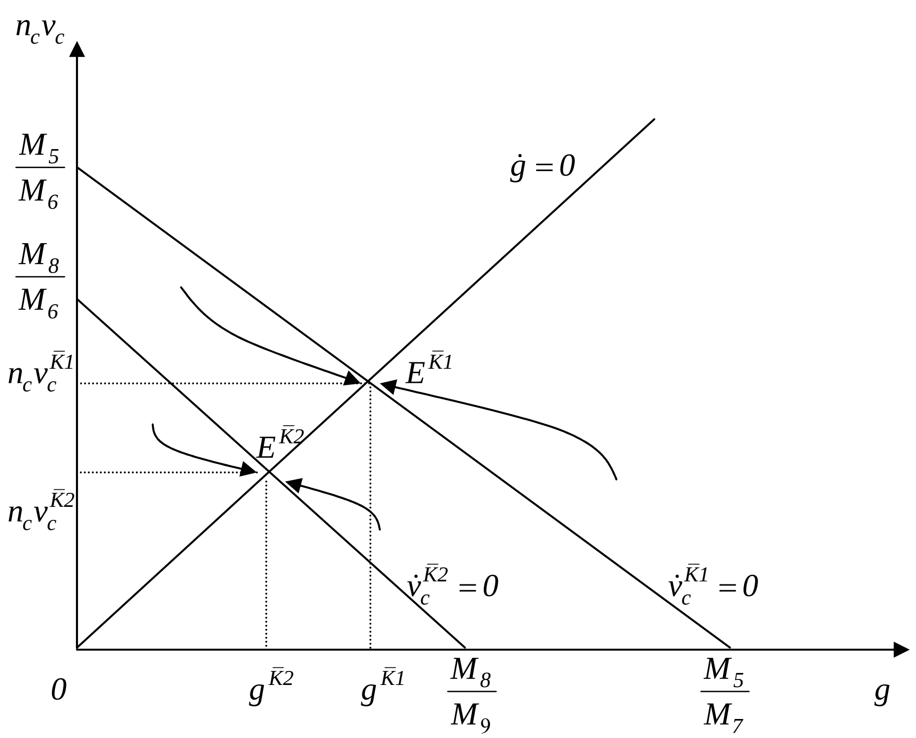
$$A_1 := a_s a_v \left(b_s d_g - b_g d_s n_s \right) n_s > 0 \qquad \Leftrightarrow \qquad \frac{b_s}{d_s} \frac{1}{n_s} > \frac{b_g}{d_g} . \tag{3.133}$$

According to (3.133), $\left(b_s / d_s\right)$ and $\left(b_g / d_g\right)$ on the RHS are the consumer-artist's instantaneous marginal willingness-to-pay for cultural services s_c, and cultural-goods stock g_c, respectively. (3.133) hence says, that the $\left(1/n_s\right)th$ part of consumer-artist's instantaneous marginal willingness-to-pay for cultural services is greater than the marginal willingness-to-pay for cultural-goods stock. If the inequality in (3.133) holds, then $A > 0$, with the consequence that $D_g > 0$ (and thus $D_v > 0$) for sufficiently large numbers of consumer-artists.

In Figure 3.19 the models $\mathrm{S\bar{K}1}$ and $\mathrm{S\bar{K}2}$ are compared, in which the inequality in (3.133) holds and therefore $D_g > 0$ and $D_v > 0$. The steady state $E^{\bar{K}2}$ lies strictly southwest of the steady state $E^{\bar{K}1}$ and a sufficient condition for that result is that the $\dot{v}_c^{\bar{K}2} = 0$ isocline is located strictly southwest of the $\dot{v}_c^{\bar{K}1} = 0$ isocline. In mathematical terms, such a situation prevails, if and only if

$$\frac{M_5}{M_7} > \frac{M_8}{M_9} \quad \text{and} \quad \frac{M_5}{M_6} > \frac{M_8}{M_6}.$$

Figure 3.19 Comparing the optimal time paths in the parameterized models $\mathrm{S\bar{K}1}$ and $\mathrm{S\bar{K}2}$

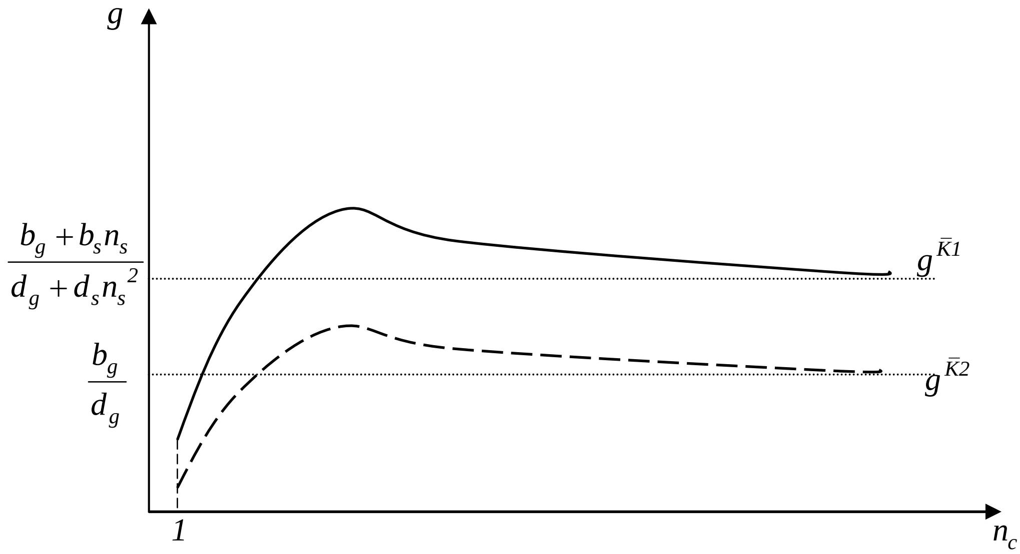


The comparison between the public-goods model and the private-goods model for very large numbers of consumer-artists is straightforward: If the inequality in (3.133) holds, then

$$\lim_{n_c \to \infty} g^{\bar{K}1} \left(= \frac{b_g + b_s n_s}{d_g + d_s n_s^{2}} \right) - \lim_{n_c \to \infty} g^{\bar{K}2} \left(= \frac{b_g}{d_g} \right) = \frac{\left(b_s d_g - b_g d_s n_s \right) n_s}{d_g \left(d_g + d_s n_s^{2} \right)} > 0 .$$

The comparison of the steady-state values of the stock of cultural goods, g, in the public and private-goods models is shown in Figure 3.20. For expository convenience we take the curve from case 1.1 of Figure 3.13 and the one from case 1 of Figure 3.17 as examples. Both curves overshoot the values $g^{\bar{K}1}$ and $g^{\bar{K}2}$, respectively.

Figure 3.20 Comparison of the curve $g^{\bar{K}1} = G^{\bar{K}1}\left(n_c, \bar{n}_s\right)$ and $g^{\bar{K}2} = G^{\bar{K}2}\left(n_c\right)$



To sum up, if the condition (3.133) holds, then the steady-state value of the optimal provision of cultural-goods stock and newly created cultural goods in the public-goods economy $S\bar{K}1$ is higher than that in the private-goods economy $S\bar{K}2$: $D_g > 0$ and $D_v > 0$.

4 Decentralization by prices of the optimal intertemporal allocation

In section 3 we focused on and characterized the efficient intertemporal allocation. As theorists we looked at the problem of allocative efficiency from the viewpoint of a social planner. If a solution to the allocation problem exists, which is posited here, the problem is how to find it. As Samuelson (1954, p.389) observed correctly, given sufficient knowledge the optimal solution can always be implemented by scanning over all attainable states of the (model) world and selecting an efficient allocation. But quite obviously, this route is not viable since one would need to process huge amounts of information which is beyond the computing capacity even of the most powerful modern computing facilities. In other words, the implementation of a socially optimal allocation through some centralized allocation mechanism is informationally infeasible (Hurwicz 1960). For an allocation mechanism to be feasible all agents must be able to pursue their objectives with limited access to and limited need of processing information implying that such mechanisms have to rely on decentralized information processing and decisions. Market systems are decentralized allocation mechanisms and hence they satisfy this requirement, in principle.

In the following we will explore how the market mechanism performs in the context of cultural economics as modeled here and, in particular, under which conditions it is possible to implement the optimal allocation through (a suitably designed) market mechanism. To answer these questions, we now employ the standard welfare economic methodology to study whether and how the optimal intertemporal allocation can be “decentralized by prices”. The market economy we envisage first exhibits a complete set of perfectly competitive markets, some of which will turn out to be purely virtual or fictitious. This is true, in particular, regarding the markets for public goods. In the theoretical literature on public goods (Lindahl, 1919; Samuelson, 1954; Roberts, 1974), Lindahl markets represent a well-established fictitious market concept for public goods that is dual to the concept of perfectly competitive markets for private goods in the sense that the role of prices and quantities is interchanged: In a private good market all demanders face the same price and demand, in general, different amounts of the good whereas in a public-good Lindahl market all demanders buy the same amount of the public good (in equilibrium, at least), and this result is brought about by appropriate personalized prices that differ across consumers, in general.

We do not claim that the market economy with a complete set of perfectly competitive markets (comprising Lindahl markets for all public goods) is an appropriate description of the real world. Rather the study of this market scenario serves as a benchmark for the later investigation of the allocative displacement effects that are generated when some of the (virtual) markets fail to exist. As in section 3 we deal with two versions of the model separately. We distinguish the cases when cultural-goods inputs and cultural services are public (BM1) and when both are private (BM2). We begin our investigation with the model BM1.

4.1 The benchmark market economy with public goods (BM1)

Suppose that cultural-goods inputs and cultural services are public. The market economy BM1 we are now going to describe is made up of five different types of agents. All of them are price takers and we characterize them, in what follows, by their market transactions and the optimization problems they solve. By doing so it will also be clarified, successively, which markets are active in the model BM1.

- Consumer-artist *i* carries out the following transactions:
  - She sells her resource endowment $\overline{r}_i$ at price p_r and buys back her own demand for the resource, r_{vi}, at the same price, to create new cultural goods.
  - She sells her newly created cultural goods, v_i, to firm G (to be specified below) at price p_v[39].
  - She buys the amount g_i of cultural goods from firm G at the personalized price p_{gi}, $i = 1,...,n_c$; in our interpretation of (2.1) we referred to the argument g_i in the utility function as reflecting the consumer-artist's passive-use of cultural goods; in the present context, g_i will be modeled as the consumer-artist's decision variable thus representing an "active use" in terms of the formal model.
  - She buys the amount s_i of cultural services for own consumption at the personalized price p_{si}, $i = 1, \ldots, n_c$, and sells the amount s_{iK} of cultural services consumed to the firm K (to be specified below) at the (uniform) price p_{sK}.

[39] In the real world, consumer-artists often sell their newly created cultural goods directly to cultural-services producers. For example, galleries purchase artwork from artists without firm G as an intermediary. We rule out such transactions to keep the analysis simple.

- She buys the amount k_i of cultural capital from firm K at the personalized price p_{ki}, $i = 1,...,n_c$; like g_i (see above) k_i is treated here as the consumer-artist's endogenous decision variable; observe also that the consumer-artist purchases cultural capital for two different reasons: for direct consumption and as an input for producing new cultural goods; at any given price p_{ki} consumer-artist *i*'s demand for cultural capital will reflect both demand motives.
- She buys private consumer goods, y_i, at price p_y.

All these transactions listed above are subject to the budget constraint

$$p_{sK} s_{iK} + p_v v_i + p_r \overline{r}_i + \pi_i \geq p_{gi} g_i + p_{ki} k_i + p_r r_{vi} + p_{si} s_i + p_y y_i, \tag{4.1}$$

where π_i is consumer-artist *i*'s share of profits, taken as constant by her.

Our preceding discussion of markets as condensed in (4.1) reveals several quite unusual features of the market economy BM1. Observe first that the markets for consumer goods (p_y), for the resource (p_r) and for new cultural goods (p_v) are conventional perfectly competitive markets for private goods. Consumed cultural services supplied by *i* to firm *K*, s_{iK}, are also private goods, but the notion of these goods being marketed (p_{sK}) has no counterpart in the real world. All other markets are Lindahl markets for public goods (g_i, k_i, s_i) with personalized prices (p_{gi}, p_{ki}, p_{si}) which need to be determined in such a way that all demanders wish to purchase exactly that amount of the good which is supplied.

Consumer-artist *i* aims at maximizing the present value of her utility

$$\underset{(g_i, k_i, r_i, s_i, s_{iK}, v_i, y_i)}{Max} \int_0^{\infty} U^i(g_i, k_i, s_i, v_i, y_i) e^{-\delta t} dt,$$

$$\text{subject to} \qquad v_i = V^i(r_{vi}, k_i), \quad s_{iK} \leq s_i \qquad \text{and} \qquad (4.1). \tag{4.2}$$

The pertinent Hamiltonian reads

$$H^C = U^i\left(g_i, k_i, s_i, v_i, y_i\right) + \beta_{vi}\left[V^i\left(r_{vi}, k_i\right) - v_i\right] + \beta_S\left(s_i - s_{iK}\right)$$
$$+\beta_i\left[p_{sK}s_{iK} + p_v v_i + p_r \overline{r_i} + \pi_i - p_{gi}g_i - p_{ki}k_i - p_r r_{vi} - p_{si}s_i - p_y y_i\right], \qquad (4.3)$$

where β_{vi}, β_S and β_i are Lagrange multipliers. In case of an interior solution, the FOCs are

$$\frac{\partial H^C}{\partial g_i} = U_g^i - \beta_i p_{gi} = 0, \qquad (4.4)$$

$$\frac{\partial H^C}{\partial k_i} = U_k^i + \beta_{vi}V_k^i - \beta_i p_{ki} = 0, \qquad (4.5)$$

$$\frac{\partial H^C}{\partial s_i} = U_s^i + \beta_S - \beta_i p_{si} = 0, \qquad (4.6)$$

$$\frac{\partial H^C}{\partial s_{iK}} = -\beta_S + \beta_i p_{sK} = 0, \qquad (4.7)$$

$$\frac{\partial H^C}{\partial v_i} = U_v^i - \beta_{vi} + \beta_i p_v = 0, \qquad (4.8)$$

$$\frac{\partial H^C}{\partial y_i} = U_y^i - \beta_i p_y = 0, \qquad (4.9)$$

$$\frac{\partial H^C}{\partial r_{vi}} = \beta_{vi}V_r^i - \beta_i p_r = 0. \qquad (4.10)$$

● <u>Firm Y</u> buys the resource, r_y, at price p_r, produces the consumer goods *y*, and sells them to the consumer-artists at price p_y. Firm Y aims at maximizing the present value of its profit,

$$\underset{\left(y, r_y\right)}{Max}\int_0^\infty \left(p_y y - p_r r_y\right)e^{-\delta t}dt, \qquad \text{subject to (2.6).} \qquad (4.11)$$

Firm Y's optimization calculus is to solve the Hamiltonian:

$$H^Y = p_y y - p_r r_y + \beta_y\left[Y\left(r_y\right) - y\right], \qquad (4.12)$$

where β_y is a Lagrange multiplier. In case of an interior solution, the FOCs read:

$$\frac{\partial H^Y}{\partial y} = p_y - \beta_y = 0, \qquad (4.13)$$

$$\frac{\partial H^Y}{\partial r_y} = -p_r + \beta_y Y_r = 0, \qquad (4.14)$$

and hence obviously

$$p_r = p_y Y_r, \qquad (4.15)$$

which is the well-known condition of pricing the resource according to its marginal productivity.

- Firm j, $j = 1,...,n_s$, produces cultural services, s_j. It buys the resource input, r_{sj}, at price p_r, and the cultural-goods input (taken from the stock of cultural goods), g_j, at the personalized (Lindahl) price p_{gj}. Since cultural services are public goods, firm *j* is able and willing to sell all of its output to all consumer-artists simultaneously. The demand price of cultural services, as introduced above, is p_{gi}, for *i* = *1*, …, n_c. Hence firm *j*'s revenue from selling one and the same unit of its output to all demanders is $\sum_i p_{si}$. In what follows it is analytically convenient to assume that firm *j*'s supply of cultural services, s_j, is (intended to be) sold to all n_c consumer-artists at some (aggregate) supply price p_s. As will be shown further below, a necessary equilibrium condition will then turn out to be $p_s = \sum_i p_{si}$. Moreover, all personalized prices p_{si} must take on values such that each consumer-artist wishes to purchase exactly the aggregate supply, $\sum_j s_j$, of all cultural-services firms. Firm *j* maximizes the present value of its profit:

$$\underset{(g_j, r_{sj}, s_j)}{Max} \int_0^\infty \left(p_s s_j - p_r r_{sj} - p_{gj} g_j \right) e^{-\delta t} dt, \qquad \text{subject to (2.4).} \qquad (4.16)$$

The pertaining optimal control is attained by solving the Hamiltonian:

$$H^j = p_s s_j - p_r r_{sj} - p_{gj} g_j + \beta_{sj} \left[S^j \left(g_j, r_{sj} \right) - s_j \right], \qquad (4.17)$$

where β_{sj} is a Lagrange multiplier. In case of an interior solution, the associated FOCs read

$$\frac{\partial H^j}{\partial s_j} = p_s - \beta_{sj} = 0\,, \tag{4.18}$$

$$\frac{\partial H^j}{\partial r_{sj}} = -p_r + \beta_{sj} S_r^j = 0\,, \tag{4.19}$$

$$\frac{\partial H^j}{\partial g_j} = -p_{gj} + \beta_{sj} S_g^j = 0\,. \tag{4.20}$$

(4.18) - (4.20) readily yield the standard conditions of marginal-productivity pricing

$$p_r = p_s S_r^j \quad \text{and} \quad p_{gj} = p_s S_g^j\,. \tag{4.21}$$

● <u>Firm G</u> purchases *new* cultural goods, v_G, at price p_v from consumer-artists and sells cultural goods, g_G, from the stock of cultural goods, *g*, to all cultural-services firms and to all consumer-artists,[40] where g_G is now firm G's decision variable and *g* is the state variable. (In equilibrium the condition $g_G = g$ needs to be satisfied). Recall that p_{gj} for $j = 1,\ldots, n_s$ is the price firm *j* pays for each unit of cultural goods purchased. Hence firm G accrues the revenue $\sum_j p_{gj}$ per unit of cultural goods sold to all cultural-service firms. Likewise consumer-artist *i* buys a unit of cultural goods from firm G at the personalized price, p_{gi}, $i = 1, \ldots, n_c$. It follows then that firm G obtains the revenue $\sum_i p_{gi}$ per unit of cultural goods sold to all consumer-artists. Hence if firm G sells a unit of cultural goods to each and every cultural-services firm *j* and to each and every consumer-artist *i*, its total revenue is $\sum_j p_{gj} + \sum_i p_{gi}$. Obviously the argument is essentially like that applied above to the market of cultural services. It suffices, therefore, to introduce an aggregate supply price, p_g, for firm G which will need to satisfy $p_g = \sum_i p_{gi} + \sum_j p_{gj}$ in equilibrium. With this set-up, firm G maximizes the present value of its profit:

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40 The sales of cultural goods from the stock of cultural goods are not meant to imply that firm G transfers the property right to the buyers. Rather, firm G only allows the buyers to use the cultural goods (without damaging or deteriorating them) at the point in time under consideration.

$$\underset{(g_G, v_G)}{Max} \int_0^\infty \left(p_g g_G - p_v v_G \right) e^{-\delta t} dt, \qquad \text{subject to} \quad \dot{g} = v_G - \alpha_g g \quad \text{and} \quad g_G \leq g. \quad (4.22)$$

The associated Hamiltonian reads:

$$H^G = p_g g_G - p_v v_G + \varphi_g \left(v_G - \alpha_g g \right) + \beta_G \left(g - g_G \right), \quad (4.23)$$

where φ_g is the co-state variable associated to the state variable g and where β_G is a Lagrange multiplier. In case of an interior solution, the FOCs are

$$\frac{\partial H^G}{\partial v_G} = -p_v + \varphi_g = 0, \quad (4.24)$$

$$\frac{\partial H^G}{\partial g_G} = p_g - \beta_G = 0, \quad (4.25)$$

$$\dot{\varphi}_g = \delta \varphi_g - \frac{\partial H^G}{\partial g} = \left(\delta + \alpha_g \right) \varphi_g - \beta_G = \left(\delta + \alpha_g \right) p_v - p_g. \quad (4.26)$$

● <u>Firm K</u> is a fictitious agent, who buys cultural services consumed by the consumer-artists, s_K, at price p_{sK} and sells the cultural capital, k_K from the stock of cultural capital, *k*, at the aggregate supply price p_k. That price needs to satisfy the condition $p_k = \sum_i p_{ki}$ in equilibrium, as argued before in the context of markets for cultural goods and cultural services. Firm K maximizes the present value of its profits,

$$\underset{(k_K, s_K)}{Max} \int_0^\infty \left(p_k k_K - p_{sK} s_K \right) e^{-\delta t} dt, \qquad \text{subject to} \quad \dot{k} = s_K - \alpha_k k \quad \text{and} \quad k_K \leq k. \quad (4.27)$$

Technically speaking, k_K belongs to firm K's control variables whereas *k* is cultural capital as a state variable. Note also that the condition $k_K = k$ needs to be satisfied in equilibrium. Firm K sells k_K to the consumer-artists. In view of (4.27), firm K can be interpreted as a public enterprise maximizing the present value of the intangible asset "cultural capital" k_K. The pertinent Hamiltonian is

$$H^K = p_k k_K - p_{sK} s_K + \varphi_k \left(s_K - \alpha_k k \right) + \beta_K \left(k - k_K \right), \quad (4.28)$$

where φ_k is the co-state variable associated to the state variable *k* and where β_K is a Lagrange multiplier. In case of an interior solution, the FOCs turn out to be

$$\frac{\partial H^K}{\partial s_K} = -p_{sK} + \varphi_k = 0\,, \tag{4.29}$$

$$\frac{\partial H^K}{\partial k_K} = p_k - \beta_K = 0\,, \tag{4.30}$$

$$\dot{\varphi}_k = \delta\varphi_k - \frac{\partial H^K}{\partial k} = \left(\delta + \alpha_k\right)\varphi_k - \beta_K = \left(\delta + \alpha_k\right)p_{sK} - p_k\,. \tag{4.31}$$

For the purpose of providing a rigorous definition of a general competitive equilibrium in the market economy BM1, it is convenient to introduce the following notation:

At any point in time, an allocation is represented by the vector

$$a_{BM1} := \left[g_G, (g_i), (g_j), (k_i), k_K, r_y, (\overline{r}_i), (r_{vi}), (r_{sj}), (s_i), (s_{iK}), (s_j), s_K, (v_i), v_G, (y_i), y\right], \tag{4.32}$$

where $(x_i) := (x_1, x_2, ..., x_{n_c})$ for $x = g, k, r, s, s_K, v, y$ and $(z_j) := (z_1, z_2, ..., z_{n_s})$ for $z = g, r_s, s$.

At any point in time, the prices (one for each and every market) are represented by the vector

$$p_{BM1} := \left[(p_{gi}), (p_{gj}), p_g, (p_{ki}), p_k, p_r, (p_{si}), p_s, p_{sK}, p_v, p_y\right], \tag{4.33}$$

where $(p_{li}) := (p_{l1}, p_{l2}, ..., p_{ln_c})$ for $l = g, k, s$ and $(p_{gj}) := (p_{g1}, p_{g2}, ..., p_{gn_s})$.

Definition 4.1

In economy BM1, for each point in time a general competitive equilibrium with a complete set of markets is constituted by an allocation a_{BM1} and prices p_{BM1} such that

(i) *the allocation a_{BM1} is a solution for prices p_{BM1} to the optimization programs of consumer-artists* (4.2), *firm* Y (4.11), *cultural-services firms* (4.16), *firm* G (4.22) *and firm* K (4.27)*;*

(ii) *the prices* p_{BM1} *have the properties:*

$$p_g = \sum_i p_{gi} + \sum_j p_{gj} \, , \tag{4.34}$$

$$p_k = \sum_i p_{ki} \, , \tag{4.35}$$

$$p_s = \sum_i p_{si} \, ; \tag{4.36}$$

(iii) *the allocation* a_{BM1} *satisfies the supply constraints* (2.7), (2.8) *and the inequalities:*

$$g_G \geq g_i \qquad i = 1,...,n_c \, , \tag{4.37}$$

$$g_G \geq g_j \qquad j = 1,...,n_s \, , \tag{4.38}$$

$$g \geq g_G \, , \tag{4.39}$$

$$k_K \geq k_i \qquad i = 1,...,n_c \, , \tag{4.40}$$

$$k \geq k_K \, , \tag{4.41}$$

$$\sum_j s_j \geq s_i \qquad i = 1,...,n_c \, , \tag{4.42}$$

$$\sum_i s_{iK} \geq s_K \, , \tag{4.43}$$

$$\sum_i v_i \geq v_G \, . \tag{4.44}$$

To sum up, in this subsection we consider a perfectly competitive market economy in which the demanders and suppliers interact through a complete set of markets. The market for cultural services between consumer-artists (suppliers) and firm K (demander) has no equivalent in the real world. The Lindahl markets for cultural services, cultural goods and cultural capital are also artificial since they presuppose the revelation of private information on individual characteristics on the part of the demanders. We will take up this issue again further below. Now we investigate how this hybrid market equilibrium fares in terms of allocative efficiency.[41] To find out we follow the standard procedure of comparing the marginal conditions of the efficient regime, the equations (2.17) - (2.29) in section 2.1, with the marginal conditions (4.4) - (4.10), (4.13) - (4.14), (4.18) - (4.20), (4.24) - (4.26) and (4.29) - (4.31) derived above. The result is summarized in

[41] It would also be important, in the first place, to secure the existence of such an equilibrium. We conjecture that an equilibrium can be shown to exist but a rigorous existence proof is beyond the scope of the present study.

Proposition 4.1 (Efficiency of equilibrium in economy BM1)

Set $p_y = \lambda_y \equiv 1$, $p_{gi} = \frac{U_g^i}{U_y^i}\ \forall\, i$, $p_{gj} = \lambda_{gj}\ \forall\, j$, $p_{ki} = \frac{U_k^i}{U_y^i} + \frac{\lambda_r V_k^i}{V_r^i}\ \forall\, i$, $p_s = \lambda_{sj}\ \forall\, j, p_{si} = \lambda_{\sigma i}\ \forall\, i$,

$p_{sK} = \mu_k$, $p_v = \mu_g$, $p_r = \lambda_r$, $p_g = \sum_i \frac{U_g^i}{U_y^i} + \sum_j \frac{\lambda_{gj}}{\lambda_y}$ *and* $p_k = \sum_i \left(\frac{U_k^i}{U_y^i} + \frac{\lambda_{vi} V_k^i}{\lambda_y} \right)$, *where all terms on the right side of the equations are evaluated at the solution of maximizing* (2.15) *subject to* (2.2) - (2.12). *Then at each point in time a general competitive equilibrium is attained in economy BM1 and the associated allocation is efficient.*

Proposition 4.1 will be proved with the help of Table 4.1. Observe first that in column 1 of Table 4.1 the optimality conditions (2.17) - (2.29) are listed except for the equations (2.18) and (2.22), which provide the information $U_y^i = \lambda_y$. This equation is implicitly considered in column 1 of Table 4.1 in that both sides of all equations in that column are divided by λ_y (or U_y^i). To avoid clutter we slightly abuse the notation by writing

$$\frac{\lambda_r}{\lambda_y} = \lambda_r\,,\ \frac{U_k^i}{U_y^i} = U_k^i \text{ and } \frac{\dot{\mu}_g}{\lambda_y} = \dot{\mu}_g \text{ etc.}$$

Consider next the second column which lists all marginal conditions in the market economy BM1 except for equation (4.9): $U_y^i = \beta_i p_y$. Similar to our treatment of the first column of Table 4.1 we divide by $\beta_i p_y$ (or U_y^i) both sides of the equations contained in the lines 4 through 7 in the second column of Table 4.1. This operation has two effects: First, U_w^i in these lines really represents the marginal rate of substitution $\left(U_w^i / U_y^i\right)$ for $w = g_i, k_i, s_i$ and v_i. Moreover, β_i vanishes or, equivalently, is set equal to one. (This normalization needs to be kept in mind in our subsequent discussion).

With these explanatory comments on Table 4.1 the proof of *Proposition 4.1* is now straightforward. It suffices to replace in the second column of Table 4.1 all prices by the Lagrange multipliers or co-state variables that have been assigned to those prices in *Proposition 4.1*. Obviously, this operation makes the second column of Table 4.1 coincide with the first column, line by line. As a consequence, the market allocation is an equilibrium allocation and it is Pareto efficient.

It remains to be shown that the equilibrium conditions (4.34) - (4.36) are also satisfied. In view of $p_{gi} = U_g^i$ and $p_{gj} = \lambda_{gj}$ we obviously have

$$p_g = \sum_i U_g^i + \sum_j \lambda_{gj} = \sum_i p_{gi} + \sum_j p_{gj} .$$

Likewise, the definition of p_k in *Proposition 4.1* yield $p_k = \sum_i p_{ki}$, and hence $p_s = \sum_i p_{si}$ follows from $\lambda_{sj} = \sum_i \lambda_{\sigma i}$, $p_s = \lambda_{sj}$ and $p_{si} = \lambda_{\sigma i}$. This completes the proof of *Proposition 4.1*.

Table 4.1: Comparison of rules governing a socially optimal allocation and an equilibrium in the market economy BM1

	GM1		BM1	
	1		2	
1	$U_k^i = \lambda_{ki} - \lambda_r \left(V_k^i / V_r^i \right)$	(2.25) (2.27)	$U_k^i = \beta_i p_{ki} - \beta_i p_r \left(V_k^i / V_r^i \right)$	(4.5) (4.10)
2	$U_g^i = \lambda_{gi}$	(2.26)	$U_g^i = \beta_i p_{gi}$	(4.4)
3	$U_v^i V_r^i = \lambda_r - \mu_g V_r^i$	(2.19) (2.25)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.10)
4	$U_s^i = \lambda_{\sigma i} - \mu_k$	(2.17)	$U_s^i = \beta_i p_{si} - \beta_i p_{sK}$	(4.6) (4.7)
5	$\lambda_r = \sum_i \lambda_{\sigma i} S_r^j$	(2.20) (2.23)	$p_r = p_s S_r^j$	(4.21)
6	$\lambda_{gj} = \sum_i \lambda_{\sigma i} S_g^j$	(2.20) (2.24)	$p_{gj} = p_s S_g^j$	(4.21)
7	$\dot{\mu}_g = \left(\delta + \alpha_g \right) \mu_g - \sum_i \lambda_{gi} - \sum_j \lambda_{gj}$	(2.28)	$\dot{\varphi}_g = \left(\delta + \alpha_g \right) p_v - p_g$	(4.26)
8	$\dot{\mu}_k = \left(\delta + \alpha_k \right) \mu_k - \sum_i \lambda_{ki}$	(2.29)	$\dot{\varphi}_k = \left(\delta + \alpha_k \right) p_{sK} - p_k$	(4.31)
9	$\lambda_r = \lambda_y Y_r$	(2.21)	$p_r = p_y Y_r$	(4.15)

Proposition 4.1 provides two important pieces of information. First, it shows that our benchmark market economy is capable to support an efficient allocation and second, it demonstrates how prices guide the allocation efficiently. All prices are positive (in case of an interior solution).

It is worth recalling how the equilibrium Lindahl prices are fixed. Conceptually the personalized price must be set equal to the agent's willingness-to-pay for the last unit of the public good under consideration. To be more specific, consider first

$$p_{gj} = p_s S_g^j = \frac{p_r S_g^j}{S_r^j},$$

from (4.21). The far right side of this equation is firm *j*'s cost savings from a marginal substitution of the resource by cultural goods which leaves the output unchanged. This cost savings exactly equals the firm's willingness-to-pay for the last unit of cultural goods. The equation

$$p_{gi} = \frac{p_y U_g^i}{U_y^i}$$

from (4.4) and (4.9) is straightforward, too. The price consumer-artist *i* pays for her passive use of the stock of cultural goods equals her marginal willingness-to-pay. As mentioned above, consumer-artists benefit from cultural capital in two different ways. Cultural capital increases the consumer-artist's utility $\left(U_k^i > 0\right)$ and her productivity in generating new cultural goods $\left(V_k^i > 0\right)$. Combining (4.5), (4.9) and (4.10) consequently yields

$$p_{ki} = \frac{p_y U_k^i}{U_y^i} + \frac{p_r V_k^i}{V_r^i}.$$

The right side of this equation indicates that her (total) marginal willingness-to-pay for cultural capital is the sum of her marginal willingness-to-pay as a consumer $\left(p_y U_k^i / U_y^i\right)$ and as a producer $\left(p_r V_k^i / V_r^i\right)$. In equilibrium, the consumer-artist's personalized price for cultural services is given by

$$p_{si} = \frac{p_y U_s^i}{U_y^i} + p_{sK} > \frac{p_y U_s^i}{U_y^i},$$

due to (4.6), (4.7) and (4.9). In other words, the price, consumer-artist *i* pays for consuming cultural services exceeds her marginal willingness-to-pay for these services implying, under standard concavity conditions, that she consumes more cultural services than she would do when following the $p_{si} = p_y \left(U_s^i / U_y^i\right)$ rule. The deviation from this rule is easily explained. After the consumer-artist has purchased and consumed cultural services she resells them to firm K. The revenue from this sale amounts to an effective reimbursement, in part, of her up-front expenditures for cultural services. Hence $p_{si} - p_{sK}$ is the net price she really pays for her consumption of cultural goods[42], and that net price is in fact equal to the marginal willingness-to-pay, U_s^i / U_y^i.

A final remark relates to the assignments $p_v = \mu_g$ and $p_{sK} = \mu_k$. In the optimal-control program of the social planner, μ_g and μ_k are co-state variables, i. e. the shadow prices, of the stock of cultural goods $\left(\mu_g\right)$ and the stock of cultural capital $\left(\mu_k\right)$ in economic interpretation. In the market economy BM1 these stocks are not directly priced. But new cultural goods serve as an investment into the stock of cultural goods. Hence $p_v = \mu_g$ means that the stock of cultural goods is valued through the price of its investment good. The same observation applies to the assignment $p_{sK} = \mu_k$.

4.2 The benchmark market economy with private goods (BM2)

We now assume cultural-goods inputs and cultural services are private goods. Consequently, each consumer-artist faces the same price for cultural services which is equal to the supply price: $p_{si} = p_s$ for all $i = 1, \ldots, n_c$. Likewise, all cultural-services firms now face the same price for cultural goods $p_{gj} = p_{gS}$ for all $j = 1, \ldots, n_s$. Note, however, that the Lindahl markets on which consumer-artists purchase cultural goods (for passive use) from firm G still prevail since in that respect cultural goods remain public. It follows that the price vector associated to the economy BM2 is given by

$$p_{BM2} := \left[\left(p_{gi}\right), p_{gS}, p_g, \left(p_{ki}\right), p_k, p_r, p_s, p_{sK}, p_v, p_y \right]. \tag{4.45}$$

[42] One could also call that reimbursement procedure a deposit-refund scheme.

Obviously, p_{BM2} is derived from p_{BM1} by adding p_{gS} and deleting $\left(p_{gj}\right)$ and $\left(p_{si}\right)$. Observe also that an allocation in economy BM1 has the same structure as that in BM1, which is not to say, of course, that a_{BM1} and a_{BM2} coincide in each of their components.

A brief review of the agent's optimization programs reveals that (4.11), (4.22) and (4.27) remain unchanged. However, the optimization calculus of consumer-artists (4.2) and cultural-services firm (4.16), respectively, are now modified. (4.2) is replaced by

$$\underset{(g_i,k_i,r_{vi},s_i,s_{iK},v_i,y_i)}{Max}\int_0^\infty U^i\left(g_i,k_i,s_i,v_i,y_i\right)e^{-\delta t}dt,$$

subject to $v_i = V^i\left(r_{vi},k_i\right)$, $\quad s_{iK} \le s_i$ and

$$p_{sK}s_{iK}+p_v v_i+p_r\overline{r}_i+\pi_i \ge p_{gi}g_i+p_{ik}k_i+p_r r_{vi}+p_s s_i+p_y y_i. \tag{4.46}$$

The associated Hamiltonian reads

$$\begin{aligned}H^C &= U^i\left(g_i,k_i,s_i,v_i,y_i\right)+\beta_{vi}\left[V^i\left(r_{vi},k_i\right)-v_i\right]+\beta_S\left(s_i-s_{iK}\right)\\ &+\beta_i\left[p_{sK}s_{iK}+p_v v_i+p_r\overline{r}_i+\pi_i-p_{gi}g_i-p_{ki}k_i-p_r r_{vi}-p_s s_i-p_y y_i\right].\end{aligned} \tag{4.47}$$

In case of an interior solution, the FOCs are (4.4), (4.5), (4,7), (4.8), (4.9), (4.10) and

$$\frac{\partial H^c}{\partial s_i}=U_s^i+\beta_S-\beta_i p_s=0. \tag{4.48}$$

For the cultural-services firm (4.16) is now replaced by

$$\underset{(g_j,r_{sj},s_j)}{Max}\int_0^\infty\left(p_s s_j-p_r r_{sj}-p_{gS}g_j\right)e^{-\delta t}dt, \qquad \text{subject to (2.4).} \tag{4.49}$$

The associated Hamiltonian is

$$H^j = p_s s_j-p_r r_{sj}-p_g g_j+\beta_{sj}\left[S^j\left(g_j,r_{sj}\right)-s_j\right]. \tag{4.50}$$

In case of an interior solution, the associated FOCs read (4.18), (4.19) and

$$\frac{\partial H^j}{\partial g_j} = -p_g + \beta_{sj} S_g^j = 0 \,. \tag{4.51}$$

Definition 4.2

In economy BM2, a general competitive equilibrium with a complete set of markets is constituted by an allocation a_{BM2} *and prices* p_{BM2} *for each point in time such that*

(i) the allocation a_{BM2} *is a solution for prices* p_{BM2} *to the optimization programs of firm* Y (4.11), *firm* G (4.22), *firm* K (4.27), *consumer-artists* (4.46) *and cultural-services firms* (4.49)*;*

(ii) the prices have the properties

$p_g = \sum_i p_{gi} + p_{gS}$,

$p_k = \sum_i p_k$;

(iii) the allocation a_{BM2} *satisfies the supply constraints* (2.7), (2.8), (2.13), (2.14), (4.37) *and* (4.39) *through* (4.41).

We proceed as with economy BM1 in the previous subsection by investigating the efficiency properties of an equilibrium:

Proposition 4.2

Set all prices as in Proposition 4.1 with the following changes:

$p_{gj} = \lambda_{gj} \forall \; j$ *is replaced by* $p_g = \lambda_g$,

$p_{si} = \lambda_{\sigma i} \forall \; i$ *is replaced by* $p_s = \lambda_\sigma$,

$p_g = \sum_i \lambda_{gi} + \sum_j \lambda_{gj}$ *is replaced by* $p_{gS} = \sum_i \lambda_{gi} + \lambda_g$.

Then at each point in time a general competitive equilibrium is attained in economy BM2 and the associated allocation is efficient.

Proposition 4.2 is proved along the same lines as *Proposition 4.1*. We solve the constrained maximization problems listed in point (i) of *Definition 4.2* and compare the marginal conditions thus derived with the efficiency conditions derived in section 2.2.

Since the calculations are standard it suffices to present the results in Table 4.2. It turns out, in fact, that the set of marginal conditions to be compared is the same as that listed in Table 4.1 except for modifying the rows 4, 5, 6 and 7 as shown in Table 4.2.

Table 4.2: Comparison of rules governing a socially optimal allocation and an equilibrium in economy BM2

	GM2		BM2	
	1		2	
1	$U_k^i = \lambda_{ki} - \lambda_r \left(V_k^i / V_r^i\right)$	(2.25) (2.27)	$U_k^i = \beta_i p_{ki} - \beta_i p_r \left(V_k^i / V_r^i\right)$	(4.5) (4.10)
2	$U_g^i = \lambda_{gi}$	(2.26)	$U_g^i = \beta_i p_{gi}$	(4.4)
3	$U_v^i V_r^i = \lambda_r - \mu_g V_r^i$	(2.19) (2.25)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.10)
4	$U_s^i = \lambda_\sigma - \mu_k$	(2.42)	$U_s^i = \beta_i p_s - \beta_i p_{sK}$	(4.7) (4.48)
5	$\lambda_r = \lambda_\sigma S_r^j$	(2.23) (2.43)	$p_r = p_s S_r^j$	(4.21)
6	$\lambda_g = \lambda_\sigma S_g^j$	(2.20) (2.44)	$p_g = p_s S_g^j$	(4.51)
7	$\dot{\mu}_g = \left(\delta + \alpha_g\right) \mu_g - \sum_i \lambda_{gi} - \lambda_g$	(2.45)	$\dot{\varphi}_g = \left(\delta + \alpha_g\right) p_v - p_g$	(4.26)
8	$\dot{\mu}_k = \left(\delta + \alpha_k\right) \mu_k - \sum_i \lambda_{ki}$	(2.29)	$\dot{\varphi}_k = \left(\delta + \alpha_k\right) p_{sK} - p_k$	(4.31)
9	$\lambda_r = \lambda_y Y_r$	(2.21)	$p_r = p_y Y_r$	(4.15)

5 Missing markets and efficiency-restoring cultural tax-subsidy policies

In section 4 we showed that the optimal intertemporal allocation of the general public-goods model (GM1) and the general private-goods model (GM2) can be "decentralized by prices" by means of market systems with competitive markets for all commodities which include, in particular, Lindahl markets for cultural goods, cultural services and cultural capital. These market systems have been denoted BM1 and BM2, respectively, and they rely on Lindahl markets with personalized prices for the public goods. The number of public goods varies across types of models but they are present in all of them. If the demand side of a Lindahl market consists of heterogeneous agents (consumers or firms) all demanders face different personalized prices, in general, since in equilibrium the Lindahl price must be equal to the individual demander's marginal willingness-to-pay for the amount of the public good supplied (to everybody). However, under the realistic assumption that information on preferences and technology is private, Lindahl markets cannot function smoothly unless one implicitly assumes that all agents reveal their characteristics (preferences or technologies) truthfully. Therefore, the important question to ask is whether truthful revelation is in the demanders' self-interest. Unfortunately the answer is no, since all agents have an incentive to misrepresent their characteristics. When they under-report their true willingness-to-pay (which cannot be detected by the supplier since individual characteristics are private information by assumption) they can consume the same amount of the public good but pay less. This so-called free-rider behavior has already forcefully been pointed out by Samuelson (1954, pp. 388n.) who observed that "…it is in the selfish interest of each person to give false signals [of her/his willingness-to-pay, the author], to pretend to have less interest in a given collective consumption activity than he really has". The issue of free riding was further pursued by Samuelson (1969), Musgrave (1959) and Roberts (1974) and others and has since then led to a large literature on preference revelation and mechanisms to induce agents to truthfully reveal their private information.[43]

To sum up, Lindahl markets as studied in section 4 presuppose that consumer-artists and cultural-services firms reveal their willingness-to-pay for public goods (cultural services and

[43] Vickrey (1961) suggested to induce individuals to reveal correct information by paying them the net increase in the sum of producer and consumer surpluses of the other persons in the market that resulted from the supply and demand curves revealed. Clarke (1971, 1972) and Groves (1970, 1973) discovered and developed this procedure independently.

cultural goods) truthfully. But since underreporting of their willingness-to-pay is in these agents' self-interest, they have an incentive not to report truthfully. Rational demanders in a Lindahl market seek to free ride implying that the market breaks down. Lindahl markets are, in fact, artificial markets that function only in a world of Kantian truth tellers which has no resemblance with the real world. In fact, Lindahl markets for public goods cannot be observed in reality. To move towards a more realistic setting we therefore modify the market economies of section 4 by assuming that there are no Lindahl markets while all other competitive markets are still active and function smoothly. In these modified market economies we can still identify states of equilibria, i.e. prices which clear all (existing) markets. But intuition leads us to conjecture that the pertaining equilibrium allocations will not be efficient.

In the following analysis we will employ the assumption that the Lindahl markets for the consumer-artists' (passive) use of cultural goods and cultural capital are absent, while the Lindahl markets for the other public goods - if any - remain active. One may cast into doubt the realism of the remaining Lindahl markets. Yet we defend our procedure on the grounds that in order not to blur the analysis by trying to deal with too many complex allocation problems simultaneously, we need to tackle them step by step. We will hence assume that the Lindahl markets for the consumer-artist's use of cultural goods and cultural capital are absent. These models will be marked by BL (**B**reakdown of **L**indahl markets). Depending on the cultural-goods inputs and cultural services being public or private, two cases are to be distinguished: BL1 stands for *public* and BL2 stands for *private*. For each model, BL1 and BL2, we distinguish two different types of consumer-artists' behavior, **I**gnorant behavior from **N**ash behavior. This distinction will be specified further below. The associated submodels are denoted BLI1, BLN1, BLI2 and BLN2, respectively. Table 5.1 provides an overview of the types of models to be scrutinized and the associated acronyms.

Table 5.1 Classification of market models

<table>
<tr><td colspan="2"></td><td>Public-goods market economy</td><td>Private-goods market economy</td></tr>
<tr><td colspan="2">Complete set of markets including Lindahl markets for all public goods</td><td>BM1</td><td>BM2</td></tr>
<tr><td rowspan="3">As above, but all Lindahl markets for consumer-artists are absent</td><td>Type of model</td><td>BL1</td><td>BL2</td></tr>
<tr><td>Model with Nash behavior</td><td>BLN1</td><td>BLN2</td></tr>
<tr><td>Model with ignorant behavior</td><td>BLI1</td><td>BLI2</td></tr>
</table>

In its first line, Table 5.1 lists the models of section 4 and then conveniently identifies the main characteristics and acronyms of the models to be investigated in the present section.

As argued above, the Lindahl market for cultural capital traded between firm K and consumer-artists, and for cultural goods traded between firm G and consumer-artists have no equivalent in the real world. Allocating cultural capital and cultural goods via any kind of price mechanism is not a practical option since due to the agents' reluctance for truthful preference revelation, correct information on marginal willingness-to-pay cannot be obtained. We therefore assume now that the Lindahl markets for the consumer-artists' (passive) use of cultural goods and cultural capital do not exist. In terms of the formal model, we set

$$p_k = p_{ki} = p_{gi} \equiv 0, \quad \text{for all } i = 1,...,n_c . \tag{5.1}$$

This assumption is to apply irrespective of whether cultural goods as inputs and cultural services are public or private goods. The models BL1 and BL2 differ from the models BM1 and BM2, respectively, only in the condition (5.1). Observe, that BL2 contains no Lindahl markets anymore but that in BL1 there are still Lindahl markets, namely the Lindahl market for cultural goods (as production inputs) traded between firm G and the cultural-services firms and the Lindahl market for cultural services traded between cultural-services firms and consumer-artists.

Before we explore the allocative displacement caused by (5.1) in more detail and analyze the options to restore efficiency by appropriate tax-subsidy schemes, it is useful to investigate the impact of (5.1) on firm K. Consider firm K's optimal-control problem

$$\underset{(k_K, s_K)}{Max} \int_0^\infty \left[\tau_k k_K - \left(p_{sK} + \tau_{sK} \right) s_K \right] e^{-\delta t} dt$$

$$\text{subject to} \quad \dot{k} = s_K - \alpha_k k \quad \text{and} \quad k_K \le k , \tag{5.2}$$

where τ_k and τ_{sK} are tax rates that are unconstrained in sign[44]. Suppose first, $\tau_k = \tau_{sK} = 0$. In this case firm K doesn't receive any revenue (subsidy) from selling its cultural capital. Since firm K's objective is to maximize the present value of its profits, any additional unit of cultural capital "produced" implies negative profit. As a result, firm K's best strategy is to cease

44 $\tau_k > 0$ is a sales subsidy and $\tau_{sK} < 0$ [$\tau_{sK} > 0$] is a subsidy [tax] on the purchase of cultural services. To avoid clutter, we use "tax" as the generic term irrespective of whether τ is positive or negative.

producing altogether. In terms of the formal model, firm K chooses $s_K = 0$, whenever $p_{sK} > 0$. But if $p_{sK} > 0$, consumer-artists will choose $s_{iK} = s_i$ for all *i*, which is positive, in general. Therefore the market for cultural services between firm K and consumer-artists is in excess supply, and it follows that $p_{sK} = 0$ is a necessary equilibrium condition. With $p_{sK} = 0$ (and $\tau_k = \tau_{sK} = 0$) firm K has neither revenues nor costs and consumer-artists are indifferent in their choice of any $s_{iK} \in [0, s_i]$. If $\tau_k = \tau_{sK} = 0$, it is therefore not restrictive to set $s_{iK} = s_i$ for all *i* and $s_K = \sum_i s_i$ such that firm K's activity is completely reduced to the differential equation $\dot{k} = s_K - \alpha_k k$, as known from (2.5).

In principle, firm K could be revitalized by introducing non-zero tax rates. In fact, one could simply set $\tau_k > 0$ and $\tau_{sK} > 0$ to replace the missing market price p_k and p_{sK}, respectively. We will refrain from pursuing this line of analysis in what follows, however, because such a tax-subsidy scheme applied to firm K is an institutional design (of a public agency or public enterprise) that doesn't appear to be in the realm of relevance for practical cultural policy. In other words, we assume in what follows that there is no market anymore for the exchange of cultural services between consumer-artists and firm K implying that firm K is no player anymore in our subsequent models. More precisely, the only "reminder" of firm K will be the differential equation (2.5). The challenge will be to find tax-subsidy schemes, not relying on (non-zero) τ_k and τ_{sK}, to correct for possible misallocations caused by the missing Lindahl markets (cf. (5.1)).

But before we address this policy issue, some other points also need to be clarified. Up to now we haven't specified the response to the missing markets of all those agents who were formerly involved in transactions on those markets.

(a) Consider first the markets for cultural services between consumer-artists and firm K. In the economies BM1 and BM2 (in section 4) the consumer-artist *i* spends the amount of money $p_{si} s_i$ and $p_s s_i$, respectively, on cultural services and receives the "reimbursement" $p_{sK} s_{iK}$. Since $s_{iK} = s_i$ is an equilibrium condition, the consumer-artist's net expenditure on cultural services amounts to $(p_{si} - p_{sK}) s_i$ in BM1 and to $(p_s - p_{sK}) s_i$ in BM2. As argued above, in case of (5.1) firm K doesn't exist anymore in the economies BL1 and BL2, implying $p_{sK} = 0$ (among other things). Yet it will turn out to be important for our subsequent analysis to allow fees for cultural services to deviate from market prices. Therefore we introduce a tax τ_s and assume consumer *i*'s expenditures for cultural services to be $(p_{si} + \tau_{si})s_i$ in BL1 and $(p_s + \tau_s)s_i$ in BL2.

(b) In the absence of a Lindahl market for cultural capital traded between firm K and the consumer-artists and with firm K's disappearance, one can envisiage two conceivable modes of behavior on the part of consumer-artists:

(1) *Ignorant consumer-artists*. All consumer-artists enjoy the "prevailing level of cultural capital" but they fail to understand and hence don't take into account the process of cultural-capital formation. In particular, they totally ignore the impact of their own contributions through consumption of cultural services to the formation of cultural capital. Quite obviously, the larger is the number of consumer-artists, the smaller is a consumer-artist's contribution to the formation of cultural capital and hence the more plausible and realistic it is for consumer-artists to adopt the behavioral pattern of ignorance (cf. Pethig and Cheng, 2002).

(2) *Nash consumer-artists*. All consumer-artists have a full understanding of the process of cultural-capital formation as specified by $\dot{k} = \sum_i s_i - \alpha_k k$. However, from the viewpoint of consumer-artist *i* the cultural-services consumption of all consumer-artists $h \neq i$ is beyond her control. Consumer-artist *i* assumes as given the sum of the consumption of cultural services by all other consumer-artists, $\sum_{h \neq i} s_h$, and accounts for her own contribution to the formation of cultural capital only. Hence in her optimization calculus she considers the differential equation $\dot{k} = s_i + \sum_{h \neq i} s_h - \alpha_k k$ and seeks to give her best reply to any given choice $\sum_{h \neq i} s_h$ of all other consumer-artists. Such a behavioral assumption has been employed in various contexts and models, most prominently in Cournot's duopoly model (1838), in Buchanan's (1968, p. 15) "model of independent adjustments", in Malinvaud's (1969) "subscription models" and, more generally, in non-cooperative game theory using the Nash equilibrium as a solution concept. This kind of behavior will be called *Nash behavior*.

The realism of ignorant or Nash consumer-artists behavior depends mainly on how large the society is. With increasing size of society, the agents are likely to change their behavior from Nash to ignorant. Olson (1965, p.53-65) argues that: "…meetings that involve too many people … cannot make decisions promptly or carefully. …When the number of participants is large, the typical participant will know that his own efforts will probably not make much difference to the outcome…"

(c) In the economies BM1 and BM2 firm G sells cultural goods to two distinct groups of demanders: to the cultural-services firms and to the consumer-artists. As outlined in (5.1) the market between firm G and consumer-artists breaks down. In the absence of the Lindahl market for cultural goods between firm G and consumer-artists, we assume again that consumer-artists behave either ignorant or Nash towards the dynamics of the stock of cultural goods:

(1) *Ignorant consumer-artists*. All consumer-artists now take the prevailing stock of cultural goods as given and enjoy its passive use for free.

(2) *Nash consumer-artists*. All consumer-artists understand the process of the growth of the stock of cultural goods specified by $\dot{g} = \sum_i v_i - \alpha_g g$. However, from the viewpoint of consumer-artist *i*, the cultural-goods creation of all consumer-artists $h \neq i$ is beyond her control. The individual *i* thus assumes as given the sum of new cultural goods created by all other consumer-artists, $\sum_{h \neq i} v_h$, and accounts for her own contribution to the change in the stock of cultural goods only. Hence in her optimization calculus she considers the differential equation $\dot{g} = v_i + \sum_{h \neq i} v_h - \alpha_g g$. She therefore seeks to give her best reply to any given choice $\sum_{h \neq i} v_h$ of all other consumer-artists.

Due to the absence of the Lindahl market for cultural goods between firm G and consumer-artists, the market is reduced to firm G selling cultural goods to the cultural-services firms only. To compensate firm G for the sales revenues foregone, $\sum_i p_{gi} g_G$, we will consider a subsidy τ_g on the price at which firm G sells cultural goods to the cultural-services firms.

Introducing those alternative modes of consumer-artist behavior renders it necessary to distinguish two submodels. As shown in Table 5.1, the economies BM1 and BM2 can be either inhabited by ignorant consumer-artists - in which case the economies are denoted by BLI1 and BLI2 (with I for **I**gnorant consumer-artists), or they are populated by Nash consumer-artists - in which case the economies are referred to as BLN1 and BLN2 (with N for **N**ash consumer-artists). We will proceed by first studying the case of ignorant consumer-artists in the economies BL1 and BL2.

5.1 Ignorant consumer-artists in the economies BL1 and BL2

In their optimization calculus ignorant consumer-artists take as given the "prevailing" stock *k*, implying that k_i and s_{iK} are no longer in the set of their decision variables. The variable s_{iK} is dropped completely (along with the constraint $s_{iK} \leq s_i$) and k_i is replaced by *k*. On the other hand, consumer-artists now get for free their use of cultural capital and their passive use of the stock of cultural goods which they had to pay for in the economies BM1 and BM2. Consequently firm K will not be paid anymore for providing cultural capital to the consumer-artists. In fact, as argued above, firm K can now be safely ignored.

Our subsequent analysis has two focal points: First we wish to demonstrate that in the absence of corrective cultural policies the breakdown of markets causes allocative inefficiency and we aim to characterize the misallocation, as far as possible. We call that situation the no-policy or *laissez-faire* scenario. After that, the natural question is to ask whether and how efficiency can be restored by suitable tax-subsidy schemes. The answer to this question can be expected to differ between public-goods and private-goods economies, and therefore both kinds of models will be taken into account and explored. To avoid repetition, we describe the analytical framework without separating the scenarios of laissez-faire and cultural policies by including in the description of all agents' optimization programs some tax and subsidy rates (that will turn out to be relevant later) right from the beginning. The no-policy scenario is then the special case where all these tax rates are set equal to zero.

5.1.1 The economy BL1 with ignorant consumer-artists (BLI1)

Suppose now cultural goods as inputs in the production of cultural services and cultural services are public goods, and consider first the economy BL1 with ignorant consumer-artists. The optimization program (4.16) of the cultural-services firms carries over from economy BM1 to BLI1, but the decision problems of firm G, consumer-artists and need to be modified as follows:

- Firm G:

$$\underset{(g_G, v_G)}{Max} \int_0^{\infty} \left[\left(p_g + \tau_g \right) g_G - p_v v_G \right] e^{-\delta t} dt \,,$$

subject to $\dot{g} = v_G - \alpha_g g$ and $g_G \le g$, (5.3)

where $p_g = \sum_j p_{gj}$. The associated Hamiltonian reads:

$$H^G = \left[\left(p_g + \tau_g \right) g_G - p_v v_G \right] + \varphi_g \left(v_G - \alpha_g g \right) + \beta_G \left(g - g_G \right), \tag{5.4}$$

where φ_g is the co-state variable in economy BLI1. In case of an interior solution, the FOCs yield $p_g + \tau_g = \beta_g$ and

$$\dot{\varphi}_g = \delta \varphi_g - \frac{\partial H^G}{\partial g} = \left(\delta + \alpha_g \right) \varphi_g - \beta_G \,. \tag{5.5}$$

$$p_v = \varphi_g \,. \tag{5.5'}$$

This specification of firm G's decision problem differs from that in section 4.1 ((4.22) - (4.26)) only through the tax rate τ_g on firm G's sales of cultural goods to the cultural-services firms.

- Consumer-artist *i*:

$$\underset{(r_{vi}, s_i, v_i, y_i)}{Max} \int_0^{\infty} U^i \left(g, k, s_i, v_i, y_i \right) e^{-\delta t} dt$$

subject to $v_i = V^i(r_{vi}, k)$ and $p_v v_i + p_r \overline{r}_i + \pi_i \geq p_r r_{vi} + (p_{si} + \tau_{si}) s_i + p_y y_i$. (5.6)

Note that consumer-artist *i*'s optimization calculus in (5.6) differs from that in (4.2) in some components: The decision variables g_i and k_i in (4.2) are substituted by the state variables *g* and *k* in (5.6), implying that the ignorant consumer-artist now takes as given the prevailing stock of cultural goods and cultural capital. The associated Hamiltonian reads:

$$H^C = U^i(g, k, s_i, v_i, y_i) + \beta_{vi}\left[V^i(r_i, k) - v_i\right] \\ + \beta_i\left[p_v v_i + p_r \overline{r}_i + \pi_i - p_r r_{vi} - (p_{si} + \tau_{si}) s_i - p_y y_i\right]. \quad (5.7)$$

In case of an interior solution, the FOCs are (4.8), (4.9), (4.10) and

$$\frac{\partial H^C}{\partial s_i} = U_s^i - \beta_i(p_{si} + \tau_{si}) = 0. \quad (5.8)$$

For later reference and comparison, the marginal conditions derived above have been enumerated in Table 5.2.

After having characterized the model BLI1 by means of listing all agents' optimization programs and the pertinent marginal conditions, we aim at defining a general competitive equilibrium of this market economy BLI1. For that purpose we introduce the following notation:

$$\tau_{BLI1} := \left[\tau_g, (\tau_{si})\right], \quad (5.9)$$

is the vector of tax rates that have been incorporated in the optimization programs (5.3) and (5.6). The relevant price vector is now

$$p_{BLI1} := \left[p_g, (p_{gj}), p_r, p_s, (p_{si}), p_v, p_y\right], \quad (5.10)$$

which is considerably less complex than p_{BM1} from (4.33). An allocation in BLI1 is given by

$$a_{BLI1} := \left[g, g_G, (g_j), k, r_y, (\overline{r}_i), (r_{vi}), (r_{sj}), (s_i), (s_j), (v_i), v_G, (y_i), y\right]. \quad (5.11)$$

Table 5.2: Comparison of rules governing a socially optimal allocation and an equilibrium in the market economy BLI1

	GM1		BM1		BLI1	
	1		2		3	
1	$U_k^i = \lambda_{ki} - \lambda_r \left(V_k^i / V_r^i\right)$	(2.25) (2.27)	$U_k^i = \beta_i p_{ki} - \beta_i p_r \left(V_k^i / V_r^i\right)$	(4.5) (4.10)	-	
2	$U_g^i = \lambda_{gi}$	(2.26)	$U_g^i = \beta_i p_{gi}$	(4.4)	-	
3	$U_v^i V_r^i = \lambda_r - \mu_g V_r^i$	(2.19) (2.25)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.10)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.10)
4	$U_s^i = \lambda_{\sigma i} - \mu_k$	(2.17)	$U_s^i = \beta_i p_{si} - \beta_i p_{sK}$	(4.6) (4.7)	$U_s^i = \beta_i p_{si} + \beta_i \tau_{si}$	(5.8)
5	$\lambda_r = \sum_i \lambda_{\sigma i} S_r^j$	(2.20) (2.23)	$p_r = p_s S_r^j$	(4.21)	$p_r = p_s S_r^j$	(4.21)
6	$\lambda_{gj} = \sum_i \lambda_{\sigma i} S_g^j$	(2.20) (2.24)	$p_{gj} = p_s S_g^j$	(4.21)	$p_{gj} = p_s S_g^j$	(4.21)
7	$\dot{\mu}_g = \left(\delta + \alpha_g\right) \mu_g - \sum_i \lambda_{gi} - \sum_j \lambda_{gj}$	(2.28)	$\dot{\varphi}_g = \left(\delta + \alpha_g\right) p_v - p_g$	(4.26)	$\dot{\varphi}_g = \left(\delta + \alpha_g\right) p_v - \tau_g - p_g$	(5.5)
8	$\dot{\mu}_k = \left(\delta + \alpha_k\right) \mu_k - \sum_i \lambda_{ki}$	(2.29)	$\dot{\varphi}_k = \left(\delta + \alpha_k\right) p_{sK} - p_k$	(4.31)	-	
9	$\lambda_r = \lambda_y Y_r$	(2.21)	$p_r = p_y Y_r$	(4.15)	$p_r = p_y Y_r$	(4.15)

Using the notation introduced in (5.9) - (5.11) we now establish the

Definition 5.1

In economy BLI1, a general competitive equilibrium is constituted by an allocation a_{BLI1}, *prices* p_{BLI1} *and taxes* τ_{BLI1} *for each point in time such that*

(i) *the allocation* a_{BLI1} *is a solution to* (4.11), (4.16) , (5.3) *and* (5.6) *for prices* p_{BLI1} *and taxes* τ_{BLI1};

(ii) *the prices* p_{BLI1} *satisfy:*

$$p_g = \sum_j p_{gj} ,$$

$$p_s = \sum_i p_{si} ;$$

(iii) *the allocation* a_{BLI1} *satisfies the resource constraints* (2.7) *through* (2.12).

To explore how the market equilibrium fares in terms of allocative efficiency we compare the marginal conditions of the efficient allocation (section 2) with the marginal conditions derived in the first part of the present subsection and report the results in

Proposition 5.1

(i) *Set*

$$p_y = \lambda_y , \quad p_r = \lambda_r , \quad p_{si} = \lambda_{\sigma i} \ \forall\, i , \quad p_v = \mu_g , \quad p_s = \sum_i \lambda_{\sigma i} , \quad p_{gj} = \sum_j \lambda_{gj} \ \forall\, j ,$$

$$p_g = \sum_j \lambda_{gj} , \ \tau_g = \sum_i \lambda_{gi} \quad \text{and} \quad \tau_{si} = -\mu_k ,$$

where $\mu_g, \mu_k, (\lambda_{gi}), \lambda_r, (\lambda_{\sigma i})$ *and* λ_y *are the values attained by the respective variables in the solution of* (2.15) *in section 2.*

Then at each point in time there exists a general competitive equilibrium in economy BLI1 and the associated allocation is efficient.

(ii) *If* τ_{BLI1} *is zero in all of its components, the general competitive equilibrium is inefficient.*

Proposition 5.1 is verified by applying the same procedure as in the proof of the previous propositions. Column 3 of Table 5.2 summarizes the first-order conditions characterizing the solutions of (4.11), (4.16), (5.3) and (5.6). With the assignment of prices and tax rates as shown in *Proposition 5.1*, column 3 of Table 5.2 is made to coincide with column 1. This match is straightforward for all rows except for the rows 1, 2 and 8.

Consider first the process of cultural-capital accumulation (row 8). To see that the subsidy $\tau_{si} = -\mu_k$ renders the accumulation of cultural capital efficient, we carry out the following thought experiment. Suppose, contrary to our setup, firm K is still active and with it the market for cultural services between firm K and the consumer-artists. Let firm K solve (5.2) assuming $p_{sK} = 0$ (as argued above) but $\tau_k = \sum_i \lambda_{ki}$ and $\tau_{sK} = \mu_k$. Hence the taxes (τ_k, τ_{sK}) exactly replace the missing prices (p_k, p_{sK}). In this scenario the net price consumer-artist *i* needs to pay for her consumption of cultural services is $p_{si} - p_{sK} = \lambda_{sj} - \mu_k$ (see above). In economy BLI1 where firm K is absent consumer-artist *i*'s net price for cultural services is $p_{si} + \tau_{si}$ and due to *Proposition 5.1* we have $p_{si} + \tau_{si} = \lambda_{sj} - \mu_k$. Hence the net price of cultural services is the same in both cases. This observation implies that the efficient accumulation of cultural capital in economy BLI1 is secured.

We now turn to the rows 1 and 2 of Table 5.2 and observe that there are entries in column 2 but no entries in column 3. The reason is, of course, the breakdown of the pertaining Lindahl markets in the economy BLI1 ($p_{ki} = p_{gi} \equiv 0$) combined with the assumption of ignorant consumer-artists. In BLI1 the intertemporal allocation of cultural capital and the stock of cultural goods are not guided by demand-side signals anymore. Note, however, that efficiency of the accumulation processes $\dot{k} = \sum_i s_i - \alpha_k k$ and $\dot{g} = \sum_i v_i - \alpha_g g$ is achieved as long as the variables s_i and v_i take on their efficient values for all *i* at each point in time. This is secured by the assignment of those prices and tax rates that are listed in *Proposition 5.1*.

The tax $\tau_g = \sum_i \lambda_{gi}$ in *Proposition 5.1* turns out to be a subsidy on firm G's sales of cultural goods. It is necessary to compensate firm G for getting no more revenues from the consumer-artists since their passive use of cultural goods is now free of charge.

The proof of *Proposition 5.1 (ii)* is simple if not trivial. Modify the first sentence of *Proposition 5.1 (i)* by setting $\tau_g = \tau_{si} \equiv 0$, then consider the modified assignments of prices and tax rates in the third column of Table 5.2, and finally juxtapose column 3, modified in this way, to column 1 of Table 5.2 for comparing all rows pairwise. The entries in the rows 4 and 7 turn out not to match anymore proving that the equilibrium allocation in economy BLI1 is bound to deviate from the Pareto-efficient allocation characterized by the marginal conditions of the first column of Table 5.2. Hence the equilibrium allocation of economy BLI1 is inefficient.

While this finding is an important piece of information one would like to know in which specific way the equilibrium allocation deviates from the efficient one. We will take up this issue later in section 5.1.3 when simplified parametric versions of the economy BLI1 are scrutinized. But first we proceed by investigating the allocative performance of the market economy BLI2.

5.1.2 The economy BL2 with ignorant consumer-artists (BLI2)

We now briefly turn to the case where cultural-goods inputs and cultural services are private goods. A closer look reveals that the optimization calculus of firm G is still given by (5.3) and firm Y's optimization calculus remains the same as (4.11), but the optimization programs of firm *j* and consumer-artist *i* now need to be modified as follows:

- Firm *j*:

$$\underset{(g_j, r_{sj}, s_j)}{Max} \int_0^\infty \left[p_s s_j - p_r r_{sj} - p_g g_j \right] e^{-\delta t} dt \qquad \text{subject to (2.4).} \qquad (5.12)$$

The pertinent Hamiltonian is

$$H^j = p_s s_j - p_r r_{sj} - p_g g_j + \beta_{sj} \left[S^j \left(g_j, r_{sj} \right) - s_j \right]. \qquad (5.13)$$

In case of an interior solution, the FOCs are: (4.18), (4.19) and

$$\frac{\partial H^j}{\partial g_j} = p_g + \beta_{sj} S_g^j = 0. \qquad (5.14)$$

- Consumer-artist *i*:

$$\underset{(r_{vi},s_i,v_i,y_i)}{Max} \int_0^\infty U^i\left(g,k,s_i,v_i,y_i\right)e^{-\delta t}dt$$

subject to $v_i = V^i\left(r_i,k\right)$ and $p_v v_i + p_r \overline{r}_i + \pi_i \geq p_r r_{vi} + \left(p_s + \tau_s\right)s_i + p_y y_i$. (5.15)

The pertinent Hamiltonian reads

$$H^C = U^i\left(g,k,s_i,v_i,y_i\right) + \beta_{vi}\left[V^i\left(r_{vi},k\right) - v_i\right]$$
$$+\beta_i\left[p_v v_i + p_r \overline{r}_i + \pi_i - p_r r_{vi} - \left(p_s + \tau_s\right)s_i - p_y y_i\right]. \quad (5.16)$$

The FOCs for an interior solution are (4.8), (4.9), (4.10) and

$$\frac{\partial H^C}{\partial s_i} = U_s^i - \beta_i\left(p_s + \tau_s\right) = 0 . \quad (5.17)$$

For later comparison we list the marginal conditions derived above in Table 5.3. The relevant vectors of tax rates and prices are now given by

$$\tau_{BLI2} := \left[\tau_g, \tau_s\right], \quad (5.18)$$

$$p_{BLI2} := \left[p_g, p_r, p_s, p_v, p_y\right]. \quad (5.19)$$

The allocation a_{BLI2} consists of the same components as a_{BLI1} from (5.11). The general competitive equilibrium in model BLI2 is specified in:

Definition 5.2

In economy BLI2, a general competitive equilibrium is constituted by an allocation a_{BLI2}*, prices* p_{BLI2} *and taxes* τ_{BLI2} *for each point in time such that*

(i) *the allocation* a_{BLI2} *is a solution to* (4.11), (5.3) , (5.12) *and* (5.15) *for prices* p_{BLI2} *and taxes* τ_{BLI2} *;*

(ii) *the allocation* a_{BLI2} *satisfies the resource constraints* (2.7) *through* (2.10), (2.13) *and* (2.14).

Table 5.3: Comparison of rules governing a socially optimal allocation and an equilibrium in the market economy BLI2

	GM2		BM2		BLI2	
	1		2		3	
1	$U_k^i = \lambda_{ki} - \lambda_r \left(V_k^i / V_r^i\right)$	(2.25) (2.27)	$U_k^i = \beta_i p_{ki} - \beta_i p_r \left(V_k^i / V_r^i\right)$	(4.5) (4.10)	-	
2	$U_g^i = \lambda_{gi}$	(2.26)	$U_g^i = \beta_i p_{gi}$	(4.4)	-	
3	$U_v^i V_r^i = \lambda_r - \mu_g V_r^i$	(2.19) (2.25)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.10)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.9)
4	$U_s^i = \lambda_\sigma - \mu_k$	(2.42)	$U_s^i = \beta_i p_s - \beta_i p_{sK}$	(4.7) (4.48)	$U_s^i = \beta_i p_s + \beta_i \tau_s$	(5.17)
5	$\lambda_r = \lambda_\sigma S_r^j$	(2.23) (2.43)	$p_r = p_s S_r^j$	(4.21)	$p_r = p_s S_r^j$	(4.21)
6	$\lambda_g = \lambda_\sigma S_g^j$	(2.20) (2.44)	$p_g = p_s S_g^j$	(4.51)	$p_g = p_s S_g^j$	(4.51)
7	$\dot{\mu}_g = \left(\delta + \alpha_g\right) \mu_g - \sum_i \lambda_{gi} - \lambda_g$	(2.45)	$\dot{\varphi}_g = \left(\delta + \alpha_g\right) p_v - p_g$	(4.26)	$\dot{\varphi}_g = \left(\delta + \alpha_g\right) p_v - \tau_g - p_g$	(5.5)
8	$\dot{\mu}_k = \left(\delta + \alpha_k\right) \mu_k - \sum_i \lambda_{ki}$	(2.29)	$\dot{\varphi}_k = \left(\delta + \alpha_k\right) p_{sK} - p_k$	(4.31)	-	
9	$\lambda_r = \lambda_y Y_r$	(2.21)	$p_r = p_y Y_r$	(4.15)	$p_r = p_y Y_r$	(4.15)

We now determine the efficiency properties of a competitive equilibrium in model BLI2 in

Proposition 5.2

(i) *Set all prices and tax rates as in Proposition 5.1 with the following changes:*

$p_{gj} = \lambda_{gj} \ \forall \ j$ *is replaced by* $p_g = \lambda_g$,

$p_{si} = \lambda_{\sigma i} \ \forall \ i$ *is replaced by* $p_s = \lambda_\sigma$,

$p_g = \sum_j \lambda_{gj}$ *is replaced by* $p_g = \lambda_g$,

$\tau_{si} = -\mu_k$ *is replaced by* $\tau_s = -\mu_k$.

Then at each point in time a general competitive equilibrium is attained in economy BLI2 and the associated allocation is efficient.

(ii) *If* τ_{BLI2} *is zero in all of its components, the general competitive equilibrium is inefficient.*

Proposition 5.2 can be proved by using the same method as in the previous propositions. Column 3 of Table 5.3 summarizes the first-order conditions pertaining to the economy BLI2. With the assignment of prices and tax rates as shown in *Proposition 5.2*, column 3 of Table 5.3 is made to coincide with column 1. To sum up, since cultural-goods input and cultural services are now private goods, the marginal conditions on the part of demanders in the rows 4 and 6 differ from those derived for the economy BLI1.

5.1.3 Laissez-faire and the transitional dynamics in simplified parametric versions of the economies BLI1 and BLI2

In the previous section we explored the allocation in market systems without Lindahl markets for the consumer-artists' passive use of cultural goods and cultural capital. We characterized these economies with ignorant consumer-artists in the models BLI1 and BLI2 and showed in *Proposition 5.1* and *Proposition 5.2* that if markets are supplemented by appropriate tax-subsidy schemes, Pareto efficiency can be achieved in those models. As noted above, the no-policy (or laissez-faire) scenarios in BLI1 and BLI2 are those special cases in which the vectors of tax rates (5.9) and (5.18) are set equal to zero:

$$\tau_{BLI1} := \left[\tau_g, (\tau_{si})\right] \equiv 0 \qquad \text{and} \qquad \tau_{BLI2} := \left[\tau_g, \tau_s\right] \equiv 0 . \tag{5.20}$$

In the absence of any taxes and subsidies, the consequence of missing prices for cultural capital is that the equilibrium allocations of the models BLI1 and BLI2 are inefficient. Clearly, to determine the scale of inefficiency is an empirical matter that is beyond the scope of our present study. Nonetheless, we aim at using our theoretical model to provide more specific qualitative information on the nature of the misallocation. Reinforcing our experience from previous sections the models BLI1 and BLI2 are too complex to allow for a more specific characterization of their equilibrium allocations by means of the phase-diagram technique. Therefore, we resort to the procedure used in section 3 again, namely to reduce the generality of the models BLI1 and BLI2:

(i) We first invoke the assumption from section 3.1 of a constant stock of cultural goods ($\dot{g} \equiv 0$) implying that the production of new cultural goods is completely shut down (economies $\mathrm{BLI\bar{G}1}$ and $\mathrm{BLI\bar{G}2}$). We then discuss the scenario from section 3.2 where the impact of cultural capital on the economy has been ignored (economies $\mathrm{BLI\bar{K}1}$ and $\mathrm{BLI\bar{K}2}$).

(ii) To simplify the analysis, we invoke again the assumptions from section 3.1 and section 3.2 that the tastes and endowments are identical across all consumer-artists[45], and that all the cultural-services firms use the same technology for producing cultural services.

Since our objective is to compare Pareto-efficient and laissez-faire allocations, we now re-examine the parametric approach developed and studied in section 3. Table 5.4 gives us an overview of the analytical agenda when the consumer-artists exhibit ignorant behavior. In the following analysis we begin with the economy $\mathrm{BLI\bar{G}1}$.

Table 5.4 Classification of market models with the ignorant consumer-artists 1

	Public-goods market economy BL1		Private-goods market economy BL2	
State variables	g: constant k: free	g: free k: no impact	g: constant k: free	g: free k: no impact
Ignorant behavior	$\mathrm{BLI\bar{G}1}$	$\mathrm{BLI\bar{K}1}$	$\mathrm{BLI\bar{G}2}$	$\mathrm{BLI\bar{K}2}$

45 For notational convenience, the subscripts i ($i = 1, \ldots, n_c$) for the identical consumer-artists will be replaced by the uniform subscript c.

5.1.3.1 The economy $\text{BLI}\overline{\text{G}}1$

For simplicity - and to secure full comparability - we make use of the assumptions already employed in section 3.1.1.2: (i) Leontief technology for producing cultural services in (3.21); (ii) linear technology for producing consumer goods in (3.22); (iii) additive separability of the representative consumer-artist's utility function in (3.23). Under these conditions the parameterized model BLI1 is characterized by the following Hamiltonians:

$$H^G = p_g g_G + \lambda_G \left(\overline{g} - g_G\right), \tag{5.21}$$

$$H^s = p_s s_s - p_r r_s - p_{gs} g_s + \lambda_{s1}\left(a_s r_s - s_s\right) + \lambda_{s2}\left(g_s - s_s\right), \tag{5.22}$$

$$H^y = p_y y - p_r r_y + \lambda_y \left(a_y r_y - y\right), \tag{5.23}$$

$$H^c = b_k k - \frac{d_k}{2} k^2 + b_s s_c - \frac{d_s}{2} s_c^{\,2} + y_c + \lambda_c \left(p_r \overline{r} + \pi - p_{sc} s_c - p_y y_c\right). \tag{5.24}$$

Focusing on interior solutions, we solve these Hamiltonians, take the resource as numeraire ($p_r \equiv 1$), and obtain, after some rearrangement of terms,

$$p_g = \lambda_G , \tag{5.25}$$

$$p_s = \lambda_{s1} + \lambda_{s2} , \tag{5.26}$$

$$a_s \lambda_{s1} = a_y \lambda_y = p_y \lambda_c = 1 , \tag{5.27}$$

$$p_{gs} = \lambda_{s2} , \tag{5.28}$$

$$p_y = \lambda_y , \tag{5.29}$$

$$p_{sc} = \frac{1}{\lambda_c}\left(b_s - d_s s_c\right). \tag{5.30}$$

Invoking the equilibrium condition $p_s = n_c p_{sc}$, (5.25) - (5.30) can be summarized as follows

$$p_s = p_{gs} + \frac{1}{a_s} , \tag{5.31}$$

$$s_c = \frac{b_s}{d_s} - \frac{a_y p_s}{d_s n_c} , \qquad \text{or, equivalently,} \qquad s_c = \frac{b_s}{d_s} - \frac{a_y p_{gs}}{d_s n_c} - \frac{a_y}{a_s d_s n_c} . \tag{5.32}$$

Note, that due to (5.27) and (5.29) λ_y, λ_c and λ_{s1} are positive and constant. Hence the corresponding Lagrange constraints hold as equalities. However, we have to distinguish two solution scenarios according to whether or not the constraint $\left(g_G \leq \bar{g}\right)$ is strictly binding. In view of the Kuhn-Tucker condition $\lambda_G\left(\bar{g}-g_G\right)=0$ the equilibrium allocation exhibits either "$\lambda_G > 0$ and $g_G = \bar{g}$" or "$\lambda_G = 0$ and $g_G < \bar{g}$" (suppressing the limiting case "$\lambda_G = 0$ and $g_G = \bar{g}$"). The first case yields $p_g > 0$ via (5.25) and $p_{gs} > 0$ due to the equilibrium condition $p_G = n_s p_{gs}$. From $p_{gs} > 0$ follows $\lambda_{s2} > 0$ via (5.30), in turn. On the other hand, if "$\lambda_G = 0$ and $g_G < \bar{g}$" applies, we conclude that $p_g = p_{gs} = 0$ and hence $\lambda_{s2} = 0$. It is obvious from these observations that the first case portrays a situation where cultural goods are a scarce input for the cultural-services firms while in the other case cultural goods are abundant.

a) The case of scarce cultural goods ($p_G = n_s p_{gs} > 0$)

In this case, the cultural-services firms buy the cultural-goods input at positive price $p_{gs} = \lambda_{s2} > 0$. Making use of (4.38), (4.39) and (4.42) as equalities, it is then straightforward that, for all *t*, the optimal allocation is given by:

$$r_s^{I\bar{G}1a} = \frac{\bar{g}}{a_s},\ s_s^{I\bar{G}1a} = \bar{g},\ r_y^{I\bar{G}1a} = n_c\bar{r} - \frac{n_s\bar{g}}{a_s},\ y^{I\bar{G}1a} = a_y r_y^{I\bar{G}1a} \text{ and } s_c^{I\bar{G}1a} = n_s\bar{g}. \qquad (5.33)$$

Hence the equation of motion (3.2) in section 3.1 readily becomes (3.34): $\dot{k} = n_c n_s \bar{g} - \alpha_k k$. As a consequence, in the economy $\text{BLI}\bar{\text{G}}1$ the motion in time of cultural capital is the same as that depicted in Figure 3.2. For $\dot{k} = 0$, equation (3.34) determines the steady-state value of cultural capital

$$k^{I\bar{G}1a} = n_c n_s\left(\frac{\bar{g}}{\alpha_k}\right). \qquad (5.33')$$

In other words, if cultural goods are scarce production inputs with positive price, the cultural-services firms have to reveal their willingness-to-pay, the intertemporal equilibrium allocation of the economy $\text{BLI}\bar{\text{G}}1$ is Pareto efficient.

b) The case of abundant cultural goods ($p_g = p_{gs} = 0$)

If cultural goods are abundant inputs, cultural-services firms use them as free inputs ($p_{gs} = \lambda_{s2} = 0$). Therefore (5.32) now reads:

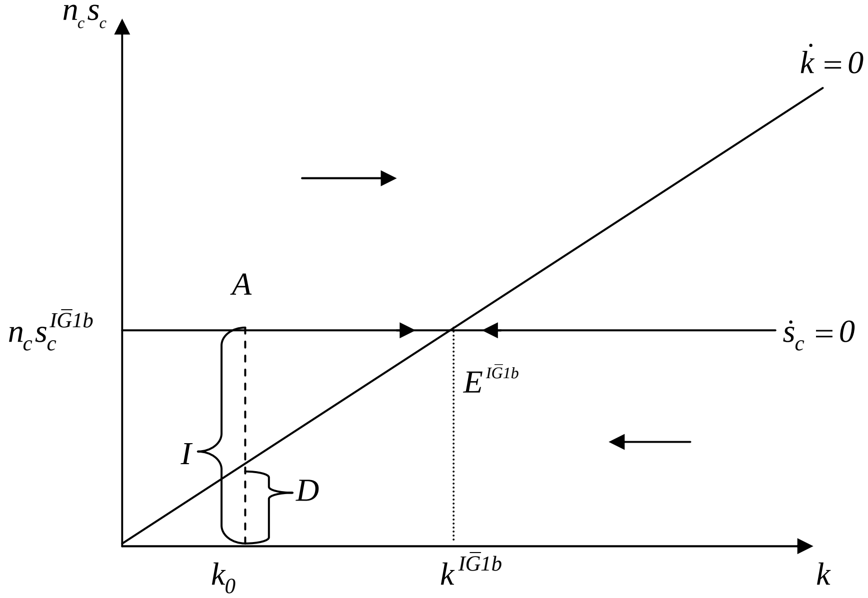
$$s_c^{I\bar{G}1b} = \frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c}. \tag{5.34}$$

Note that (5.34) holds at each point in time along the entire time path (including the steady state), i. e. $\dot{s}_c^{I\bar{G}1b} = 0$. Hence (5.34) represents the $\dot{s}_c = 0$ isocline whose graph is a horizontal line parallel to the *k*-axis in Figure 5.1. When combined with the $\dot{k} = 0$ isocline derived from (3.2) we obtain Figure 5.1 as a phase diagram for the parametric version of the economy $\text{BLI}\bar{\text{G}}1$. A steady state of this market economy $\text{BLI}\bar{\text{G}}1$ is defined by $\dot{s}_c^{I\bar{G}1b} = \dot{k} = 0$ and hence

$$\left.\begin{aligned} s_c^{I\bar{G}1b} &= \frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c}, \\ n_c s_c - \alpha_k k &= 0. \end{aligned}\right\} \tag{5.35}$$

In view of (5.35) the $\dot{s}_c^{I\bar{G}1b} = 0$ and $\dot{k} = 0$ isoclines in Figure 5.1 partition the space into four regions. The point of intersection of both isoclines, $E^{I\bar{G}1b}$, is the unique interior steady state.

Figure 5.1 Phase diagram for the parametric version of the laissez-faire economy $\text{BLI}\bar{\text{G}}1$ ($\dot{g} \equiv 0$)



Suppose that the initial stock of cultural capital is k_0 in Figure 5.1. Since at each point in time one has $n_c s_c = n_c s_c^{I\bar{G}1b}$ the economy's initial allocation is represented by point *A* in Figure 5.1. The associated change in cultural capital is given by the vertical distance between the isocline $\dot{k} = 0$ and $\dot{s}_c = 0$. Graphically, the difference is *I* – *D*, where *I* is the gross investment through the consumption of cultural services, and *D* is the depreciation of the cultural-capital stock k_0. Since $\dot{k} = I - D > 0$ the point *A* is shifted strictly to the left and eventually reaches the steady state $E^{I\bar{G}1b}$.

By solving (5.35) the pertaining steady-state value of cultural capital is calculated as

$$k^{I\bar{G}1b} = \frac{n_c}{\alpha_k}\left(\frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c}\right) = \frac{a_s b_s n_c - a_y}{a_s d_s \alpha_k}. \tag{5.36}$$

Clearly, $k^{I\bar{G}1b}$ (and likewise $s_c^{I\bar{G}1b}$) is strictly growing in a_s, b_s and n_c and is strictly declining in a_y, d_s and α_k.

The allocative (in)efficiency of the economy $\mathrm{BLI\bar{G}1}$

As already observed in the case a) if cultural goods are scarce inputs for cultural-services firms, the equilibrium allocation of the economy $\mathrm{BLI\bar{G}1}$ is Pareto efficient. Otherwise (for case b) we obtain from comparing (3.41) with (5.36) and (3.42) with (5.34)

$$\begin{aligned} D_k &:= k^{\bar{G}1b} - k^{I\bar{G}1b} = \frac{M_1}{\alpha_k M_2 + M_3} - \frac{n_c}{\alpha_k}\left(\frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c}\right) \\ &= \frac{(-a_s b_s d_k) n_c^3 + (a_y d_k + a_s b_k d_s \alpha_k) n_c^2}{a_s d_k d_s \alpha_k n_c^2 + a_s d_s^2 \alpha_k^2 (\alpha_k + \delta)}. \end{aligned} \tag{5.37}$$

$$\begin{aligned} D_s &:= s_c^{\bar{G}1b} - s_c^{I\bar{G}1b} = \frac{\alpha_k M_1}{n_c(\alpha_k M_2 + M_3)} - \left(\frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c}\right) \\ &= \frac{(-a_s b_s d_k) n_c^2 + (a_y d_k + a_s b_k d_s \alpha_k) n_c}{a_s d_k d_s n_c^2 + a_s d_s^2 \alpha_k (\alpha_k + \delta)}. \end{aligned} \tag{5.38}$$

It follows that $D_k \gtreqless 0$ and $D_s \gtreqless 0$, if and only if

$$n_c \lesseqgtr n_c^{\bar{G}1} := \frac{a_s b_k d_s \alpha_k + a_y d_k}{a_s b_s d_k}. \tag{5.39}$$

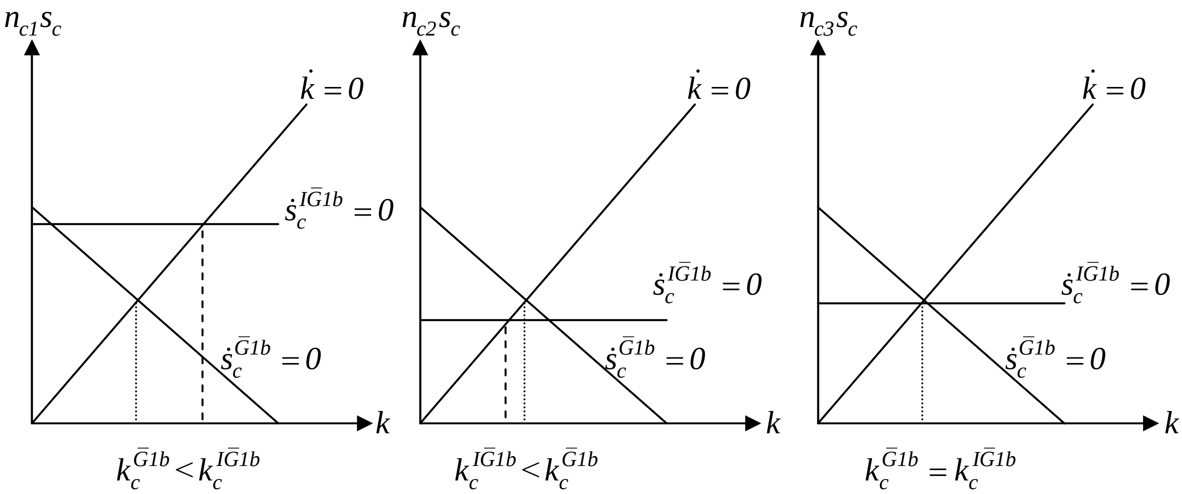
The comparison between the market result and the Pareto-efficient allocation needs further discussion. At the first glance, it is counterintuitive that cultural capital and cultural services are overprovided in the market economy, if the number of consumer-artists is sufficiently large, because one would have expected underprovision of the cultural capital and cultural services for any number of consumer-artists.

To better understand this result we recall that in section 3.1.1.3 (case b) we discussed in depth how the optimal steady-state value of the stock of cultural capital depends on n_c. We found, in particular, that this value was bounded from above and converged to $k^{\bar{G}1b} = b_k / d_k$ when n_c becomes large because expanding *k* beyond that value reduces the instantaneous utility of the consumer-artists. This result is due, in turn, to the assumption (3.23) of quadratic utility. In contrast, in the market economy presently under consideration, the consumer-artists ignore the impact of their consumption of cultural services on the formation of cultural capital. They keep their consumption of cultural services at a high level even if, due to their large number, the resultant steady-state value of cultural capital exceeds the value b_k / d_k reducing the instantaneous utility from cultural capital below that which consumer-artists could have had derived from a smaller stock of cultural capital. In fact, since $k^{\bar{G}1b}$ is linear increasing in n_c, there is $\tilde{n}_c > 0$ such that the instantaneous utility from cultural capital is negative for all $n_c > \tilde{n}_c$.

We now illustrate the phase diagrams with three alternative value of n_c in Figure 5.2.

Figure 5.2 (In)efficiency of the market economy $BLI\bar{G}1$ **(comparison between** $S\bar{G}1$ **and** $BLI\bar{G}1$**)**

Panel a) Overprovision $n_c > n_c^{\bar{G}1}$ **Panel b) Underprovision** $n_c < n_c^{\bar{G}1}$ **Panel c) Allocative efficiency** $n_c = n_c^{\bar{G}1}$



5.1.3.2 The economy $BLI\bar{G}2$

Consider next the parameterized model BLI2 with a constant stock of cultural goods in which cultural-goods inputs and cultural services are private. The optimization programs of firm G in (5.21) and firm Y in (5.23) remain the same, while the representative consumer-artist's and the cultural-services firm's decision problem change slightly. The equilibrium conditions $p_g = n_s p_{gs}$ and $p_s = n_c p_{sc}$ are now replaced by $p_g = p_{gs}$ and $p_s = p_{sc}$, respectively. With these modifications the solution of (5.21) - (5.24) yield

$$p_s = p_g + \frac{1}{a_s} \qquad \text{and} \qquad s_c = \frac{b_s - a_y p_s}{d_s}. \tag{5.40}$$

As in case of the public-goods economy $BLI\bar{G}1$, we need to distinguish solutions with either $p_g = p_{gs} > 0$ or $p_g = p_{gs} = 0$.

a) The case of scarce cultural goods ($p_g = p_{gs} > 0$ **)**

Applying the same procedure as above, we easily find

$$\bar{g} = n_s g_s^{I\bar{G}2a} = n_s s_s^{I\bar{G}2a} = n_c s_c^{I\bar{G}2a} . \tag{5.41}$$

The differential equation (3.2) turns out to be $\dot{k} = \bar{g} - \alpha_k k$, hence in the steady state the value of cultural capital is

$$k^{I\bar{G}2a} = \frac{\bar{g}}{\alpha_k} . \tag{5.41'}$$

The allocative (in)efficiency of the economy BLI$\bar{\text{G}}$2

We conclude that when cultural goods as inputs are positively priced, the equilibrium allocation of the private-goods market economy BLI$\bar{\text{G}}$2 in (5.41) coincides with the outcome in (3.53) of the Pareto-efficient allocation in 3.1.2.2 owing to the revealed willingness-to-pay of the cultural-services firms and of the consumer-artists.

b) The case of abundant cultural goods ($p_g = p_{gs} = 0$)

In this scenario (5.40) turns out to be

$$s_c^{I\bar{G}2b} = \frac{b_s}{d_s} - \frac{a_y}{a_s d_s} , \tag{5.42}$$

and therefore the steady-state value of cultural capital is

$$k^{I\bar{G}2b} = \frac{n_c}{\alpha_k}\left(\frac{b_s}{d_s} - \frac{a_y}{a_s d_s} \right) . \tag{5.43}$$

The allocative (in)efficiency of the economy BLI$\bar{\text{G}}$2

We now compare the results (5.42) and (5.43) for the market economy with the condition for allocative efficiency (3.60) and (3.61). The differences between the respective steady-state values of cultural capital are

$$D_k := k^{\bar{G}2b} - k^{I\bar{G}2b} = \frac{M_4}{\alpha_k M_2 + M_3} - \frac{n_c}{\alpha_k}\left(\frac{b_s}{d_s} - \frac{a_y}{a_s d_s}\right)$$

$$= \frac{\left(a_y d_k - a_s b_s d_k\right) n_c^{3} + a_s b_k d_s \alpha_k n_c^{2}}{a_s d_s \alpha_k \left[d_s \alpha_k \left(\delta + \alpha_k\right) + d_k n_c^{2}\right]}, \tag{5.44}$$

$$D_s := s^{\bar{G}2b} - s^{I\bar{G}2b} = \frac{\alpha_k M_4}{n_c\left(\alpha_k M_2 + M_3\right)} - \frac{a_s b_s - a_y}{a_s d_s}$$

$$= \frac{\left(a_y d_k - a_s b_s d_k\right) n_c^{2} + a_s b_k d_s \alpha_k n_c}{a_s d_s \left[d_s \alpha_k \left(\delta + \alpha_k\right) + d_k n_c^{2}\right]}. \tag{5.45}$$

We infer from (5.44) and (5.45) that $D_k \gtreqless 0$ and $D_s \gtreqless 0$, if and only if

$$n_c \lesseqgtr n_c^{\bar{G}2} := \frac{a_s b_k d_s \alpha_k + a_y d_k}{a_s b_s d_k}.$$

Since this condition is identical to (5.39), the discussion about the comparison between the private-goods market economy and Pareto efficiency is the same as the discussion in the previous subsection.

The steady-state allocation of the economies BLI$\bar{\text{G}}$1 **and** BLI$\bar{\text{G}}$2 **in comparison**

We now compare the steady-state allocations of the public-goods market economy and the private-goods economy.

a) The case of scarce cultural goods ($p_g = p_{gs} > 0$ **)**

Subtracting $k^{I\bar{G}2a}$ in (5.41') from $k^{I\bar{G}1a}$ in (5.33') and $s_c^{I\bar{G}2a}$ in (5.41) from $s_c^{I\bar{G}1a}$ in (5.33) yields

$$k^{I\bar{G}1a} - k^{I\bar{G}2a} = \left(n_c n_s - 1\right)\frac{\bar{g}}{\alpha_k}\begin{cases} > 0 & \text{for} \quad n_c n_s > 1, \\ = 0 & \text{for} \quad n_c n_s = 1. \end{cases} \tag{5.46}$$

$$s_c^{I\bar{G}1a} - s_c^{I\bar{G}2a} = \left(n_c n_s - 1\right)\frac{\bar{g}}{n_c}\begin{cases} > 0 & \text{for} \quad n_c n_s > 1, \\ = 0 & \text{for} \quad n_c n_s = 1. \end{cases} \tag{5.47}$$

Our discussion of the comparison of the models $\mathrm{S\bar{G}1}$ and $\mathrm{S\bar{G}2}$ ($g_s = \bar{g}$ and $\lambda_g > 0$) in section 3.1.3 also applies to (5.46) and (5.47). With only one agent ($n_c = n_s = 1$), jointly consumable goods are not jointly consumed and therefore the difference between public goods and private goods disappears. For the case that $n_c n_s > 1$, the optimal provision of the stock of cultural capital and cultural services is higher in the public-goods model than in the private-goods model.

b) The case of abundant cultural goods ($p_g = p_{gs} = 0$)

In the optimal steady state, the differences between the values of cultural services and cultural capital in the economies $\mathrm{BLI\bar{G}1}$ (cf. (5.34) and (5.36)) and $\mathrm{BLI\bar{G}2}$ (cf. (5.42) and (5.43)) are

$$D_k := k^{I\bar{G}1b} - k^{I\bar{G}2b} = \frac{n_c}{\alpha_k}\left(\frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c}\right) - \frac{n_c}{\alpha_k}\left(\frac{b_s}{d_s} - \frac{a_y}{a_s d_s}\right) = \frac{a_y\left(n_c - 1\right)}{a_s d_s \alpha_k}, \tag{5.48}$$

$$D_s := s^{I\bar{G}1b} - s^{I\bar{G}2b} = \frac{a_s b_s n_c - a_y}{a_s d_s n_c} - \frac{a_s b_s - a_y}{a_s d_s} = \frac{a_y\left(n_c - 1\right)}{a_s d_s n_c}. \tag{5.49}$$

(5.48) and (5.49) show that if there were only one consumer-artist ($n_c = 1$), the steady states of the economies $\mathrm{BLI\bar{G}1}$ and $\mathrm{BLI\bar{G}2}$ would coincide. For $n_c > 1$ society's provision with both cultural capital and cultural services is greater in the public-goods economy $\mathrm{BLI\bar{G}1}$ than in the private-goods economy $\mathrm{BLI\bar{G}2}$. Moreover, the gap in provision widens with increasing numbers of consumer-artists. However, that gap does not widen indefinitely since: $\lim_{n_c \to \infty}\left(s_c^{I\bar{G}1b} - s_c^{I\bar{G}2b}\right) = a_y / a_s d_s$ and $\lim_{n_c \to \infty}\left(k^{I\bar{G}1b} - k^{I\bar{G}2b}\right) = a_y n_c / a_s d_s \alpha_k$.

5.1.3.3 The economy $\mathrm{BLI\bar{K}1}$

We will continue to explore the parameterized models BLI1 and BLI2 but now we disregard cultural capital while considering as endogenous the stock of cultural goods. For that purpose we invoke the Hamiltonians (5.22) and (5.23) together with (5.4) and

$$H^c = b_g g - \frac{d_g}{2} g^2 + b_s s_c - \frac{d_s}{2} s_c^{\ 2} + b_v v_c - \frac{d_v}{2} v_c^{\ 2} + y_c$$
$$+\lambda_v \left(a_v r_c - v_c \right) + \lambda_c \left(p_r \overline{r} + \pi + p_v v_c - p_r r_c - p_{sc} s_c - p_y y_c \right). \qquad (5.50)$$

Again, we choose the resource as numeraire and consider interior solutions of these Hamiltonians only. Thus we obtain (5.5), (5.5'), (5.26) through (5.30) as well as

$$p_g = \beta_G , \qquad (5.51)$$

$$\lambda_v = \frac{a_y}{a_v} , \qquad (5.52)$$

$$v_c = \frac{b_v}{d_v} - \frac{\lambda_v}{d_v} + \frac{\lambda_c}{d_v} p_v . \qquad (5.53)$$

Making use of (5.5'), (5.27), (5.29) and (5.52) we transform (5.53) to get

$$v_c = \frac{b_v}{d_v} - \frac{a_y}{a_v d_v} + \frac{a_y}{d_v} \varphi_g , \qquad \text{or, alternatively,} \qquad \varphi_g = \frac{d_v}{a_y} v_c - \frac{b_v}{a_y} + \frac{1}{a_v} . \qquad (5.54)$$

Recall that in the economies $\mathrm{BLI\overline{G}1}$ and $\mathrm{BLI\overline{G}2}$ we had to distinguish two scenarios differing with respect to the scarcity or abundance of cultural goods used as an input for producing cultural services. We need to check now whether such a distinction is also necessary in the economy $\mathrm{BLI\overline{K}1}$ currently under review. For that purpose consider (5.4) and suppose that it is optimal for firm G to choose $g > g_G$. Then $\beta_G = p_G = 0$ follows to the effect that firm G doesn't receive any revenues from selling g_G. If $p_v > 0$, firm G would choose $v_G = 0$ which contradicts the presupposition of an interior solution. Hence $p_v = 0$ and therefore $\varphi_G = 0$. The stock of cultural goods is indeed abundant. Consumer-artists cannot earn money from selling their newly created cultural goods, v_c, but they may nevertheless choose $v_c > 0$ since they derive pleasure from their own creative work ($U_v > 0$). At $p_v = 0$ firm G is willing to "purchase" $v_G = n_c v_c > 0$ to the effect that the stock of cultural goods may even grow although it is zero-priced. As a consequence, persistent abundance of cultural goods cannot be ruled out in the economy $\mathrm{BLI\overline{K}1}$. On the other hand such a scenario is not very convincing. For that reason - and to avoid tedious repetition - our subsequent analysis will be restricted to the case that, for all *t*, $\beta_G > 0$ (and hence $g = g_G$) is optimal for firm G in the market equilibrium.

We now consider (5.26) through (5.30) and (5.54), and make use of the equilibrium conditions $p_g = n_s p_{gs}$ and $p_s = n_c p_{sc}$ with the consequence that

$$p_g = n_s \left[\frac{n_c}{a_y}\left(b_s - d_s s_c\right) - \frac{1}{a_s} \right] = \frac{b_s}{a_y} n_c n_s - \frac{d_s}{a_y} n_c n_s s_c - \frac{1}{a_s} n_s \,.$$

To make further progress we rearrange (5.5) and obtain

$$\dot{\varphi}_g = \left[\left(\delta + \alpha_g\right)\left(\frac{1}{a_v} - \frac{b_v}{a_y}\right) - \frac{b_s}{a_y} n_c n_s + \frac{n_s}{a_s} \right] + \left(\delta + \alpha_g\right)\frac{d_v}{a_y} v_c + \frac{d_s}{a_y} n_c n_s s_c \,. \tag{5.55}$$

Next we invoke the equilibrium conditions $s_c = n_s s_s = n_s g_s = n_s g$ and consider the derivative of (5.54) with respect to time, $\dot{\varphi}_g = \left(d_v / a_y\right)\dot{v}_c$, to transform (5.55) into

$$\dot{v}_c = \frac{a_y}{d_v}\left[\left(\delta + \alpha_g\right)\left(\frac{1}{a_v} - \frac{b_v}{a_y}\right) - \frac{b_s}{a_y} n_c n_s + \frac{n_s}{a_s} \right] + \left(\delta + \alpha_g\right) v_c + \frac{d_s}{d_v} n_c n_s^{2} g \,. \tag{5.56}$$

After some rearrangement of terms (5.56) yields

$$\dot{v}_c = -R_1 + M_6 n_c v_c + R_2 g \,, \tag{5.57}$$

where

$$R_1 := \frac{\left[\left(\delta + \alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) + a_s a_v b_s n_c n_s - a_v a_y n_s \right]}{a_s a_v d_v} \quad \text{and} \quad R_2 := \frac{d_s}{d_v} n_c n_s^{2} > 0 \,.$$

The sign of R_1 is positive, if and only if

$$n_c > n_{c0} := \frac{\left(\delta + \alpha_g\right)\left(a_s a_y - a_s a_v b_v\right) + a_v a_y n_s}{a_s a_v b_s n_s} \,,$$

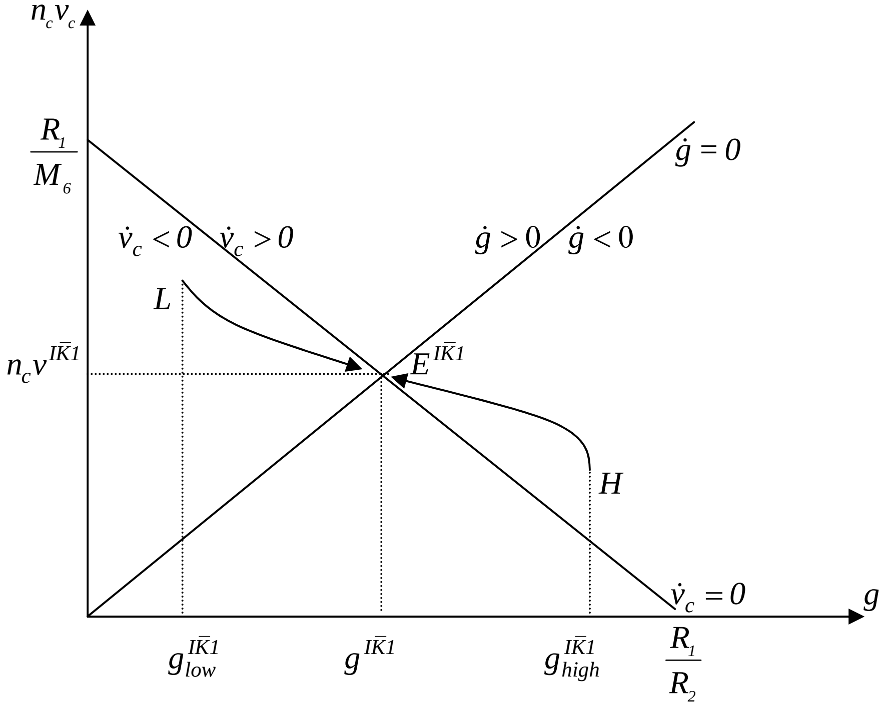
Since our interest is focused on economies in which the number of consumer-artists n_c is sufficiently large and the number of cultural-services firms n_s is more limited, we will assume that R_1 is positive. Moreover, the condition $n_c > max\left[1, n_{c0}\right]$ needs to be satisfied.

(3.75) and (5.57) represent a system of two differential equations which jointly determine the steady state through the equations:

$$\left.\begin{aligned} -R_1 + M_6 n_c v_c + R_2 g = 0, \\ n_c v_c - \alpha_g g = 0. \end{aligned}\right\} \tag{5.58}$$

The phase diagram of the economy $\mathrm{BLI\bar{K}1}$ is shown in Figure 5.3.

Figure 5.3 Phase diagram for the market economy $\mathrm{BLI\bar{K}1}$



Our comment on Figure 3.12 also applies to Figure 5.3. This figure illustrates that if the initial stock of cultural goods is smaller/greater than its steady-state value ($g_{low}^{I\bar{K}1} / g_{high}^{I\bar{K}1}$ in Figure 5.3), the amount of new cultural goods produced is higher/lower than its steady-state level, $n_c v^{I\bar{K}1}$

(the points L and H in Figure 5.3) in order to “jump” on the trajectory towards the steady state $E^{I\bar{K}1}$.

Solving the equations (5.58) give us the following steady-state values

$$g^{I\bar{K}1} := G^{I\bar{K}1}(n_c, n_s) = \frac{R_1}{\alpha_g M_6 + R_2}$$

$$= \frac{a_s a_v b_s n_s n_c^2 - \left[a_v a_y n_s - \left(a_s a_v b_v - a_s a_y\right)\left(\delta + \alpha_g\right)\right] n_c}{a_s a_v d_s n_s^2 n_c^2 + a_s a_v d_v \alpha_g \left(\delta + \alpha_g\right)}, \tag{5.59}$$

$$n_c v^{I\bar{K}1} = \frac{\alpha_g R_1}{\alpha_g M_6 + R_2} = \alpha_g \left\{ \frac{a_s a_v b_s n_s n_c^2 - \left[a_v a_y n_s - \left(a_s a_v b_v - a_s a_y\right)\left(\delta + \alpha_g\right)\right] n_c}{a_s a_v d_s n_s^2 n_c^2 + a_s a_v d_v \alpha_g \left(\delta + \alpha_g\right)} \right\}. \tag{5.60}$$

Comparing (5.59) with (3.108) and (5.60) with (3.109) reveals that, in qualitative terms, the impact of the model parameters on the respective steady-state values is the same. We therefore refer to our discussion of (3.108) and (3.109) in section 3.2.1.3 and refrain from repeating those calculations and interpretations in the present section.

The steady-state allocations of $\text{BLI}\bar{\text{K}}1$ and $\text{S}\bar{\text{K}}1$ in comparison

To specify the allocative (in)efficiency of the economy $\text{BLI}\bar{\text{K}}1$ we compare the market allocation of the model $\text{BLI}\bar{\text{K}}1$ from (5.59) with the Pareto-efficient allocation of the model $\text{S}\bar{\text{K}}1$ as determined in (3.108) and (5.60) with (3.109) to obtain

$$D_g := g^{\bar{K}1} - g^{I\bar{K}1}$$

$$= \frac{\left\{\left[a_s a_v n_s \left(b_g d_s n_s - b_s d_g\right)\right] n_c^4\right.}{a_s a_v \left[d_s n_s^2 n_c^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]\left[d_g n_c^2 + d_s n_c^2 n_s^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]}$$

$$+ \frac{\left[\left(\delta + \alpha_g\right)\left(a_s a_y d_g - a_s a_v b_v d_g\right) + a_v a_y d_g n_s\right] n_c^3 + a_s a_v b_g d_v \alpha_g \left(\delta + \alpha_g\right) n_c^2 \left.\right\}}{a_s a_v \left[d_s n_s^2 n_c^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]\left[d_g n_c^2 + d_s n_c^2 n_s^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]}, \tag{5.61}$$

$$D_v := v_c^{\bar{K}1} - v_c^{I\bar{K}1} = \frac{\alpha_g}{n_c}\left(g^{\bar{K}1} - g^{I\bar{K}1}\right). \tag{5.62}$$

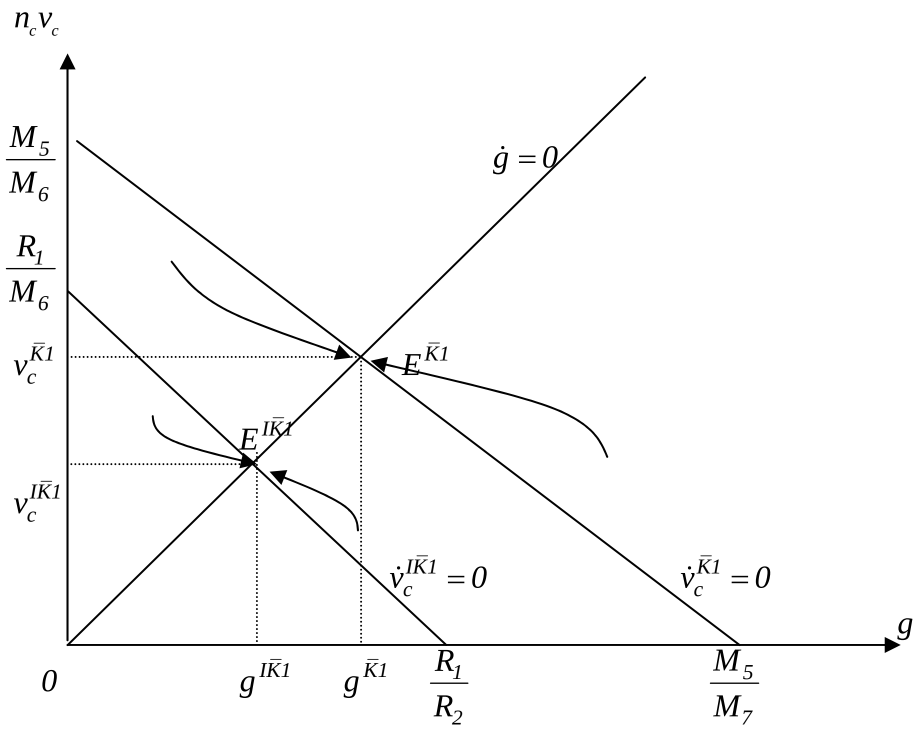
Unfortunately, the signs of D_g and D_v are ambiguous. Observe also that

$$\lim_{n_c \to \infty} D_g := \frac{\left(b_g d_s n_s - b_s d_g\right)}{d_s n_s \left(d_g + d_s n_s^{\,2}\right)},$$

which is positive, if and only if the term $\left(b_g d_s n_s - b_s d_g\right)$ is positive (cf. (3.133)). Since in the real world n_s is positive but "not too large"[46], the inequality $b_g d_s n_s > b_s d_g$ appears to be plausible and will therefore be assumed to hold. With this weak restriction we conclude that $D_g > 0$ and $D_v > 0$ for sufficiently large numbers of consumer-artists. Hence the steady-state stock of cultural goods in economy $\mathrm{S\bar{K}1}$ is greater than in the market economy $\mathrm{BLI\bar{K}1}$. The market result is inefficient.

After having compared the steady-state values of cultural capital and cultural services in the efficient state and in the market economy, we are now in the position to compare the entire phase diagrams of the Figure 3.12 and 5.3 in Figure 5.4.

Figure 5.4 Inefficiency of the market economy $\mathrm{BLI\bar{K}1}$



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46 However, it is interesting to observe, that in case the number of cultural-services firm is sufficiently large, we then have: $\lim_{n_s \to \infty}\left[\lim_{n_c \to \infty}\left(g^{\bar{K}1} - g^{I\bar{K}1}\right)\right] = 0$.

In Figure 5.4 the inequality $b_g d_s n_s > b_s d_g$ is satisfied implying $g^{\bar{K}1} - g^{I\bar{K}1} > 0$ and $v_c^{\bar{K}1} - v_c^{I\bar{K}1} > 0$.

5.1.3.4 The economy BLI$\bar{K}$2

We now briefly turn to the case, in which cultural-goods inputs for cultural-services firms and cultural services for consumer-artists are private goods. In view of (3.21), (3.49) and (3.117) the equations $s_c = n_s s_s = n_s g_s = n_s g$ characterizing the economy BLI$\bar{K}$1 are replaced by:

$$n_c s_c = n_s s_s = n_s g_s = g .$$

In addition, the equilibrium conditions $p_g = n_s p_{gs}$ and $p_s = n_c p_{sc}$ are substituted by $p_g = p_{gs}$ and $p_s = p_{sc}$, respectively. As a consequence,

$$p_g = \left[\frac{1}{a_y} \left(b_s - d_s s_c \right) - \frac{1}{a_s} \right] = \frac{b_s}{a_y} - \frac{d_s}{a_y n_c} g - \frac{1}{a_s} .$$

We insert this equation into (5.5) to get, after some rearrangement of terms,

$$\dot{\varphi}_g = \left[\left(\delta + \alpha_g \right) \left(\frac{1}{a_v} - \frac{b_v}{a_y} \right) - \frac{b_s}{a_y} + \frac{1}{a_s} \right] + \left(\delta + \alpha_g \right) \frac{d_v}{a_y} v_c + \frac{d_s}{a_y n_c} g . \tag{5.63}$$

The next step is to take the derivative of (5.54) with respect to time, $\dot{\varphi}_g = \left(d_v / a_y \right) \dot{v}_c$ and re-write (5.63) as

$$\dot{v}_c = \frac{a_y}{d_v} \left[\left(\delta + \alpha_g \right) \left(\frac{1}{a_v} - \frac{b_v}{a_y} \right) - \frac{b_s}{a_y} + \frac{1}{a_s} \right] + \left(\delta + \alpha_g \right) v_c + \frac{d_s}{d_v n_c} g , \tag{5.64}$$

which yields, after some rearrangement of terms,

$$\dot{v}_c = -R_3 + M_6 n_c v_c + R_4 g , \tag{5.65}$$

where

$$R_3 := \frac{\left[\left(\delta+\alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) + a_s a_v b_s - a_v a_y\right]}{a_s a_v d_v} \quad \text{and} \quad R_4 := \frac{d_s}{d_v n_c} > 0 .$$

The sign of R_3 is positive, if and only if

$$\left(\delta+\alpha_g\right) a_s \left(a_v b_v - a_y\right) + a_v \left(a_s b_s - a_y\right) > 0 .$$

Moreover,

$$a_y < min\left[a_s b_s , a_v b_v\right], \tag{5.66}$$

is obviously sufficient for $R_3 > 0$. To interpret the requirement (5.66) recall that a_v, a_s and a_y are technological coefficients while b_v and b_s are preference parameters. Increasing b_v and b_s raises the consumer-artist's utility, respectively, derived from the consumption of cultural services and from creating new cultural goods. As a consequence, for any given set of technologies represented by a_v, a_s and a_y, the inequality (5.66) is satisfied, if the preference parameters b_v and b_s are sufficiently large, i.e. if the consumer-artists' preferences for newly created cultural goods, b_v, and for cultural services, b_s, are sufficiently strong.

We invoke the differential equations (3.75) and (5.65), to characterize a steady state of the economy BLI$\bar{\text{K}}$2 by:

$$\left.\begin{aligned} -R_3 + M_6 n_c v_c + R_4 g = 0, \\ n_c v_c - \alpha_g g = 0. \end{aligned}\right\} \tag{5.67}$$

Straightforward calculation yields the steady-state values

$$\begin{aligned} g^{I\bar{K}2} := G^{I\bar{K}2}\left(n_c\right) &= \frac{R_3}{\alpha_g M_6 + R_4} \\ &= \frac{\left[\left(\delta+\alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) + a_s a_v b_s - a_v a_y\right] n_c}{a_s a_v d_s + a_s a_v d_v \alpha_g \left(\delta+\alpha_g\right)}, \end{aligned} \tag{5.68}$$

$$n_c v^{I\bar{K}2} = \frac{\alpha_g R_3}{\alpha_g M_6 + R_4} = \alpha_g \left\{ \frac{\left[\left(\delta+\alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) + a_s a_v b_s - a_v a_y\right] n_c}{a_s a_v d_s + a_s a_v d_v \alpha_g \left(\delta+\alpha_g\right)} \right\}. \tag{5.69}$$

Assuming (5.66) to hold, the phase diagram of the economy $\mathrm{BLI\bar{K}2}$ is qualitatively the same as that of $\mathrm{BLI\bar{K}1}$ (cf. Figure 5.3). To avoid repetition, we therefore proceed immediately to the comparison of the steady states of the models $\mathrm{BLI\bar{K}2}$ and $\mathrm{S\bar{K}2}$.

The steady-state allocations of $\mathrm{BLI\bar{K}2}$ and $\mathrm{S\bar{K}2}$ in comparison

To determine the allocative (in)efficiency of the economy $\mathrm{BLI\bar{K}2}$ we compare the market allocation of the model $\mathrm{BLI\bar{K}2}$ from (5.68) and (5.69) with the Pareto-efficient allocation of the model $\mathrm{S\bar{K}1}$ from (3.124) and (3.125) to obtain

$$\begin{aligned} D_g &:= g^{\bar{K}2} - g^{I\bar{K}2} \\ &= \frac{\left[\left(\delta+\alpha_g\right)a_s a_v d_g\left(1-b_v\right)+a_v d_g\left(a_y-a_s b_s\right)\right]n_c^3}{a_s a_v\left[d_s+d_v\alpha_g\left(\delta+\alpha_g\right)\right]\left[d_s+d_g n_c^2+d_v\alpha_g\left(\delta+\alpha_g\right)\right]} \\ &\quad+\frac{\left[\left(\delta+\alpha_g\right)a_s a_v b_g d_v\alpha_g+a_s a_v b_g d_s\right]n_c^2}{a_s a_v\left[d_s+d_v\alpha_g\left(\delta+\alpha_g\right)\right]\left[d_s+d_g n_c^2+d_v\alpha_g\left(\delta+\alpha_g\right)\right]}, \end{aligned} \tag{5.70}$$

$$D_v := v_c^{\bar{K}2} - v_c^{I\bar{K}2} = \frac{\alpha_g}{n_c}\left(g^{\bar{K}2} - g^{I\bar{K}2}\right). \tag{5.71}$$

Unfortunately, the signs of D_g and D_v are still indeterminate. Yet closer inspection of (5.70) yields:

$$D_g > 0 \text{ and } D_v > 0 \iff n_c > n_c^{\bar{K}2} := \frac{\left(\delta+\alpha_g\right)a_s a_v b_g d_v\alpha_g+a_s a_v b_g d_s}{\left[\left(\delta+\alpha_g\right)a_s a_y d_g\left(1-b_v\right)+a_v d_g\left(a_y-a_s b_s\right)\right]}. \tag{5.72}$$

Clearly, it is plausible to assume that this inequality holds since we are interested in economies with a large number of consumer-artists. As a consequence, the steady-state stock of cultural goods and the amount of newly created cultural goods in $\mathrm{S\bar{K}2}$ is likely to be greater than in the market economy $\mathrm{BLI\bar{K}2}$.

The steady-state allocations of the economies $\mathrm{BLI\bar{K}1}$ and $\mathrm{BLI\bar{K}2}$ in comparison

We subtract (5.68) from (5.59) to obtain

$$D_g := g^{I\bar{K}1} - g^{I\bar{K}2}$$

$$= \frac{n_c}{a_s a_v} \left\{ \frac{\left[\left(\delta + \alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) + a_s a_v b_s - a_v a_y \right]}{d_s + d_v \alpha_g \left(\delta + \alpha_g\right)} - \frac{a_s a_v b_s n_s n_c + \left[\left(\delta + \alpha_g\right)\left(a_s a_v b_v - a_s a_y\right) - a_v a_y n_s \right]}{d_s n_c^2 n_s^2 + d_v \alpha_g \left(\delta + \alpha_g\right)} \right\} \tag{5.73}$$

$$= \frac{\left[\left(\delta + \alpha_g\right)\left(a_s a_y d_s - a_s a_v b_v d_s\right) + a_v a_y d_s - a_s a_v b_s d_s \right] n_c^3}{\left[d_s + d_v \alpha_g \left(\delta + \alpha_g\right)\right]\left[d_s n_c^2 n_s^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]}$$

$$+ \frac{\left[\left(\delta + \alpha_g\right) a_s a_v b_s d_v \alpha_g + a_s a_v b_s d_s \right] n_c^2}{\left[d_s + d_v \alpha_g \left(\delta + \alpha_g\right)\right]\left[d_s n_c^2 n_s^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]}$$

$$- \frac{\left\{ \left[\left(\delta + \alpha_g\right) a_v a_y \left(d_v \alpha_g + d_s\right)\right] n_s - \left[\left(\delta + \alpha_g\right)\left(a_v a_y - a_s a_v b_s\right) + \left(a_s a_v b_v - a_s a_y\right) d_s \right] \right\} n_c}{\left[d_s + d_v \alpha_g \left(\delta + \alpha_g\right)\right]\left[d_s n_c^2 n_s^2 + d_v \alpha_g \left(\delta + \alpha_g\right)\right]}, \tag{5.73'}$$

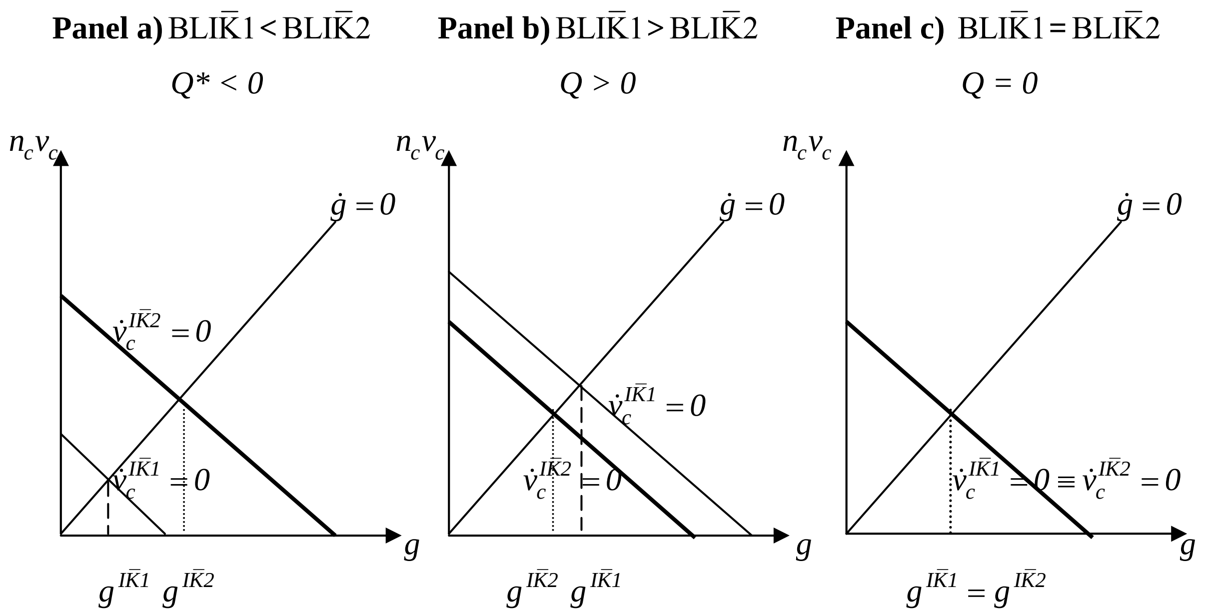
$$D_v := v_c^{I\bar{K}1} - v_c^{I\bar{K}2} = \frac{\alpha_g}{n_c}\left(g^{I\bar{K}1} - g^{I\bar{K}2}\right). \tag{5.74}$$

Observe first that if $n_c = n_s \equiv 1$ in (5.73), $D_g = 0$ and $D_v = 0$ follows which confirms one's intuition that with only one consumer-artist and only one cultural-services firm, there is no difference between the public-goods economy $\mathrm{BLI\bar{K}1}$ and the private-goods economy $\mathrm{BLI\bar{K}2}$ because in that limiting case none of the public goods can be jointly consumed.

Next, for $n_c n_s > 1$ and n_c sufficiently large, the terms D_g and D_v are positive, if and only if the term attached to n_c^3 in (5.73'), $\left[\left(\delta + \alpha_g\right) a_s d_s \left(a_y - a_v b_v\right) + a_v d_s \left(a_y - a_s b_s\right)\right]$, is positive. This condition is satisfied if $a_y > max\left[a_s b_s, a_v b_v\right]$. Note, however that this requirement is the opposite of the condition (5.66). We conclude, therefore, that in the public-goods model $\mathrm{BLI\bar{K}1}$ the steady-state provision of cultural-goods stock and newly created cultural goods is not unambiguously greater than in the private-goods economy $\mathrm{BLI\bar{K}2}$.

The result is illustrated in Figure 5.5.

Figure 5.5 Comparing the time paths in the parameterized models $\mathrm{BLI\bar{K}1}$ **and** $\mathrm{BLI\bar{K}2}$ (**for** $n_c n_s > 1$)



* $Q := \left[\left(\delta + \alpha_g \right) a_s d_s \left(a_y - a_v b_v \right) + a_v d_s \left(a_y - a_s b_s \right) \right]$

In qualitative terms, Figure 5.5 is very similar to Figure 5.2. We hence refrain from further interpretations.

5.2 Nash consumer-artists in the economies BLN1 and BLN2

Suppose now the consumer-artists exhibit Nash behavior, i.e. at each point in time consumer-artist *i* maximizes her utility taking as given the other consumer-artists' consumption of cultural services. As in section 5.1 we will distinguish two submodels according to whether cultural-goods inputs and cultural services are public or private. For convenience of reference the models BL1 and BL2 with Nash consumer-artists will be denoted by BLN1 and BLN2, respectively. The only difference regarding the economies BLI1 and BLI2 from the previous section 5.1 is the assumption on the consumer-artists' behavior.

Therefore (5.3) and (4.11) carry over unchanged as well as (5.7) for the public-goods economy and (5.15) for the private-goods economy, so that the focus of the subsequent analysis can be restricted to the consumer-artists' optimization calculus. We will proceed as in section 5.1: We determine a tax-subsidy scheme capable to restore the efficiency of the market allocation when there is no Lindahl market for cultural capital. Having done that we investigate the displacement of the market allocation that occurs in the absence of taxes and subsidies.

5.2.1 The economy BL1 with Nash consumer-artists (BLN1)

Since in the models BLI1 and BLN1 cultural-goods inputs as well as cultural services are public goods, the optimization programs of firm Y in (4.11), firm G in (5.3) and of the cultural-services firms in (4.16) are not affected by switching from economy BLI1 to economy BLN1. However, the consumer-artist *i*'s decision problem is now modified as follows:

$$\underset{(r_i, s_i, v_i, y_i)}{Max} \int_0^\infty U^i\left(g_i, k_i, s_i, v_i, y_i\right) e^{-\delta t} dt, \qquad \text{subject to}$$

$$v_i = V^i\left(r_i, k_i\right),$$

$$\dot{g}_i = v_i + v_{-i} - \alpha_g g_i,$$

$$\dot{k}_i = s_i + s_{-i} - \alpha_k k_i,$$

$$\left(p_v + \tau_v\right) v_i + p_r \overline{r}_i + \pi_i \geq \tau_{gi} g_i + \tau_{ki} k_i + p_r r_i + p_{si} s_i + p_y y_i, \qquad (5.75)$$

where $v_{-i} := \sum_{h \neq i} v_h$ and $s_{-i} := \sum_{h \neq i} s_h$. Consumer-artist *i* maximizes the present value of her utility subject to her budget constraint for given actions v_{-i} and s_{-i} of all other consumer-artists. She also takes into account the impact of her own choice v_i and s_i on the change in the stock of cultural goods and on the formation of cultural capital, respectively. The pertinent Hamiltonian reads:

$$H^c = U^i\left(g_i, k_i, s_i, v_i, y_i\right) + \beta_{vi}\left[V^i\left(r_i, k_i\right) - v_i\right] + \eta_{gi}\left(v_i + v_{-i} - \alpha_g g\right) + \eta_{ki}\left(s_i + s_{-i} - \alpha_k k\right)$$
$$+\beta_i\left[\left(p_v + \tau_v\right)v_i + p_r\bar{r}_i + \pi_i - \tau_{gi}g_i - \tau_{ki}k_i - p_r r_i - p_{si}s_i - p_y y_i\right], \qquad (5.76)$$

where η_{gi} and η_{ki} are the co-state variables associated to cultural goods and cultural capital, respectively. β_i and β_{vi} are Lagrange multipliers. In case of an interior solution the FOCs are

$$U_s^i = \beta_i p_{si} - \eta_{ki}, \qquad (5.77)$$

$$U_v^i = \beta_{vi} - \beta_i\left(p_v + \tau_v\right) - \eta_{gi}, \qquad (5.78)$$

$$U_y^i = \beta_i p_y, \qquad (5.79)$$

$$\beta_{vi}V_r^i = \beta_i p_r, \qquad (5.80)$$

$$\dot{\eta}_{gi} = \left(\delta + \alpha_g\right)\eta_{gi} - U_g^i + \beta_i\tau_{gi}, \qquad (5.81)$$

$$\dot{\eta}_{ki} = \left(\delta + \alpha_k\right)\eta_{ki} - U_k^i - \beta_{vi}V_k^i + \beta_i\tau_{ki}. \qquad (5.82)$$

We now define the vector of tax rates

$$\tau_{BLN1} := \left[\tau_g, \left(\tau_{gi}\right), \left(\tau_{ki}\right), \tau_v\right], \qquad (5.83)$$

and note that the price vector p_{BLN1} is the same as p_{BLI1}. An allocation in BLN1 is given by

$$a_{BLN1} := \left[g, g_G, \left(g_i\right), \left(g_j\right), \left(k_i\right), k, r_y, \left(r_i\right), \left(r_j\right), \left(s_i\right), \left(s_j\right), \left(v_i\right), v_G, \left(y_i\right), y\right]. \qquad (5.84)$$

With this notation, we now introduce the

Definition 5.3

In economy BLN1 a general competitive equilibrium is constituted by an allocation a_{BLN1}*, prices* p_{BLN1} *and taxes* τ_{BLN1} *for any point in time such that*

(i) *the allocation* a_{BLN1} *is a solution to* (4.11), (4.16), (5.3) *and* (5.75) *for prices* p_{BLN1} *and taxes* τ_{BLN1}*;*

(ii) *the allocation* a_{BLN1} *satisfies the resource constraints* (2.7) *through* (2.12) *and* $g_G = g_i = g \;\; \forall\, i = 1,..., n_c$, $k_i = k \;\; \forall\, i = 1,..., n_c$.

We establish the efficiency properties of a competitive equilibrium in model BLN1 in

Proposition 5.3

(i) *Set the prices*

$$p_y = \lambda_y,\; p_r = \lambda_r,\; p_s = \sum_i \lambda_{\sigma i},\; p_{si} = \lambda_{\sigma i} \;\forall\, i,\; p_g = \sum_j \lambda_{gj},\; p_{gj} = \lambda_{gj} \;\forall\, j,\; p_v = \mu_g,$$

and the tax rates

$$\tau_g = \sum_i \lambda_{gi},\; \tau_{gi} = -\sum_{h \neq i} \lambda_{gh} - \sum_j \lambda_{gj},\; \tau_{ki} = -\sum_{h \neq i} \lambda_{kh},\; \tau_v = -\mu_g,$$

where μ_g, (λ_{gi}), (λ_{gj}), λ_r, $(\lambda_{\sigma i})$ *and* λ_y *are the values attained by the respective variables in the solution of (2.15) in section 2.*

Then at each point in time a general competitive equilibrium is attained in economy BLN1 and the associated allocation is efficient.

(ii) *If* τ_{BLN1} *is zero in all of its components, the general competitive equilibrium in economy BLN1 is inefficient.*

The marginal conditions governing, respectively, the Pareto-efficient allocation of the model GM1, the Lindahl market economy BM1, and the market economy BLN1 are listed in Table 5.5. *Proposition 5.3* is proved by using the same method as in the proof of the previous propositions. The marginal conditions from solving (4.11), (4.16), (5.3) and (5.75) are presented in column 3 of Table 5.5. We insert in column 3 of Table 5.5 all prices, taxes and subsidies as specified in *Proposition 5.3* to find that column 3 then coincides with column 1.

Table 5.5: Comparison of rules governing a socially optimal allocation and an equilibrium in the market economy BLN1

	GM1		BM1		BLN1	
	1		2		3	
1	$U_k^i = \lambda_{ki} - \lambda_r \left(V_k^i / V_r^i \right)$	(2.25) (2.27)	$U_k^i = \beta_i p_{ki} - \beta_i p_r \left(V_k^i / V_r^i \right)$	(4.5) (4.10)	-	
2	$U_g^i = \lambda_{gi}$	(2.26)	$U_g^i = \beta_i p_{gi}$	(4.4)	-	
3	$U_v^i V_r^i = \lambda_r - \mu_g V_r^i$	(2.19) (2.25)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.10)	$U_v^i V_r^i = \beta_i p_r - \beta_i \left(p_v + \tau_v \right) V_r^i - \eta_{gi} V_r^i$	(5.78) (5.79) (5.80)
4	$U_s^i = \lambda_{\sigma i} - \mu_k$	(2.17)	$U_s^i = \beta_i p_{si} - \beta_i p_{sK}$	(4.6) (4.7)	$U_s^i = \beta_i p_{si} - \eta_{ki}$	(5.77)
5	$\lambda_r = \sum_i \lambda_{\sigma i} S_r^j$	(2.20) (2.23)	$p_r = p_s S_r^j$	(4.21)	$p_r = p_s S_r^j$	(4.21)
6	$\lambda_{gj} = \sum_i \lambda_{\sigma i} S_g^j$	(2.20) (2.24)	$p_{gj} = p_s S_g^j$	(4.21)	$p_{gj} = p_s S_g^j$	(4.21)
7	$\dot{\mu}_g = \left(\delta + \alpha_g \right) \mu_g - \sum_i \lambda_{gi} - \sum_j \lambda_{gj}$	(2.28)	$\dot{\varphi}_g = \left(\delta + \alpha_g \right) p_v - p_g$	(4.26)	$\dot{\varphi}_g = \left(\delta + \alpha_g \right) p_v - \tau_g - p_g$ $\dot{\eta}_{gi} = \left(\delta + \alpha_g \right) \eta_{gi} - U_g^i + \beta_i \tau_{gi}$	(5.5) (5.81)
8	$\dot{\mu}_k = \left(\delta + \alpha_k \right) \mu_k - \sum_i \lambda_{ki}$	(2.29)	$\dot{\varphi}_k = \left(\delta + \alpha_k \right) p_{sK} - p_k$	(4.31)	$\dot{\eta}_{ki} = \left(\delta + \alpha_k \right) \eta_{ki} - \left(U_k^i + \beta_i p_r \frac{V_k^i}{V_r^i} \right) + \beta_i \tau_{ki}$	(5.71) (5.76)
9	$\lambda_r = \lambda_y Y_r$	(2.21)	$p_r = p_y Y_r$	(4.15)	$p_r = p_y Y_r$	(4.15)

For the rows 5, 6 and 9, the assignment of the prices p_{gj}, p_r, p_s and p_y in *Proposition 5.3* gives a perfect match between the columns 1 and 3 to Table 5.5. The other matching pairs are less straightforward. However, closer inspection shows that a match for the rows 3, 4, 7 (lower line) and 8 is secured, if and only if for all *i*

$$\frac{\eta_{gi}}{\beta_i} = \mu_g \quad \text{and} \quad \frac{\eta_{ki}}{\beta_i} = \mu_k$$

is satisfied. Hence the market-equilibrium allocation coincides with the Pareto-efficient allocation.

To get a better understanding of the impact of consumer behavior on cultural policy that aims at achieving Pareto efficiency we now compare the assignment of prices and taxes in the *Proposition 5.1* and *Proposition 5.3*. We first observe that all market prices are the same. Moreover, the subsidy τ_g firm G receives for selling cultural goods to the cultural-services firms is also the same in both models. But all other tax parameters are different. In BLI1 the consumer-artists' consumption of cultural services is subsidized with the rate τ_{si} while there is no such subsidy in BLN1. In the latter economy consumer-artists are affected by two other subsidies, τ_{gi} and τ_{ki}, on their (passive) use of cultural goods and cultural capital, respectively. In addition, a tax τ_v is imposed on the revenues from selling their newly created cultural goods. In fact, that tax is confiscatory, $\tau_v = -p_v$, implying that the consumer-artists' net revenue from "selling" their cultural goods is zero. It is therefore equivalent to set $p_v = \tau_v = 0$ in (5.75) and to replace p_v in (5.3) by $\tau_{vG} = \mu_g$.

5.2.2 The economy BL2 with Nash consumer-artists (BLN2)

We now consider the case that both cultural-goods inputs and cultural services are private goods. The optimization programs of firm Y (4.11), firm G (5.3) and of the cultural-services firms (5.12) still apply. However, the consumer-artist *i*'s decision problem in (5.75) differs slightly. It now reads:

$$\underset{(r_i, s_i, v_i, y_i)}{Max} \int_0^{\infty} U^i\left(g_i, k_i, s_i, v_i, y_i\right) e^{-\delta t} dt, \qquad \text{subject to}$$

$$v_i = V^i\left(r_i, k_i\right),$$

$$\dot{g}_i = v_i + v_{-i} - \alpha_g g_i,$$

$$\dot{k}_i = s_i + s_{-i} - \alpha_k k_i,$$

$$\left(p_v + \tau_v\right) v_i + p_r \overline{r}_i + \pi_i \geq \tau_{gi} g_i + \tau_{ki} k_i + p_r r_i + p_s s_i + p_y y_i, \tag{5.85}$$

Consumer-artist *i*'s optimal control problem is solved by applying the Hamiltonian:

$$H^c = U^i\left(g_i, k_i, s_i, v_i, y_i\right) + \beta_{vi}\left[V^i\left(r_i, k_i\right) - v_i\right] + \eta_{gi}\left(v_i + v_{-i} - \alpha_g g_i\right) + \eta_{ki}\left(s_i + s_{-i} - \alpha_k k_i\right)$$
$$+ \beta_i\left[\left(p_v + \tau_v\right) v_i + p_r \overline{r}_i + \pi_i - \tau_{gi} g_i - \tau_{ki} k_i - p_r r_i - p_s s_i - p_y y_i\right]. \tag{5.86}$$

The pertinent FOCs for an interior solution are (5.77) - (5.82) and

$$U_s^i = \beta_i p_s - \eta_{ki}. \tag{5.87}$$

We list the associated marginal conditions in Table 5.6. The relevant vectors of tax rates vector and allocation are now given by

$$\tau_{BLN2} := \left[\tau_g, \left(\tau_{gi}\right), \left(\tau_{ki}\right), \tau_v\right], \tag{5.88}$$

$$a_{BLN2} := \left[g, g_G, \left(g_i\right), \left(g_j\right), \left(k_i\right), k, r_y, \left(r_i\right), \left(r_j\right), \left(s_i\right), \left(s_j\right), \left(v_i\right), v_G, \left(y_i\right), y\right]. \tag{5.89}$$

With this notation we define a general competitive equilibrium in model BLN2:

Definition 5.4

In economy BLN2, a general competitive equilibrium is constituted by an allocation a_{BLN2}, *prices* $p_{BLN2} = p_{BLI2}$ *and by taxes* τ_{BLN2} *for any point in time such that*

(i) *the allocation* a_{BLN2} *is a solution to* (4.11), (5.3), (5.12) *and* (5.85) *for prices* p_{BLN2} *and taxes* τ_{BLN2};

(ii) *the allocation* a_{BLN2} *satisfies the resource constraints* (2.7) *through* (2.10), (2.13), (2.14) *and* $g_G = g_i = g \;\; \forall\, i = 1,..., n_c$, $k_i = k \;\; \forall\, i = 1,..., n_c$.

Table 5.6: Comparison of rules governing a socially optimal allocation and an equilibrium in the market economy BLN2

	GM2		BM2		BLN2	
	1		2		3	
1	$U_k^i = \lambda_{ki} - \lambda_r \left(V_k^i / V_r^i \right)$	(2.25) (2.27)	$U_k^i = \beta_i p_{ki} - \beta_i p_r \left(V_k^i / V_r^i \right)$	(4.5) (4.10)	-	
2	$U_g^i = \lambda_{gi}$	(2.26)	$U_g^i = \beta_i p_{gi}$	(4.4)	-	
3	$U_v^i V_r^i = \lambda_r - \mu_g V_r^i$	(2.19) (2.25)	$U_v^i V_r^i = \beta_i p_r - \beta_i p_v V_r^i$	(4.8) (4.10)	$U_v^i V_r^i = \beta_i p_r - \beta_i \left(p_v + \tau_v \right) V_r^i - \eta_{gi} V_r^i$	(5.78) (5.79) (5.80)
4	$U_s^i = \lambda_\sigma - \mu_k$	(2.42)	$U_s^i = \beta_i p_s - \beta_i p_{sK}$	(4.7) (4.48)	$U_s^i = \beta_i p_s - \eta_{ki}$	(5.87)
5	$\lambda_r = \lambda_\sigma S_r^j$	(2.23) (2.43)	$p_r = p_s S_r^j$	(4.21)	$p_r = p_s S_r^j$	(4.21)
6	$\lambda_g = \lambda_\sigma S_g^j$	(2.20) (2.44)	$p_g = p_s S_g^j$	(4.51)	$p_g = p_s S_g^j$	(4.51)
7	$\dot{\mu}_g = \left(\delta + \alpha_g \right) \mu_g - \sum_i \lambda_{gi} - \lambda_g$	(2.45)	$\dot{\varphi}_g = \left(\delta + \alpha_g \right) p_v - p_g$	(4.26)	$\dot{\varphi}_g = \left(\delta + \alpha_g \right) p_v - \tau_g - p_g$ $\dot{\eta}_{gi} = \left(\delta + \alpha_g \right) \eta_{gi} - U_g^i + \beta_i \tau_{gi}$	(5.5) (5.81)
8	$\dot{\mu}_k = \left(\delta + \alpha_k \right) \mu_k - \sum_i \lambda_{ki}$	(2.29)	$\dot{\varphi}_k = \left(\delta + \alpha_k \right) p_{sK} - p_k$	(4.31)	$\dot{\eta}_{ki} = \left(\delta + \alpha_k \right) \eta_{ki} - \left(U_k^i + \beta_i p_r \frac{V_k^i}{V_r^i} \right) + \beta_i \tau_{ki}$	(5.71) (5.76)
9	$\lambda_r = \lambda_y Y_r$	(2.21)	$p_r = p_y Y_r$	(4.15)	$p_r = p_y Y_r$	(4.15)

The efficiency properties of a competitive equilibrium in model BLN2 are spelled out in

Proposition 5.4:

(i) *Set the prices*

$p_y = \lambda_y$, $p_r = \lambda_r$, $p_s = \lambda_\sigma$, $p_g = \lambda_g$, $p_v = \mu_g$,

and the tax rates

$\tau_g = \sum_i \lambda_{gi}$, $\tau_{gi} = -\sum_{h \neq i} \lambda_{gh} - \sum_j \lambda_{gj}$, $\tau_{ki} = -\sum_{h \neq i} \lambda_{kh}$, $\tau_v = -\mu_g$,

where μ_g, λ_g, λ_r, λ_σ *and* λ_y *are the values attained by the respective variables in the solution of (2.41) in section 2.*

Then at each point in time a general competitive equilibrium is attained in economy BLN2 and the associated allocation is efficient.

(ii) *If* τ_{BLN2} *is zero in all of its components, a general competitive equilibrium in economy BLN2 is inefficient.*

Column 3 of Table 5.6 exhibits all relevant FOCs from solving (4.11), (5.3), (5.12) and (5.85). The assignments made in *Proposition 5.4* turn the equations in column 3 into those of column 1.

The tax-subsidy scheme of *Proposition 5.4* that restores efficiency in the economy BLN2 has similar properties as in *Proposition 5.3*. Since cultural-goods inputs and cultural services are now private goods, the marginal conditions on the part of demanders in the rows 4 and 6 differ from those derived from the model BLN1.

5.2.3 Laissez-faire in the economies BLN1, BLN2 and transitional dynamics in simplified versions of these economies with Nash consumer-artists

We now proceed to study the no-policy scenario (laissez-faire) in the economies BLN1 and BLN2 by setting equal to zero the vector of tax rates from (5.83) and (5.88) respectively:

$$\tau_{BLN1} = \tau_{BLN2} = \left[\tau_g, \left(\tau_{gi} \right), \left(\tau_{ki} \right), \tau_v \right] \equiv 0 .$$

An immediate consequence is that the columns 1 and 3 in Table 5.5 and in Table 5.6, respectively, do not coincide anymore, implying that the market allocations are inefficient. Similar as in section 5.1.3 we wish to further specify this intertemporal misallocation. As before, we will address this issue by constructing a phase diagram. And we proceed as in section 3:

(i) First, we maintain the assumption, that the formation of cultural capital is endogenous but keep the stock of cultural goods constant ($\dot{g} \equiv 0$); as a result, new cultural goods are not created anymore. We then treat the formation of cultural goods as an endogenous process while the stock of cultural capital has no impact on the economy, implying that the consumption of cultural services does not play any role either.

(ii) To further simplify, all consumer-artists and cultural-services producers are assumed to be identical.

(iii) The parametric functions introduced in section 3 will be taken over in the subsequent analysis.

Table 5.7 provides an overview of the analytical agenda when the consumer-artists exhibit Nash behavior. We begin with the case $\mathrm{BLN\bar{G}1}$.

Table 5.7 Classification of market models with Nash consumer-artists

	Public-goods market economy BL1		Private-goods market economy BL2	
State variables	*g*: constant *k*: free	*g*: free *k*: no impact	*g*: constant *k*: free	*g*: free *k*: no impact
Nash behavior	$\mathrm{BLN\bar{G}1}$	$\mathrm{BLN\bar{K}1}$	$\mathrm{BLN\bar{G}2}$	$\mathrm{BLN\bar{K}2}$

5.2.3.1 The economy $\mathrm{BLN\bar{G}1}$

The optimization programs of firm Y, of firm G, and of the cultural-services firms remain unchanged by switching from the economy BLI1 to BLN1. Therefore the parameterized optimization programs (5.21), (5.22) and (5.23) apply, but the optimization calculus of consumer-artists now reads:

$$H^c = b_k k_c - \frac{d_k}{2} k_c{}^2 + b_s s_c - \frac{d_s}{2} s_c{}^2 + y_c + \lambda_c \left(p_r \bar{r} + \pi - p_{sc} s_c - p_y y_c \right)$$
$$+\eta_{ki} \left(s_c + \bar{s}_{-c} - \alpha_k k_c \right). \tag{5.90}$$

We keep taking the resource as numeraire ($p_r \equiv 1$) and consider the equilibrium conditions for public goods: $p_g = n_s p_{gs}$ and $p_s = n_c p_{sc}$. The FOCs of an interior solution to the Hamiltonian yield (5.25) - (5.29), (5.31) and:

$$s_c = \frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c} - \frac{a_y}{d_s n_c} p_{gs} + \frac{1}{d_s} \eta_{ki}, \tag{5.91}$$

$$\dot{\eta}_{ki} = \left(\delta + \alpha_k \right) \eta_{ki} - b_k + d_k k_c. \tag{5.92}$$

As in section 5.1.3.1 we need to distinguish two classes of solutions depending on whether $\lambda_G > 0$ or $\lambda_G = 0$. We know from section 5.1.3.1 that the case $\lambda_G > 0$ is quite trivial, in analytical terms. Therefore we will not investigate it here any further but rather restrict our focus on $\lambda_G = 0$. Since $\lambda_G = 0$ implies $p_g = n_s p_{gs} = 0$, (5.91) turns into

$$s_c^{N\bar{G}1} = \frac{b_s}{d_s} - \frac{a_y}{a_s d_s n_c} + \frac{1}{d_s} \eta_{ki}, \text{ or, equivalently, } \qquad \eta_{ki} = d_s s_c^{N\bar{G}1} - b_s + \frac{a_y}{a_s n_c}. \tag{5.93}$$

We now consider (5.25) - (5.29), (5.91) to reorganize (5.92) as follows:

$$\dot{\eta}_{ki} = \left(\delta + \alpha_k \right) \left(d_s s_c - b_s + \frac{a_y}{a_s n_c} \right) - b_k + d_k k_c ,. \tag{5.94}$$

Next we equate $\dot{\eta}_{ki}$ from (5.94) with the derivative of (5.93) with respect to time and obtain, after some rearrangement of terms,

$$\dot{s}_c^{N\bar{G}1} = -N_1 + M_2 n_c s_c^{N\bar{G}1} + N_2 k_c^{N\bar{G}1}, \tag{5.95}$$

where

$$N_1 := \frac{\left(\delta + \alpha_k \right) \left(a_s b_s n_c - a_y \right) + a_s b_k n_c}{a_s d_s n_c} \qquad \text{and} \qquad N_2 := \frac{d_k}{d_s}.$$

The term N_1 is positive, if and only if

$$n_c > n_{c0} := \frac{(\delta + \alpha_k) a_y}{(\delta + \alpha_k) a_s b_s + a_s b_k}.$$

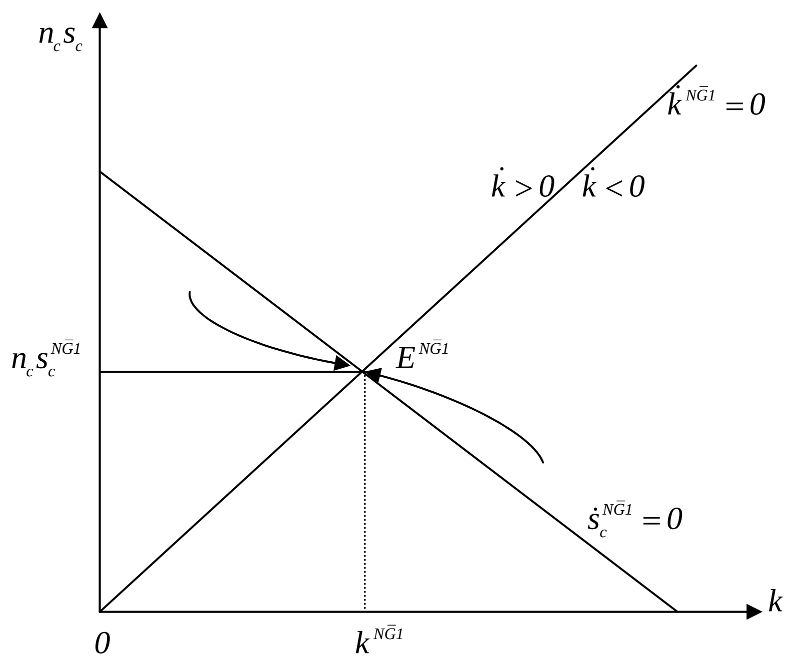
Our assumption that this inequality holds does not seem to be severely restrictive, since our focus is on economies with large numbers of consumer-artists. In addition, the condition $n_c > max[1, n_{c0}]$ needs to be satisfied.

We now apply the steady-state conditions $\dot{s}_c^{N\bar{G}1} = \dot{k} = 0$ to the two differential equations (5.95) and (3.2) to obtain

$$\left.\begin{aligned} -N_1 + M_2 n_c s_c^{N\bar{G}1} + N_2 k_c^{N\bar{G}1} = 0, \\ n_c s_c - \alpha_k k = 0. \end{aligned}\right\} \qquad (5.96)$$

The associated phase diagram of the economy BLN$\bar{\text{G}}$1 is depicted in Figure 5.6.

Figure 5.6 Phase diagram for model BLN$\bar{\text{G}}$1 **when cultural goods are abundant (** $p_{gs} = 0$ **)**



Since Figure 5.6 is similar in structure to Figure 3.4, the interpretation given in the context of Figure 3.4 will not be repeated here.

We now determine the steady-state values by solving (5.96)

$$k_c^{N\bar{G}1} := K_c^{N\bar{G}1}(n_c) = \frac{N_1}{\alpha_k M_2 + N_2} = \frac{\left[a_s b_k + a_s b_s (\delta + \alpha_k)\right] n_c - a_y (\delta + \alpha_k)}{a_s d_k n_c + a_s d_s \alpha_k (\delta + \alpha_k)}, \tag{5.97}$$

$$n_c s_c^{N\bar{G}1} = \frac{\alpha_k N_1}{\alpha_k M_2 + N_2} = \alpha_k \left\{ \frac{\left[a_s b_k + a_s b_s (\delta + \alpha_k)\right] n_c - a_y (\delta + \alpha_k)}{a_s d_k n_c + a_s d_s \alpha_k (\delta + \alpha_k)} \right\}. \tag{5.98}$$

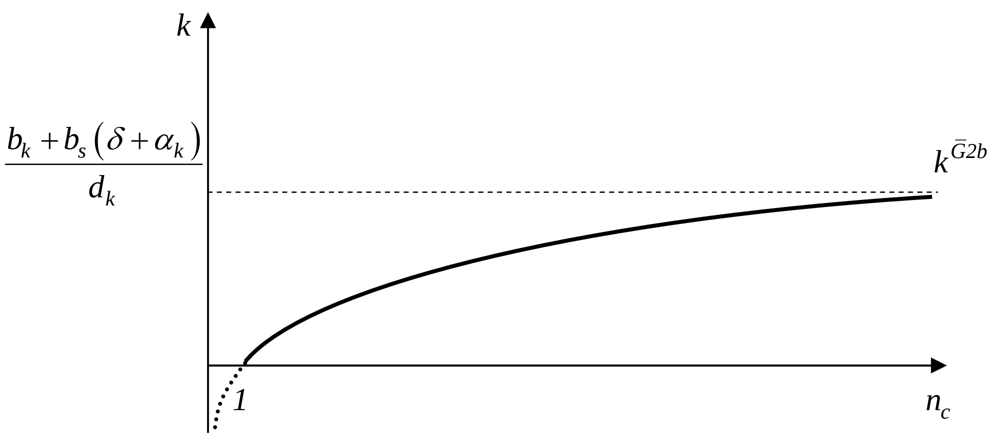
It is easy to establish that $k_c^{N\bar{G}1}$ and $s_c^{N\bar{G}1}$ are increasing in a_s, b_k and b_s, strictly declining in a_y, d_k and d_s, and ambiguous in δ and α_k. As in the previous model, a closer look at the link between the number of consumer-artists and the steady-state cultural capital is desirable. From (5.97) we readily infer

$$\frac{d\, k_c^{N\bar{G}1}}{dn_c} = \frac{(\delta + \alpha_k)\left[a_y d_k + a_s b_k d_s \alpha_k + a_s b_s d_s \alpha_k (\delta + \alpha_k)\right]}{a_s \left[d_k n_c + d_s \alpha_k (\delta + \alpha_k)\right]^2} > 0,$$

$$\text{and} \qquad \lim_{n_c \to \infty} k_c^{N\bar{G}1} = \frac{b_k + b_s (\delta + \alpha_k)}{d_k} > 0.$$

Hence $k_c^{N\bar{G}1}$ is strictly increasing in n_c and converges to the level $\left[b_k + b_s (\delta + \alpha_k)\right] / d_k$ for sufficiently large numbers of consumer-artists. This result is illustrated in Figure 5.7.

Figure 5.7 Cultural capital $k_c^{N\bar{G}1}$ and the number of consumer-artists n_c



Numerical examples

We apply the same specifications of parameters as in section 3.1.1.4:

$$a_s = 2,\ b_k = 3,\ d_k = 2,\ \alpha_k = 0{,}5,\ a_y = 1,\ b_s = 3,\ d_s = 2 \text{ and } \delta = 0{,}5,$$

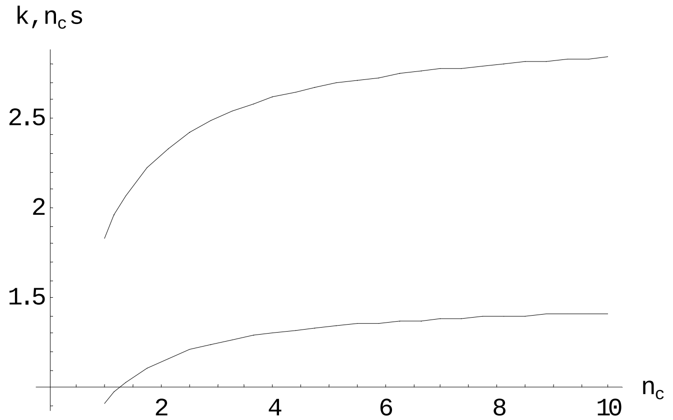
and obtain the result listed in Table 5.8.

Table 5.8 Dependence of $k_c^{N\bar{G}1}$ and $s_c^{N\bar{G}1}$ on n_c

n_c	*10*	*100*	10^6
$k_c^{N\bar{G}1}$	*2.83*	*2.98*	*3.00*
$n_c s_c^{N\bar{G}1}$	*1.42*	*1.49*	*1.50*

In Table 5.8 $k_c^{N\bar{G}1}$ and $n_c s_c^{N\bar{G}1}$ from (5.97) and (5.98), respectively, are calculated for three different values of n_c. Confirming our preceding conclusion, $k_c^{N\bar{G}1}$ converges to $\left\{\left[b_k + b_s\left(\delta + \alpha_k\right)\right]/d_k\right\} = 3$ for very large numbers of consumer-artists. Figure 5.8 illustrates the steady-state value of $k_c^{N\bar{G}1}$ and $n_c s_c^{N\bar{G}1}$ for all $n_c \in [1,10]$.

Figure 5.8 Numerical example for the dependence of $k_c^{N\bar{G}1}$ and $n_c s_c^{N\bar{G}1}$ on n_c



Comparison between the models $\mathrm{BLN\bar{G}1}$ **and** $\mathrm{S\bar{G}1}$

In view of (3.41) and (5.97), the difference between the steady-state values of cultural capital in the Pareto optimum $\mathrm{S\bar{G}1}$ and in the market economy $\mathrm{BLN\bar{G}1}$ is

$$D_k := k_c^{\bar{G}1b} - k_c^{N\bar{G}1} = \frac{M_1}{\alpha_k M_2 + M_3} - \frac{N_1}{\alpha_k M_2 + N_2}$$

$$= \frac{(n_c - 1) n_c (\delta + \alpha_k)\left(a_y d_k - a_s b_s d_k n_c + a_s b_k d_s \alpha_k\right)}{a_s\left[d_k n_c^{\,2} + d_k \alpha_k (\delta + \alpha_k)\right]}. \tag{5.99}$$

$$D_s := s_c^{\bar{G}1b} - s_c^{N\bar{G}1} = \frac{\alpha_k}{n_c}\left(\frac{M_1}{\alpha_k M_2 + M_3} - \frac{N_1}{\alpha_k M_2 + N_2}\right)$$

$$= \frac{\alpha_k}{n_c}\left\{\frac{(n_c - 1) n_c (\delta + \alpha_k)\left(a_y d_k - a_s b_s d_k n_c + a_s b_k d_s \alpha_k\right)}{a_s\left[d_k n_c^{\,2} + d_k \alpha_k (\delta + \alpha_k)\right]}\right\}. \tag{5.100}$$

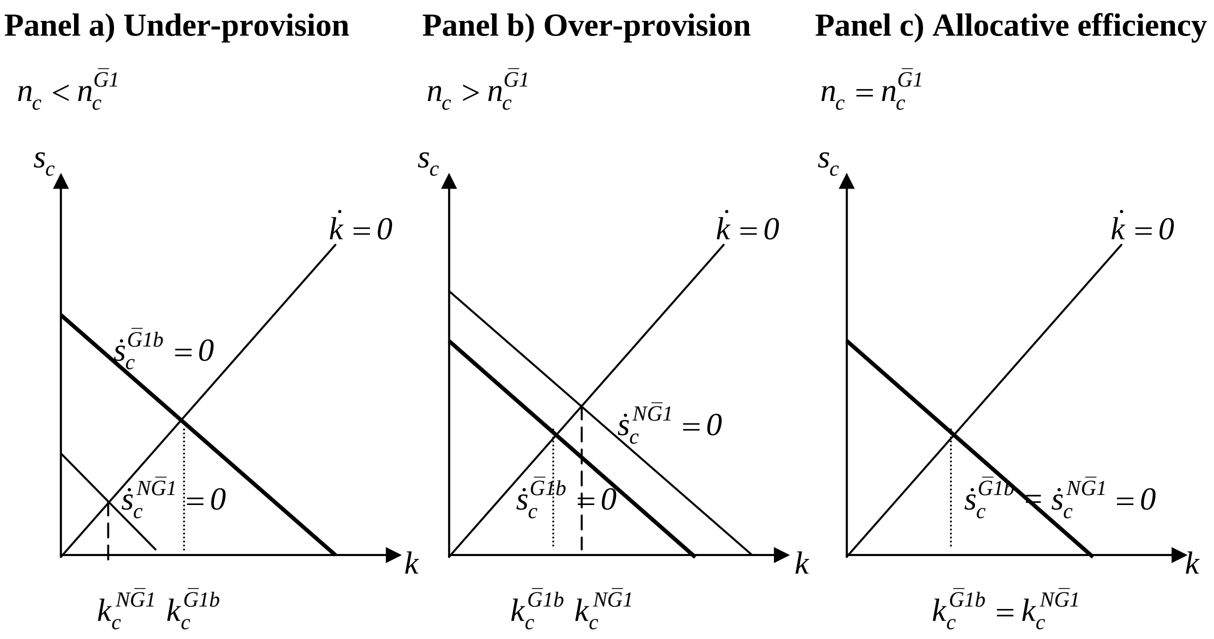
It follows that (for $n_c > 1$) $D_k \gtreqless 0$ and $D_s \gtreqless 0$, if and only if the condition (5.39) holds

$$n_c \lesseqgtr n_c^{\bar{G}1} := \frac{a_y d_k + a_s b_k d_s \alpha_k}{a_s b_s d_k}.$$

To avoid repetition we therefore omit detailed discussions of (5.99) and (5.100). It suffices to summarize that if (5.39) holds, then the market allocation is characterized by under-provision of both cultural capital and cultural services.

Figure 5.9 contains the phase diagrams for these different parameter constellations discussed above. Qualitatively, Figure 5.9 is very similar to Figure 5.2 except for the isocline $\dot{s}_c^{N\bar{G}1} = 0$ which is horizontal in Figure 5.2 but negatively sloped in Figure 5.9. However, since the comparison between the market allocation and the efficient allocation is analogous, we refrain from further interpretations of Figure 5.9 and turn to investigate the private-goods market economy $\mathrm{BLN\bar{G}2}$, instead.

Figure 5.9 **(In)efficiency of the market economy (Comparison of the models** $S\bar{G}1$ **and** $BLN\bar{G}1$)



5.2.3.2 The economy $BLN\bar{G}2$

Consider now the parameterized model BLN2 with a constant stock of cultural goods, in which cultural-goods inputs and cultural services are private. The optimization programs of firm G (5.21) and firm Y (5.23) remain the same in the private-goods model, while the optimization programs of cultural-services firm (5.22) and consumer-artist (5.90) are modified due to the change in the equilibrium conditions of private goods: $p_g = p_{gs}$ and $p_s = p_{sc}$. Taking the condition $p_{gs} = 0$ into account the demand for cultural services by consumer-artists now reads

$$s_c^{N\bar{G}2} = \frac{b_s}{d_s} - \frac{a_y}{a_s d_s} + \frac{1}{d_s}\eta_{ki}, \quad \text{or, equivalently,} \quad \eta_{ki} = d_s s_c^{N\bar{G}2} - b_s + \frac{a_y}{a_s}. \tag{5.101}$$

Applying the same procedure as before we take (5.101) into account and reorganize the differential equation (5.92) to obtain

$$\dot{s}_c^{N\bar{G}2} = -N_3 + M_2 n_c s_c^{N\bar{G}1} + N_2 k_c^{N\bar{G}1}, \tag{5.102}$$

where

$$N_3 := \frac{(\delta + \alpha_k)(a_s b_s - a_y) + a_s b_k}{a_s d_s}.$$

The term N_3 is positive, if and only if

$$(\delta + \alpha_k)(a_s b_s - a_y) + a_s b_k > 0. \tag{5.103}$$

Since the inequality (5.103) appears to be a weak restriction, we assume it to hold and thus take N_3 to be positive. Next we invoke the differential equations (5.102) and (3.2) that in steady state take the form

$$\left.\begin{aligned} -N_2 + M_2 n_c s_c^{N\bar{G}2} + N_2 k_c^{N\bar{G}2} = 0, \\ n_c s_c - \alpha_k k = 0. \end{aligned}\right\} \tag{5.104}$$

The steady-state values are

$$k_c^{N\bar{G}2} := K_c^{N\bar{G}2}(n_c) = \frac{N_3}{\alpha_k M_2 + N_2} = \frac{\left[a_s b_k + (a_s b_s - a_y)(\delta + \alpha_k)\right] n_c}{a_s d_k n_c + a_s d_s \alpha_k (\delta + \alpha_k)}, \tag{5.105}$$

$$n_c s_c^{N\bar{G}2} = \frac{\alpha_k N_3}{\alpha_k M_2 + N_2} = \alpha_k \left\{ \frac{\left[a_s b_k + (a_s b_s - a_y)(\delta + \alpha_k)\right] n_c}{a_s d_k n_c + a_s d_s \alpha_k (\delta + \alpha_k)} \right\}. \tag{5.106}$$

Inspection of (5.105) and (5.106) shows that $k_c^{N\bar{G}2}$ and $s_c^{N\bar{G}2}$ are strictly increasing in a_s, b_k and b_s, and strictly declining in a_y, d_k and d_s.

The phase diagram associated to the economy $\mathrm{BLN\bar{G}2}$ has a very similar structure as that of the model $\mathrm{BLN\bar{G}1}$. We hence omit the phase diagram, and proceed to focusing on the difference between the economies $\mathrm{S\bar{G}2}$ and $\mathrm{BLN\bar{G}2}$.

Comparison between the models $\mathrm{S\bar{G}2}$ **and** $\mathrm{BLN\bar{G}2}$

We now compare the steady-state values for cultural capital in the Pareto optimum $\mathrm{S\bar{G}2}$ and in the market economy $\mathrm{BLN\bar{G}2}$. Invoking (3.60), (3.61), (5.105) and (5.106) yields

$$D_k := k_c^{\bar{G}2b} - k_c^{N\bar{G}2} = \frac{M_4}{\alpha_k M_2 + M_3} - \frac{N_3}{\alpha_k M_2 + N_2}$$

$$= \frac{(n_c - 1) n_c (\delta + \alpha_k)(a_y d_k n_c - a_s b_s d_k n_c + a_s b_k d_s \alpha_k)}{a_s \left[d_k n_c^2 + d_s \alpha_k (\delta + \alpha_k) \right] \left[d_k n_c + d_s \alpha_k (\delta + \alpha_k) \right]}, \tag{5.107}$$

$$D_s := s_c^{\bar{G}2b} - s_c^{N\bar{G}2}$$

$$= \frac{\alpha_k}{n_c} \left\{ \frac{(n_c - 1) n_c (\delta + \alpha_k)(a_y d_k n_c - a_s b_s d_k n_c + a_s b_k d_s \alpha_k)}{a_s \left[d_k n_c^2 + d_s \alpha_k (\delta + \alpha_k) \right] \left[d_k n_c + d_s \alpha_k (\delta + \alpha_k) \right]} \right\}. \tag{5.108}$$

From (5.107) and (5.108) obviously follows:

$$\begin{cases} \text{If } (a_s b_s - a_y) \le 0, \text{ then } D_k > 0 \text{ and } D_s > 0, \ \forall n_c \ge 1, \\ \text{If } (a_s b_s - a_y) > 0 \text{ and } n_c^{\bar{G}2} := \dfrac{a_s b_k d_s \alpha_k}{a_s b_s - a_y} < 1, \text{ then } D_k > 0 \text{ and } D_s > 0, \ \forall n_c \ge 1, \\ \text{If } (a_s b_s - a_y) > 0 \text{ and } n_c^{\bar{G}2} > 1, \text{ then } D_k \gtreqless 0 \text{ and } D_s \gtreqless 0, \text{ if and only if } n_c \lesseqgtr n_c^{\bar{G}2}. \end{cases} \tag{5.109}$$

(5.109) hence concludes, if the consumer-artist's maximum marginal willingness-to-pay for cultural services (at $U_s|_{s=0}$) is greater [smaller] than the marginal rate of transformation between consumer goods and cultural services, $(a_s b_s - a_y) > 0$ [$(a_s b_s - a_y) < 0$] and the number of consumer-artists n_c is smaller [greater] than $n_c^{\bar{G}2}$, the market economy $\mathrm{BLN\bar{G}2}$ is Pareto inefficient in the sense that the cultural capital and cultural services are underprovided. Next we proceed the comparison between models $\mathrm{BLN\bar{G}1}$ and $\mathrm{BLN\bar{G}2}$.

Comparison between $\mathrm{BLN\bar{G}1}$ **and** $\mathrm{BLN\bar{G}2}$

By subtracting (5.97) from (5.105) and (5.98) from (5.106), the difference between the steady-state values of the public- goods model $\mathrm{BLN\bar{G}1}$ and the private-goods model $\mathrm{BLN\bar{G}2}$ is calculated as

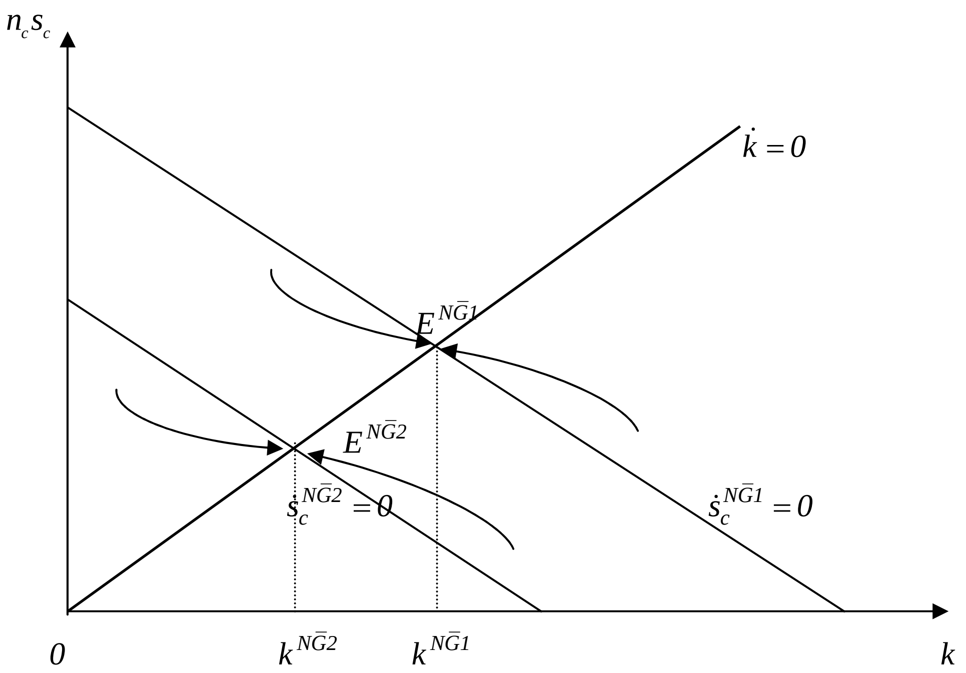
$$k_c^{N\bar{G}1} - k_c^{N\bar{G}2} = \frac{(\delta + \alpha_k) a_y (n_c - 1)}{a_s d_k n_c + a_s d_s \alpha_k (\delta + \alpha_k)}, \tag{5.110}$$

$$s_c^{N\bar{G}1} - s_c^{N\bar{G}2} = \frac{\alpha_k}{n_c} \left[\frac{(\delta + \alpha_k) a_y (n_c - 1)}{a_s d_k n_c + a_s d_s \alpha_k (\delta + \alpha_k)} \right]. \tag{5.111}$$

If $n_c = 1$, the jointly consumable consumer goods are not jointly consumed. As expected, in that case (5.110) and (5.111) imply $k^{N\bar{G}1} - k^{N\bar{G}2} = 0$ and $s_c^{N\bar{G}1} - s_c^{N\bar{G}2} = 0$, i.e. the outcomes of the public-goods market economy BLN$\bar{\text{G}}$1 and the private-goods market economy BLN$\bar{\text{G}}$2 coincide. However, if $n_c > 1$, (5.110) and (5.111) imply $k^{N\bar{G}1} - k^{N\bar{G}2} > 0$ and $s_c^{N\bar{G}1} - s_c^{N\bar{G}2} > 0$, i.e. the steady-state provision of cultural capital and cultural services in the public-goods market economy BLN$\bar{\text{G}}$1 is greater than in the private-goods market economy BLN$\bar{\text{G}}$2. The greater is the number of consumer-artists, the greater is the allocative deviation between the public-goods and the private-goods model.

The comments on (5.46) and (5.47) also apply here. We illustrate this result in Figure 5.10.

Figure 5.10 Comparison of the models BLN$\bar{\text{G}}$1 **and** BLN$\bar{\text{G}}$2



Since the interpretation of Figure 5.10 is similar to the previous ones (Figures 3.10, 5.4) it needs no further repetitive interpretation.

5.2.3.3 The economy $\mathrm{BLN\bar{K}1}$

Our objective now is to investigate the parameterized models BLN1 and BLN2 with the simplifying assumption that the cultural capital has no impact on the economy. Hence the stock of cultural goods is the only state variable left. We begin with analyzing the public-goods model. The optimization programs of firm G (5.3), cultural-services firms (5.22) and firm Y (5.23) remain the same as in section 5.1, whereas consumer-artist *i*'s optimization calculus must be modified. The associated Hamiltonian is now:

$$H^c = b_g g_c - \frac{d_g}{2} g_c{}^2 + b_s s_c - \frac{d_s}{2} s_c{}^2 + b_v v_c - \frac{d_v}{2} v_c{}^2 + y_c + \eta_{gi}\left(v_c + \bar{v}_{-c} - \alpha_g g_c\right)$$
$$+\lambda_v\left(a_v r_c - v_c\right) + \lambda_c\left(p_r \bar{r} + \pi + p_v v_c - p_r r_c - p_{sc} s_c - p_y y_c\right). \quad (5.112)$$

With the resource as numeraire, the solution of the Hamiltonians yields (4.24), (4.25), (5.5), (5.5') and (5.26) - (5.30) and

$$s_c = \frac{b_s}{d_s} - \frac{\lambda_c}{d_s} p_{sc}, \quad (5.113)$$

$$v_c = \frac{b_v}{d_v} - \frac{\lambda_v}{d_v} + \frac{\lambda_c}{d_v} p_v + \frac{\eta_{gi}}{d_v} \quad \text{or} \quad \eta_{gi} = d_v v_c - b_v + \lambda_v - \lambda_c p_v, \quad (5.114)$$

$$\dot{\eta}_{gi} = \left(\delta + \alpha_g\right)\eta_{gi} - b_g + d_g g_c. \quad (5.115)$$

Making use of (4.24), (4.25), (5.5'), (5.32), (5.34) and (5.52), we rearrange (5.114) to obtain

$$v_c = \frac{b_v}{d_v} - \frac{a_y}{a_v d_v} + \frac{a_y}{d_v}\varphi_g + \frac{\eta_{gi}}{d_v}. \quad (5.116)$$

Since in equilibrium the marginal condition (4.23) characterizing the optimal production plan of firm G must equal (5.115), we infer

$$\eta_{gi} = p_v \quad \text{and} \quad p_g = b_g - d_g g_c. \quad (5.117)$$

The equations imply $\varphi_g = \eta_{gi}$ with the consequence that

$$\varphi_g = \frac{1}{1+a_y}\left(d_v v_c - b_v + \frac{a_y}{d_v} \right). \tag{5.118}$$

By taking the equilibrium conditions $s_c = n_s s_s = n_s g_s = n_s g_c$ into account, one calculates the equilibrium price for the stock of cultural goods as

$$p_g = \frac{b_s}{a_y} n_c n_s - \frac{d_s}{a_y} n_c n_s^{2} g_c - \frac{1}{a_s} n_s . \tag{5.119}$$

By inserting (5.117) - (5.119) into (5.116) one obtains, after some rearrangement of terms,

$$\dot{\varphi}_g = \left(\delta + \alpha_g\right)\left[\frac{1}{1+a_y}\left(d_v v_c - b_v + \frac{a_y}{a_v} \right)\right] - \left(\frac{b_s}{a_y} n_c n_s - \frac{d_s}{a_y} n_c n_s^{2} g_c - \frac{1}{a_s} n_s \right). \tag{5.120}$$

Differentiation of (5.116) with respect to time yields $\dot{\varphi}_g = \left[d_v / \left(1+a_y\right)\right] \dot{v}_c$. The equation is plugged into (5.119) which can thus be turned into:

$$\dot{v}_c = \frac{1+a_y}{d_v}\left\{\left[\left(\delta+\alpha_g\right)\left(\frac{1}{1+a_y}\right)\left(-b_v + \frac{a_y}{a_v}\right) - \frac{b_s}{a_y} n_c n_s + \frac{1}{a_s} n_s \right]\right\}$$
$$+\left(\delta + \alpha_g\right) v_c + \frac{\left(1+a_y\right) d_s}{a_y d_v} n_c n_s^{2} g_c , \tag{5.121}$$

or, more compactly, into

$$\dot{v}_c = -N_4 + M_6 n_c v_c + N_5 g_c , \tag{5.122}$$

where

$$N_4 := \left[\frac{\left(\delta+\alpha_g\right)\left(a_s a_v a_y b_v - a_s a_y^{2}\right) + a_s a_v \left(1+a_y\right) b_s n_c n_s - a_v a_y \left(1+a_y\right) n_s}{a_s a_v a_y d_v} \right],$$

$$N_5 := \frac{\left(1+a_y\right) d_s}{a_y d_v} n_c n_s^{2} > 0 .$$

N_4 is positive, if and only if

$$n_c > n_{c0} := \frac{a_y}{a_s b_s} - \frac{\left(\delta + \alpha_g\right)\left(a_v a_y b_v - a_y^2\right)}{a_v b_s \left(1 + a_y\right) n_s}.$$

Since we are interested in economies with large number of consumer-artists we assume that N_4 is positive. In addition, the condition $n_c > max\left[1, n_{c0}\right]$ needs to be satisfied.

The combination of (3.72) and (5.122) forms a system of two differential equations which yields in the steady state:

$$\left.\begin{aligned} &-N_4 + M_6 n_c v_c + N_5 g_c = 0, \\ &n_c v_c - \alpha_g g = 0. \end{aligned}\right\} \tag{5.123}$$

The phase diagram of economy $\mathrm{BLN\bar{K}1}$ has the similar structure as that shown in Figures 3.11 and 5.3. We hence do not reproduce it here.

The solution of (5.123) gives rise to the steady state values

$$g^{N\bar{K}1} := G^{N\bar{K}1}\left(n_c, n_s\right) = \frac{N_4}{\alpha_g M_6 + N_5}$$

$$= \frac{a_s a_v \left(1 + a_y\right) b_s n_s n_c^2 + \left[a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) - a_v a_y \left(1 + a_y\right) n_s\right] n_c}{a_s a_v \left[\left(1 + a_y\right) d_s n_s^2 n_c^2 + a_y d_v \alpha_g \left(\delta + \alpha_g\right)\right]}, \tag{5.124}$$

$$= \frac{a_v \left(1 + a_y\right)\left(a_s b_s n_c - a_y\right) n_c n_s - a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) n_c}{a_s a_v \left[\left(1 + a_y\right) d_s n_c^2 n_s^2 + a_y d_v \alpha_g \left(\delta + \alpha_g\right)\right]}, \tag{5.124'}$$

$$n_c v^{N\bar{K}1} = \frac{\alpha_g N_4}{\alpha_g M_6 + N_5}$$

$$= \alpha_g \left\{ \frac{a_s a_v \left(1 + a_y\right) b_s n_s n_c^2 + \left[a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) - a_v a_y \left(1 + a_y\right) n_s\right] n_c}{a_s a_v \left[\left(1 + a_y\right) d_s n_s^2 n_c^2 + a_y d_v \alpha_g \left(\delta + \alpha_g\right)\right]} \right\}. \tag{5.125}$$

The dependence of $g^{N\bar{K}1}$ and $v^{N\bar{K}1}$ on the parameters n_c, n_s, a_s, a_v, a_y, b_s, b_v, d_s, d_v, δ and α_g is very complex. However, since the equations (5.124), (5.124') and (5.125) have similar structures as the equations (3.108), (3.108') and (3.109) of section 3.2.1.3, we apply the same procedure as in section 3.2.1.3 and avoid the repetitive interpretations. Now we pay our attention merely to the dependence of $g^{N\bar{K}1}$ and $v^{N\bar{K}1}$ on n_c and n_s. The derivative of $g^{N\bar{K}1}$ with respect to n_c reads

$$\frac{dg^{N\bar{K}1}}{dn_c} = \frac{-\left(1+a_y\right)d_s n_s^2 E n_c^2 + 2a_s a_v a_y \left(1+a_y\right) b_s d_v \alpha_g \left(\delta+\alpha_g\right) n_s n_c}{a_s a_v \left[\left(1+a_y\right) d_s n_s^2 n_c^2 + a_y d_v \alpha_g \left(\delta+\alpha_g\right)\right]^2}$$

$$\frac{+a_y d_v \alpha_g \left(\delta+\alpha_g\right) E}{a_s a_v \left[\left(1+a_y\right) d_s n_s^2 n_c^2 + a_y d_v \alpha_g \left(\delta+\alpha_g\right)\right]^2}, \tag{5.126}$$

where $E := a_s a_y \left(a_v b_v - a_y\right)\left(\delta+\alpha_g\right) - a_v a_y \left(1+a_y\right) n_s$. The derivative of $g^{N\bar{K}1}$ with respect to n_s is

$$\frac{dg^{N\bar{K}1}}{dn_s} = \frac{n_c \left(1+a_y\right)\left[a_v \left(1+a_y\right) d_s F n_c^2 n_s^2 - 2a_s a_y \left(a_y - a_v b_v\right)\left(\delta+\alpha_g\right) n_c^2 n_s\right.}{a_s a_v \left[\left(1+a_y\right) d_s n_c^2 n_s^2 + a_y d_v \alpha_g \left(\delta+\alpha_g\right)\right]^2}$$

$$-\frac{\left. a_v a_y d_v \alpha_g \left(\delta+\alpha_g\right) F\right]}{a_s a_v \left[\left(1+a_y\right) d_s n_c^2 n_s^2 + a_y d_v \alpha_g \left(\delta+\alpha_g\right)\right]^2}, \tag{5.127}$$

where $F := \left(a_y - a_s b_s n_c\right)$. To determine the signs of $dg^{N\bar{K}1} / dn_c$ we have to distinguish three cases depending on the sign of *E*.

Case 1.1 $E < 0$:

Under this condition (5.126) implies

$$\frac{dg^{N\bar{K}1}}{dn_c} \gtreqless 0 \iff n_c \lesseqgtr n_{M1}^{N\bar{K}1},$$

where

$$n_{M1}^{N\bar{K}1} := \frac{a_s a_v a_y \left(1+a_y\right) b_s d_v \alpha_g \left(\delta+\alpha_g\right) - \left\{\left[a_s a_v a_y \left(1+a_y\right) b_s d_v \alpha_g \left(\delta+\alpha_g\right)\right]^2\right.}{\left(1+a_y\right) d_s n_s E}$$

$$+ \frac{\left. E^2 a_y \left(1+a_y\right) d_s d_v \alpha_g \left(\delta+\alpha_g\right)\right\}^{\frac{1}{2}}}{\left(1+a_y\right) d_s n_s E} .$$

Case 1.2 $E = 0$:

(5.126) becomes

$$\frac{dg^{N\bar{K}1}}{dn_c} = \frac{2a_y \left(1+a_y\right) b_s d_v \alpha_g \left(\delta+\alpha_g\right) n_s n_c}{\left[\left(1+a_y\right) d_s n_s^2 n_c^2 + a_y d_v \alpha_g \left(\delta+\alpha_g\right)\right]^2} > 0 .$$

Case 1.3 $E > 0$:

Now (5.126) implies

$$\frac{dg^{N\bar{K}1}}{dn_c} \lesseqgtr 0 \Leftrightarrow n_c \lesseqgtr n_{M3}^{N\bar{K}1} ,$$

where

$$n_{M3}^{N\bar{K}1} := \frac{a_s a_v a_y \left(1+a_y\right) b_s d_v \alpha_g \left(\delta+\alpha_g\right) + \left\{\left[a_s a_v a_y \left(1+a_y\right) b_s d_v \alpha_g \left(\delta+\alpha_g\right)\right]^2\right.}{\left(1+a_y\right) d_s n_s E}$$

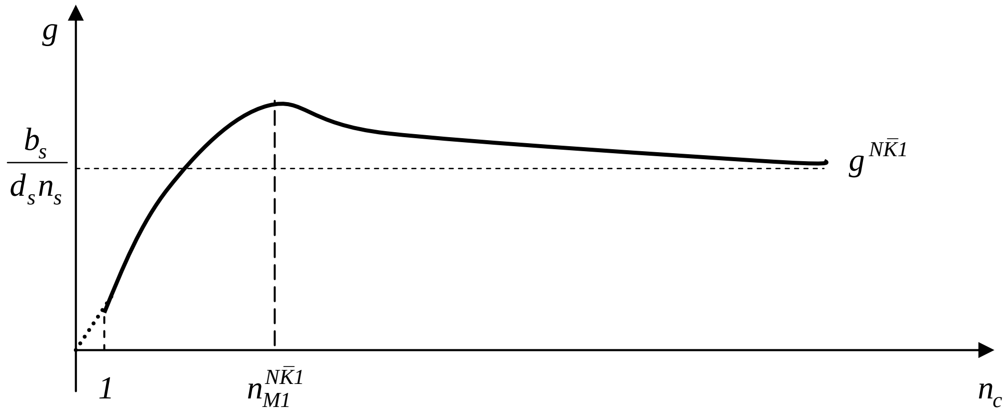
$$+ \frac{\left. E^2 a_y \left(1+a_y\right) d_s d_v \alpha_g \left(\delta+\alpha_g\right)\right\}^{\frac{1}{2}}}{\left(1+a_y\right) d_s n_s E} .$$

In addition, the following observation is straightforward

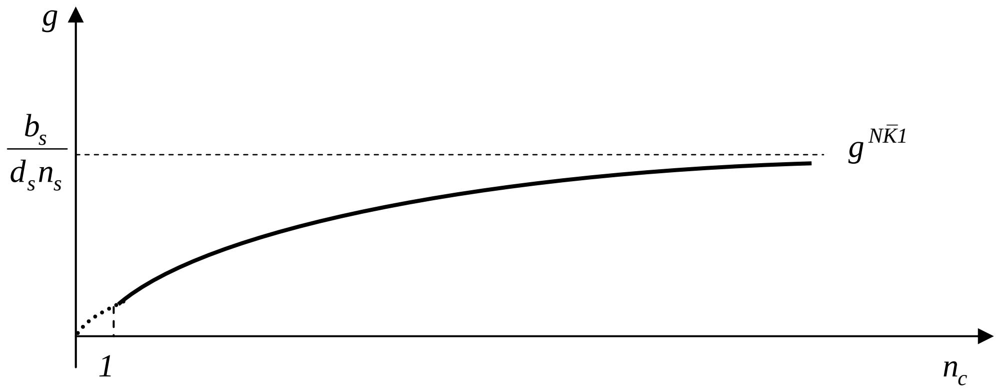
$$\lim_{n_c \to \infty} g^{N\bar{K}1} = \frac{b_s}{d_s n_s} .$$

We now illustrate those cases in Figure 5.11.

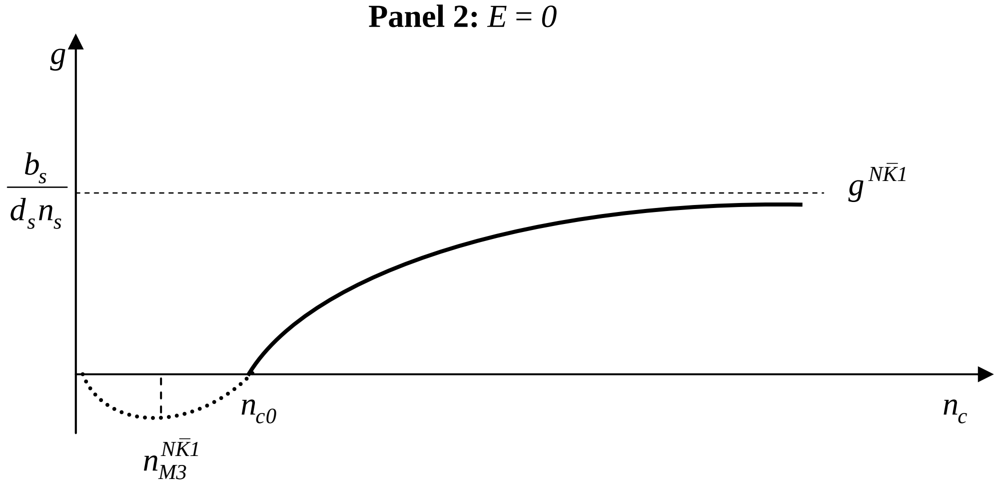
Figure 5.11 Different shapes of the curve $g^{N\bar{K}1} = G^{N\bar{K}1}\left(n_c, \bar{n}_s\right)$



Panel 1: $E < 0$



Panel 2: $E = 0$



Panel 3: $E > 0$

We now turn to the discussion of the sign of $dg^{N\bar{K}1} / dn_s$. For the sake of simplicity we suppose that $\left(a_y - a_v b_v\right) > 0$. It suffices then to only distinguish three cases depending on the sign of the *F*.

Case 2.1 *F < 0*:

(5.127) implies

$$\frac{dg^{N\bar{K}1}}{dn_s} \gtreqless 0 \Leftrightarrow n_s \lesseqgtr n_{sM1}^{N\bar{K}1},$$

where

$$n_{sM1}^{N\bar{K}1} := -\frac{a_s a_y \left(a_y - a_v b_v\right)\left(\delta + \alpha_g\right) n_c}{a_v \left(1 + a_y\right) F d_s n_c}$$
$$-\frac{\left\{\left[a_s a_y \left(a_y - a_v b_v\right)\left(\delta + \alpha_g\right) n_c\right]^2 + a_v{}^2 a_y \left(1 + a_y\right) d_s d_v \alpha_g \left(\delta + \alpha_g\right) F^2\right\}^{\frac{1}{2}}}{a_v \left(1 + a_y\right) F d_s n_c}.$$

Case 2.2 *F = 0*:

Under this condition (5.127) becomes

$$\frac{dg^{N\bar{K}1}}{dn_s} = \frac{-2a_y \left(1 + a_y\right)\left(a_y - a_v b_v\right)\left(\delta + \alpha_g\right) n_c{}^3 n_s}{a_v \left[\left(1 + a_y\right) d_s n_c{}^2 n_s{}^2 + a_y d_v \alpha_g \left(\delta + \alpha_g\right)\right]^2} < 0 .$$

Note that

$$g^{N\bar{K}1} = \left.\frac{-a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) n_c}{a_s a_v \left[\left(1 + a_y\right) d_s n_c{}^2 n_s{}^2 + a_y d_v \alpha_g \left(\delta + \alpha_g\right)\right]}\right|_{n_s = 0} > 0 .$$

Hence $g^{N\bar{K}1}$ is strictly decreasing in n_s for $n_s \in [1, \infty[$.

Case 2.3 *F > 0*:

The equation (5.127) implies

$$\frac{dg^{N\bar{K}1}}{dn_s} \lesseqgtr 0 \Leftrightarrow n_s \lesseqgtr n_{sM3}^{N\bar{K}1},$$

where

$$n_{sM3}^{N\bar{K}1} := -\frac{a_s a_y \left(a_y - a_v b_v\right)\left(\delta + \alpha_g\right) n_c}{a_v \left(1 + a_y\right) F d_s n_c}$$

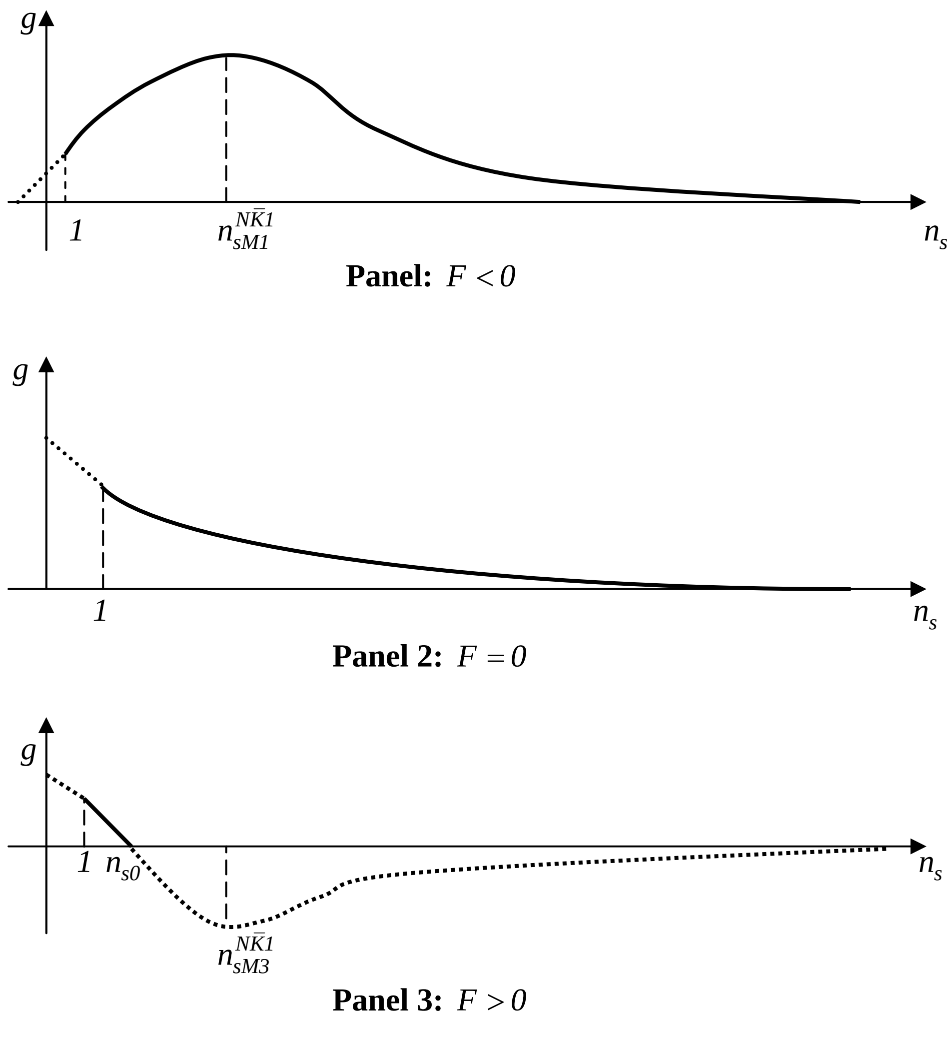
$$+ \frac{\left\{\left[a_s a_y \left(a_y - a_v b_v\right)\left(\delta + \alpha_g\right) n_c\right]^2 + {a_v}^2 a_y \left(1 + a_y\right) d_s d_v \alpha_g \left(\delta + \alpha_g\right) F^2\right\}^{\frac{1}{2}}}{a_v \left(1 + a_y\right) F d_s n_c}.$$

In addition, the following attribute of $g^{N\bar{K}1} = G^{N\bar{K}1}\left(\bar{n}_c, n_s\right)$ is straightforward:

$$\lim_{n_s \to \infty} g^{N\bar{K}1} = 0.$$

We now depict those different cases in Figure 5.12.

Figure 5.12 Different shapes of the curve $g^{N\bar{K}1} = G^{N\bar{K}1}\left(\bar{n}_c, n_s\right)$



Comparison between the steady states of the models $\text{BLN}\bar{\text{K}}1$ **and** $\text{S}\bar{\text{K}}1$

To compare the provision of newly created cultural goods and the stock of cultural goods in the market economy model $\text{BLN}\bar{\text{K}}1$ and the model $\text{S}\bar{\text{K}}1$**,** we invoke (3.108), (3.109), (5.124) and (5.125) to obtain the differences:

$$D_g\left(n_c,n_s\right):=g_c^{\bar{K}1}-g_c^{N\bar{K}1}=\frac{M_5\left(\alpha_g M_6+N_5\right)-N_4\left(\alpha_g M_6+M_7\right)}{\left(\alpha_g M_6+M_7\right)\left(\alpha_g M_6+N_5\right)}, \tag{5.128}$$

$$D_v\left(n_c,n_s\right):=v_c^{\bar{K}1}-v_c^{N\bar{K}1}=\frac{\alpha_g}{n_c}\left(g_c^{\bar{K}1}-g_c^{N\bar{K}1}\right). \tag{5.129}$$

The signs of D_g and D_v are indeterminate. By expanding the numerator of (5.128) we find after some rearrangement of terms that $D_g > 0$ and $D_v > 0$, if and only if $A > 0$, where

$$A:=A_1 n_c^3+A_2 n_c^2+A_3 n_c+A_4, \tag{5.130}$$

and where

$$A_1:=\left[a_s a_v\left(1+a_y\right)b_g d_s n_s^2-a_s a_v\left(1+a_y\right)b_s d_g n_s\right],$$

$$A_2:=-\left[a_v a_y\left(1+a_y\right)d_s n_s^3+a_s\left(a_y-a_v b_v\right)d_s\left(\delta+\alpha_g\right)n_s^2+a_s a_v a_y b_v d_g\delta-a_s a_y^2 d_g\left(\delta+\alpha_g\right)\right],$$

$$A_3:=-\left[a_s a_v b_s d_v\alpha_g\left(\delta+\alpha_g\right)n_s-a_s a_v a_y b_g d_v\alpha_g\left(\delta+\alpha_g\right)\right],$$

$$A_4:=a_s a_y^2 d_v\alpha_g\left(\delta+\alpha_g\right)n_s.$$

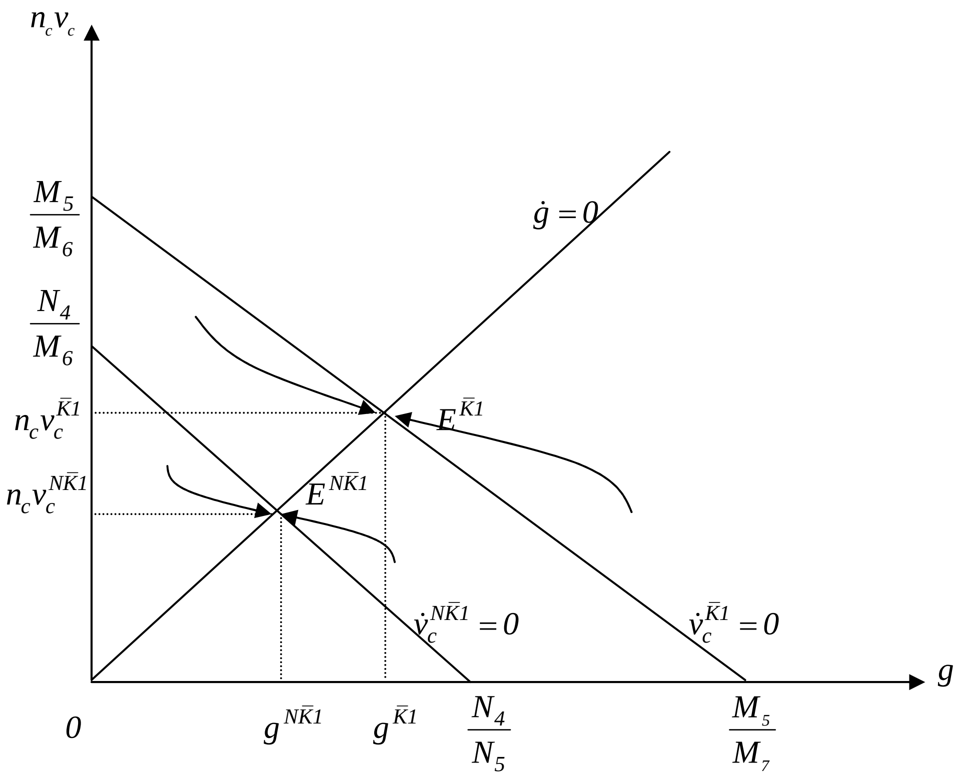
Though the sign of A is ambiguous, we can determine, at least,

$$\text{sign}\left(\lim_{n_c\to\infty}A\right)=\text{sign}\lim_{n_c\to\infty}\left(A_1 n_c^3+A_2 n_c^2+A_3 n_c^1+A_4\right)$$

$$=\text{sign}\lim_{n_c\to\infty}\left[n_c^3\left(A_1+\frac{A_2}{n_c}+\frac{A_3}{n_c^2}+\frac{A_4}{n_c^3}\right)\right]=\text{sign}\ A_1.$$

Obviously it is true that $\left(b_g d_s n_s-b_s d_g\right)>0$ (or $n_s>\left(b_s d_g/b_g d_s\right)$). This inequality is hence identical to (3.133). Furthermore, we have also discussed this inequality in (5.61) and (5.62) by comparing the models $\mathrm{BLI\bar{K}1}$ and $\mathrm{S\bar{K}1}$. We therefore avoid to repeat the interpretations and conclude that, $D_g > 0$ and $D_v > 0$, under the condition $n_s > \left(b_s d_g / b_g d_s\right)$. In other words, the steady-state provision of newly created cultural goods and the stock of cultural goods in the economy $\mathrm{S\bar{K}1}$ is greater than in the market economy $\mathrm{BLN\bar{K}1}$. The market result is inefficient. The result is presented in Figure 5.13.

Figure 5.13 Inefficiency of the market economy $\mathrm{BLN\bar{K}1}$



5.2.3.4 The economy $\mathrm{BLN\bar{K}2}$

For the case that cultural-goods inputs for cultural-services firms and cultural services for consumer-artists are private goods, the optimization problems of firm G (5.3) and firm Y (5.23) remain the same. However, the supply constraints (3.21), (3.49) and (3.73) now read

$$n_c s_c = n_s s_s = n_s g_s = g \; ,$$

and the equilibrium conditions $p_g = n_s p_{gs}$ and $p_s = n_c p_{sc}$ are replaced by $p_g = p_{gs}$ and $p_s = p_{sc}$. As a consequence, the optimization programs of the cultural-services firm and the consumer-artist now read

$$H^c = b_g g_c - \frac{d_g}{2} g_c^{2} + b_s s_c - \frac{d_s}{2} s_c^{2} + b_v v_c - \frac{d_v}{2} v_c^{2} + y_c + \eta_{gi}\left(v_c + \overline{v}_{-c} - \alpha_g g_c\right)$$

$$+\lambda_v\left(a_v r_c - v_c\right) + \lambda_c\left(p_r \overline{r} + \pi + p_v v_c - p_r r_c - p_s s_c - p_y y_c\right), \tag{5.131}$$

$$H^s = p_s s_s - p_r r_s - p_g g_s + \lambda_{s1}\left(a_s r_s - s_s\right) + \lambda_{s2}\left(g_s - s_s\right). \tag{5.132}$$

The FOCs of an interior solution to the Hamiltonians (5.3), (5.23), (5.131) and (5.132) are (5.25) through (5.30), (5.52), (5.114) and

$$s_c = \frac{b_s}{d_s} - \frac{\lambda_c}{d_s} p_s.$$

Taking those FOCs into account yields

$$p_g = \frac{b_s}{a_y} - \frac{1}{a_s} - \frac{d_s n_s}{a_y n_c} g_c.$$

We insert this equation into (5.5), use the same argument as in (5.117) and apply the same procedure as in the public-goods model $\mathrm{BLN\overline{K}1}$, to obtain after some rearrangements of terms,

$$\dot{\varphi}_g = \left(\delta + \alpha_g\right)\left[\frac{1}{1+a_y}\left(d_v v_c - b_v + \frac{a_y}{a_v}\right)\right] - \left(\frac{b_s}{a_y} - \frac{d_s}{a_y}\frac{n_s}{n_c} g_c - \frac{1}{a_s}\right). \tag{5.133}$$

Combining (5.133) with the derivative of (5.118) with respect to time, $\dot{\varphi}_g = \left[d_v / \left(1 + a_y\right)\right]\dot{v}_c$, gives us

$$\dot{v}_c = \frac{1+a_y}{d_v}\left\{\left[\left(\delta + \alpha_g\right)\left(\frac{1}{1+a_y}\right)\left(-b_v + \frac{a_y}{a_v}\right) - \frac{b_s}{a_y} + \frac{1}{a_s}\right]\right\}$$

$$+\left(\delta + \alpha_g\right) v_c + \frac{\left(1+a_y\right) d_s n_s}{a_y d_v n_c} g_c, \tag{5.134}$$

or, in a more compact form,

$$\dot{v}_c = -N_6 + M_6 n_c v_c + N_7 g_c, \tag{5.135}$$

where

$$N_6 := \left[\frac{a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) + a_v \left(1 + a_y\right)\left(a_s b_s - a_y\right)}{a_s a_v a_y d_v} \right] \text{ and } N_7 := \frac{\left(1 + a_y\right) d_s n_s}{a_y d_v n_c} > 0 .$$

Sufficient for $N_6 > 0$ is the inequality (5.66) which we will assume to hold.

The equations (3.72) and (5.135) constitute a system of two differential equations which reads, after imposing the steady-state conditions $\dot{v}_c = \dot{g} = 0$,

$$\left. \begin{array}{l} -N_6 + M_6 n_c v_c + N_7 g_c = 0, \\ n_c v_c - \alpha_g g = 0. \end{array} \right\} \tag{5.136}$$

The phase diagram of the economy $\mathrm{BLN\bar{K}2}$ is qualitatively the same as that of Figures 3.11 and 5.3. We hence refrain from reprinting it. In view of (5.136) the steady-state values of the stock of cultural goods and newly created cultural goods are

$$\begin{aligned} g^{N\bar{K}2} &:= G^{N\bar{K}2}\left(n_c, n_s\right) = \frac{N_6}{\alpha_g M_6 + N_7} \\ &= \frac{\left[a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) + a_v \left(a_s b_s - a_y\right)\left(1 + a_y\right)\right] n_c}{a_s a_v \left[\left(1 + a_y\right) d_s n_s + a_y d_v \alpha_g \left(\delta + \alpha_g\right)\right]}, \end{aligned} \tag{5.137}$$

$$n_c v^{N\bar{K}2} = \frac{\alpha_g N_6}{\alpha_g M_6 + N_7} . \tag{5.138}$$

In view of (5.137) and (5.138) we infer that $g^{N\bar{K}2}$ and $n_c v^{N\bar{K}2}$ are strictly increasing in a_s, a_v, b_s and b_v, and strictly decreasing in a_y, d_s and d_v. However, the impact of changes in n_c and n_s on $g^{N\bar{K}2}$ depends on the sign of the term $\left[a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right) + a_v \left(a_s b_s - a_y\right)\left(1 + a_y\right)\right]$ in the numerator on the RHS of (5.137). Since we have assumed the inequality in (5.66) to be satisfied, the sign is positive. We then turn to study the dependence between n_c, n_s and $g^{N\bar{K}2}$. The derivative of $g^{N\bar{K}2}$ with respect to n_c is

$$\frac{dg^{N\bar{K}2}}{dn_c}=\frac{\left[a_s a_y\left(a_v b_v-a_y\right)\left(\delta+\alpha_g\right)+a_v\left(a_s b_s-a_y\right)\left(1+a_y\right)\right]}{a_s a_v\left[\left(1+a_y\right)d_s n_s+a_y d_v \alpha_g\left(\delta+\alpha_g\right)\right]}>0\,. \tag{5.139}$$

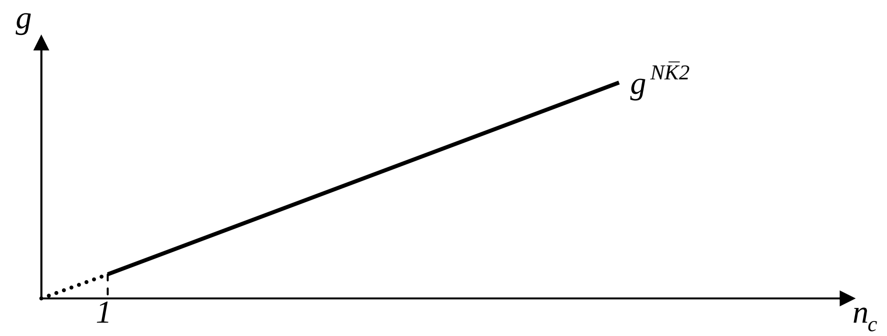
Hence $g^{N\bar{K}2}$ is strictly increasing in n_c. The derivative of $g^{N\bar{K}2}$ with respect to n_s is

$$\frac{dg^{N\bar{K}2}}{dn_s}=\frac{-\left(1+a_y\right)\left[a_s a_y\left(a_v b_v-a_y\right)\left(\delta+\alpha_g\right)+a_v\left(a_s b_s-a_y\right)\left(1+a_y\right)\right]n_c}{a_s a_v\left[\left(1+a_y\right)d_s n_s+a_y d_v \alpha_g\left(\delta+\alpha_g\right)\right]^2}<0\,, \tag{5.140}$$

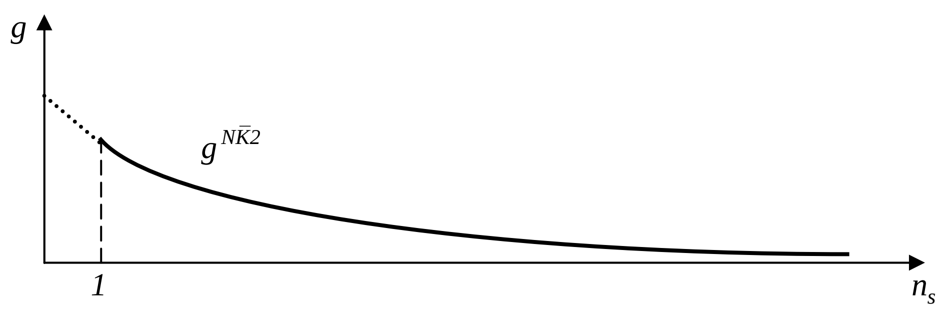
$g^{N\bar{K}2}$ is strictly decreasing in n_s, and we note, in addition, that $\lim_{n_s\to\infty} g^{N\bar{K}2}=0$.

We now illustrate the results in Figure 5.14.

Figure 5.14 Dependence between n_c, n_s and $g^{N\bar{K}2}$



Panel: $g^{N\bar{K}2}=G^{N\bar{K}2}\left(n_c,\bar{n}_s\right)$



Panel 2: $g^{N\bar{K}2}=G^{N\bar{K}2}\left(\bar{n}_c,n_s\right)$

Since the curve shown in Panel 1 is straightforward, and the one drawn in Panel 2 is very similar to the previous one (Figure 3.14, Panel 2), we refrain from repeating the interpretation and proceed directly to explore the difference between the Pareto-efficient economy $\mathrm{S\bar{K}2}$ and the market economy $\mathrm{BLN\bar{K}2}$.

The steady-state allocation of the economies $\mathrm{BLN\bar{K}2}$ and $\mathrm{S\bar{K}2}$ in comparison

We now compare, pairwise, (3.124) with (5.137) and (3.125) with (5.138) to get

$$D_g := g_c^{\bar{K}2} - g_c^{N\bar{K}2} = \frac{M_8\left(\alpha_g M_6 + N_7\right) - N_6\left(\alpha_g M_6 + M_9\right)}{\left(\alpha_g M_6 + M_9\right)\left(\alpha_g M_6 + N_7\right)}, \tag{5.141}$$

$$D_v := v_c^{\bar{K}2} - v_c^{N\bar{K}2} = \frac{\alpha_g}{n_c}\left(g_c^{\bar{K}2} - g_c^{N\bar{K}2}\right). \tag{5.142}$$

The signs of D_g and D_v are ambiguous. We expand the numerator of (5.141) and find, after some rearrangement of terms, that $D_g > 0$ and $D_v > 0$, if and only if $E > 0$, where

$$E := E_1 n_c^2 + E_2 n_c + E_3, \tag{5.143}$$

and where

$$E_1 := \left[a_s a_y\left(a_y - a_v b_v\right)\left(\delta + \alpha_g\right) + a_v\left(a_y - a_s b_s\right)\left(1 + a_y\right)\right] d_g,$$

$$E_2 := a_s a_v b_g\left[d_s\left(1 + a_y\right)n_s + a_y d_v \alpha_g\left(\delta + \alpha_g\right)\right],$$

$$E_3 := \left(1 + a_y\right)d_s\left[a_v\left(a_v b_s - a_y\right) + a_s\left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right)\right]n_s + a_s a_y d_s\left(\delta + \alpha_g\right)\left(a_y - a_v b_v\right)$$
$$+ a_v d_s\left(1 + a_y\right)\left(a_y - a_v b_s\right) + a_v d_v \alpha_g\left(\delta + \alpha_g\right)\left(a_y - a_s b_s\right).$$

Suppose $E_1 > 0$:

(i) If $E_2^2 < 4E_1E_3$, then $D_g > 0$ for all $n_c \geq 1$.

(ii) If $E_2^2 > 4E_1E_3$, then

(a) $D_g < 0$ for $n_c \in \left[max\left(n_{c1}^{\bar{K}2}, 1\right), max\left(n_{c2}^{\bar{K}2}, 1\right)\right]$,

(b) $D_g > 0$ for $n_c \in \left[1, max\left[n_{c1}^{\bar{K}2}, 1\right]\right]$

and for $n_c \in \left[max\left(n_{c2}^{\bar{K}2}, 1\right), \infty\right[$.

Suppose $E_1 < 0$:

(i) If $E_2^{\,2} < 4E_1E_3$, then $D_g < 0$ for all $n_c \geq 1$.

(ii) If $E_2^{\,2} > 4E_1E_3$, then

(a) $D_g > 0$ for $n_c \in \left[max\left(n_{c1}^{\bar{K}2}, 1\right), max\left(n_{c2}^{\bar{K}2}, 1\right)\right]$,

(b) $D_g < 0$ for $n_c \in \left[1, max\left[n_{c1}^{\bar{K}2}, 1\right]\right]$

and for $n_c \in \left[max\left(n_{c2}^{\bar{K}2}, 1\right), \infty\right[$,

where

$$n_{c1}^{\bar{K}2} := \frac{-E_2 + \sqrt{E_2^{\,2} - 4E_1E_3}}{2E_1} \quad \text{and} \quad n_{c2}^{\bar{K}2} := \frac{-E_2 - \sqrt{E_2^{\,2} - 4E_1E_3}}{2E_1}.$$

Unfortunately, the signs of D_g and D_v are indeterminate, however, since we have assumed, in Nash economy the number of consumer-artists are relatively small, comparing to ignorant economy, which can support our intuition, that the market(Nash) economy is thus characterized by underprovision of the stock of cultural goods and newly created cultural goods, market economy $\mathrm{BLN\bar{K}2}$ is Pareto inefficient.

This result is presented in Figure 5.15.

Figure 5.15 Inefficiency of the market economy $\mathrm{BLN\bar{K}2}$

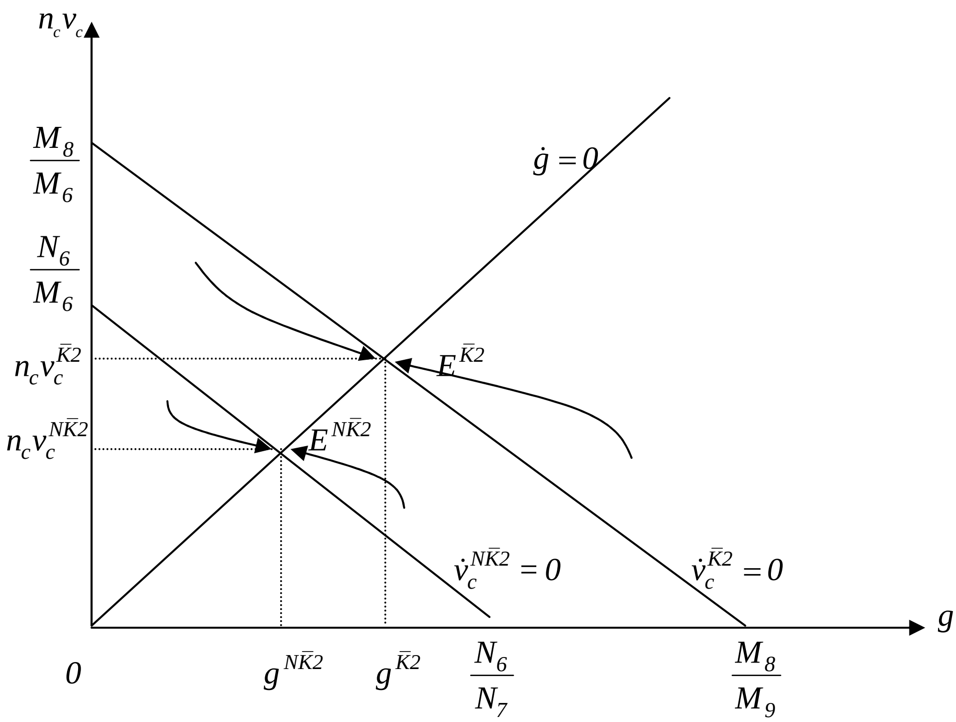


Figure 5.15 is qualitatively very similar to previous ones (e.g. Figure 3.10), we therefore make no further comment.

Comparison between the models $\mathrm{BLN\bar{K}1}$ **and** $\mathrm{BLN\bar{K}2}$

By subtracting (5.124) from (5.137) and (5.125) from (5.138), we obtain the difference between public- goods model $\mathrm{BLN\bar{G}1}$ and private-goods model $\mathrm{BLN\bar{G}2}$:

$$D_g := g_c^{N\bar{K}1} - g_c^{N\bar{K}2} = \frac{N_4\left(\alpha_g M_6 + N_7\right) - N_6\left(\alpha_g M_6 + N_5\right)}{\left(\alpha_g M_6 + N_5\right)\left(\alpha_g M_6 + N_7\right)}, \tag{5.144}$$

$$D_v := v_c^{N\bar{K}1} - v_c^{N\bar{K}2} = \frac{\alpha_g}{n_c}\left(g_c^{N\bar{K}1} - g_c^{N\bar{K}2}\right). \tag{5.145}$$

To determine the sign of D_g and D_v we need to expand the numerator on the RHS of (5.144), and conclude, after some algebraic manipulation, that $D_g > 0$ and $D_v > 0$, if and only if $H > 0$, where

$$H := H_1 n_c^2 + H_2 n_c + H_3 = \left\{ d_s \left[a_v \left(1+a_y\right)\left(a_s b_s - a_y\right) + a_s a_y \left(a_v b_v - a_y\right)\left(\delta + \alpha_g\right)\right] n_s^2 \right\} n_c^2$$

$$-\left\{ a_s a_v b_s \left[\left(1+a_y\right) d_s n_s^2 + a_y d_v \alpha_g \left(\delta + \alpha_g\right) n_s \right] \right\} n_c$$

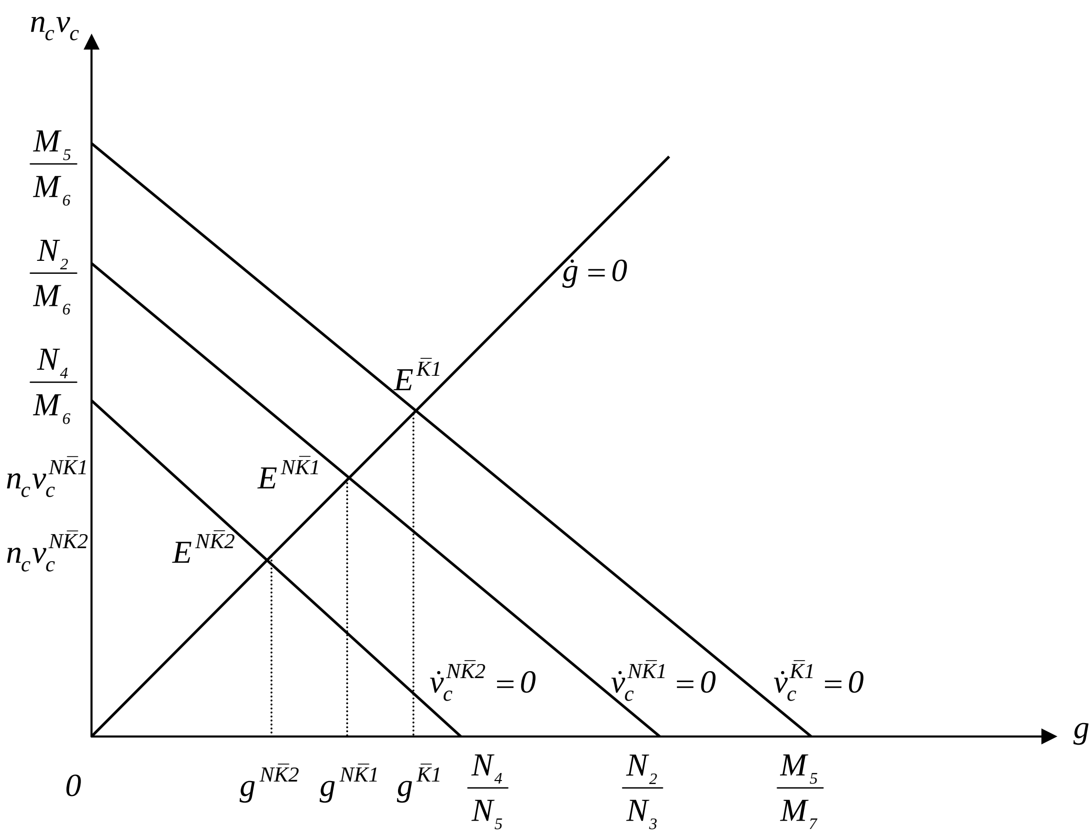
$$+\left\{ \left[a_v a_y \left(1+a_y\right) d_s \right] n_s^2 + a_y \left[a_v a_y d_v \alpha_g - a_s d_s \left(a_v b_v - a_y\right) \right] \left(\delta + \alpha_g\right) n_s \right.$$

$$\left. + a_v a_y d_v \left(a_s b_s - a_y\right) \alpha_g \left(\delta + \alpha_g\right) \right\}. \tag{5.146}$$

According to (5.146) it is obvious that in economies with only one consumer-artist and one cultural-services firm ($n_c = 1$ and $n_s = 1$), both steady-state values coincide. Since we have extensively discussed the rationale in previous sections (e.g. sections 3.1.3, 3.2.3), we refrain from repeating the discussion and turn to study situations where both the numbers of consumer-artists and cultural-services firms are greater than one ($n_c > 1$ and $n_s > 1$). Since we are interested in economies with very large numbers of consumer-artists, H can be shown to be positive, if H_1 is positive. Sufficient for $H_1 > 0$ is $a_y \le min\left[a_s b_s, a_v b_v\right]$ which is identical to the requirement of (5.66). Provided that $a_y \le min\left[a_s b_s, a_v b_v\right]$, we conclude that $D_g > 0$ and $D_v > 0$ for a sufficiently large number of consumer-artists. In other words, in the public-goods economy $\mathrm{BLN\bar{K}1}$ the steady-state values of the stock of cultural goods and of newly created cultural goods are greater than in the public-goods economy $\mathrm{BLN\bar{K}2}$.

This result is presented in Figure 5.16 without further comment.

Figure 5.16 Comparison of the public-goods and private-goods market economies (Comparison of the models $\mathrm{BLN\bar{K}1}$ **and** $\mathrm{BLN\bar{K}2}$)



6 Concluding remarks

This study provides a theoretical framework in the cultural context that can, as realistically as possible, describe the real world (conditions were discussed in sections 2, 3 and 4). This description, in turn, can help us to better understand the reality. If the performance of the observed subjects in the reality is shown to be unsatisfactory, such understanding then serves as the basis for devising a cultural policy that can improve the unsatisfactory performance (conditions were discussed in section 5). In our descriptive analysis, we first established a reference market model in which the economy is endowed with a full set of perfectly competitive markets including Lindahl markets whose equilibrium has been shown to be Pareto efficient. If it is assumed that the collective decisions should be based on the economic agents' welfare, and that the agents are likely to know better than the government what makes them happy, the achievement of Pareto efficiency through the market system then rules out the necessity of government intervention on efficiency grounds. Under these conditions it would be highly recommendable to leave the supply, demand and pricing of the agents' cultural activities to the market system. However, acknowledging that Lindahl markets don't emerge in the real world for reasons well understood by economists, we found that the *laissez-faire* market allocation without Lindahl markets becomes inefficient. To correct such misallocation and internalize the externalities governmental intervention in the agents' cultural activities is inevitable and justified. We hence explored cultural policies in form of appropriate subsidy/tax schemes that are capable to restore Pareto efficiency. In other words, the provision of cultural capital and cultural goods in the policy-supported market economy coincides with their efficient provision in the benchmark model. Now we summarize the principal findings of our study in the following four theses.

Thesis 1: In the *laissez-faire* market economy, consumers tend to ignore the beneficial external effects of their cultural-services consumption on the other consumers through accumulating cultural capital. The result is an underprovision of cultural capital.

The reason for that underprovision of public goods is the "free-rider problem". This was originally discussed in Samuelson's seminal paper (1954, p. 888-9), where he observed that "it is in the selfish interest of each person to give false signals, to pretend to have less interest in a given collective consumption activity than he really has".

Thesis 2: In the *laissez-faire* market economy, consumers tend to ignore the beneficial external effects of their creation of cultural goods on the other consumers through accumulating the stock of cultural goods. The result is the underprovision of cultural goods.

The reason for this inefficiency is again the consumer's free-riding behavior.

Thesis 3: Allocative efficiency can be restored by appropriate subsidies on the consumption of cultural services and on the creation of cultural goods. These subsidies stimulate the consumers' demand for cultural services and the supply of cultural goods which promotes the accumulation of both cultural capital and cultural goods.

The theses 1 and 2 present the typical cases of market failure, which is considered a justification of governmental regulation described in thesis 3. Conceptually, this kind of regulatory approach was first introduced by Pigou. An appropriate subsidy on cultural activities increases the individuals' consumption of cultural activities to the point where the (positive) externalities are internalized. In summary, Pigouvian subsidy/tax schemes render the efficient allocation of cultural activities and all other consumption activities.

Thesis 4: If the cultural services for consumers and the cultural-goods inputs for cultural-services firms are public goods, the stocks of cultural capital and cultural goods tend to be greater than in the case where cultural services and cultural-goods inputs for cultural-services firms are private goods.

Essentially, the four theses were driven by our basic hypotheses that the consumption of cultural services and the creations of cultural goods are not only beneficial for the individual consumers but also contribute to form a "better" or a "more cultivated" society that is valued by all members of society irrespective of their own cultural-services consumption and cultural-goods creations. Therefore, the empirical relevance of our approach depends heavily on the concepts of "cultural capital" and "cultural goods", and their measurability. Similarly, as with the related notion of “social capital” or “human capital”, empirical measurement turns out to be difficult. We are therefore left without straightforward evidence for the hypotheses that members of society appreciate the accumulation of cultural capital and are proud of the cultural goods created by themselves and their ancestors. Though the hypotheses may seem trivial, they present a demanding challenge, which urges us to address the difficult problems, and provide clear-cut suggestions for the future.

Apart from the difficulties of empirical tests of our hypotheses, our treatment of individual preferences and technologies has been highly simplified and stylized. The discussion of the technology of producing has been very circumscribed. It is also plausible that, among other inputs, cultural capital should enter the production functions of cultural goods and of cultural services. As a production factor, cultural capital would then create a feedback effect that renders more complex the dynamics of cultural growth or decline. Another extension that has not been considered in the present study is the heterogeneity of consumers and cultural-services firms. There are many possibilities to specify the production functions in alternative ways. No account has been taken of the implications of price-excludable public goods in our analysis though they play a crucial role in real-world cultural activities. We have not explicitly modeled the role of the electronic media, which can deliver and multiply cultural services and cultural goods quickly and extensively. Tackling these additional aspects in a rigorous analysis would be a highly relevant and rewarding task on the agenda for future research, since such analyses would promise to offer further new insights into other issues, such as the cultural activities through the internet.

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