Hecke Operators on Jacobi Forms of Lattice Index and the Relation to Elliptic Modular Forms

DISSERTATION
zur Erlangung des Grades eines Doktors
der Naturwissenschaften

vorgelegt von

Ali Ajouz

eingereicht bei der Naturwissenschaftlich-Technischen
Fakultät der Universität Siegen

Siegen 2015

This page is intentionally left blank.

Author's declaration

Ich erkläre hiermit an Eides statt, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer, nicht angegebener Hilfs mittel angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.
Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.
Es wurden keine Dienste eines Promotionsvermittlung sinstituts oder einer ähnlichen Organisation in Anspruch genommen.

SIGNED: .... DATE: ....

Erstgutachter: Prof. Dr. Nils-Peter Skoruppa
Zweigutachter: Prof. Dr. Lynne Walling
Tag der mündlichen Prüfung: 10-07-2015

This page is intentionally left blank.

Abstract

Jacobi forms of lattice index, whose theory can be viewed as extension of the theory of classical Jacobi forms, play an important role in various theories, like the theory of orthogonal modular forms or the theory of vertex operator algebras. Every Jacobi form of lattice index has a theta expansion which implies, for index of odd rank, a connection to half integral weight modular forms and then via Shimura lifting to modular forms of integral weight, and implies a direct connection to modular forms of integral weight if the rank is even. The aim of this thesis is to develop a Hecke theory for Jacobi forms of lattice index extending the Hecke theory for the classical Jacobi forms, and to study how the indicated relations to elliptic modular forms behave under Hecke operators. After defining Hecke operators as double coset operators, we determine their action on the Fourier coefficients of Jacobi forms, and we determine the multiplicative relations satisfied by the Hecke operators, i.e. we study the structural constants of the algebra generated by the Hecke operators. As a consequence we show that the vector space of Jacobi forms of lattice index has a basis consisting of simultaneous eigenforms for our Hecke operators, and we discover the precise relation between our Hecke algebras and the Hecke algebras for modular forms of integral weight. The latter supports the expectation that there exist equivariant isomorphisms between spaces of Jacobi forms of lattice index and spaces of integral weight modular forms. We make this precise and prove the existence of such liftings in certain cases. Moreover, we give further evidence for the existence of such liftings in general by studying numerical examples.

This page is intentionally left blank.

Zusammenfassung

Jacobiformen von Gitterindex, deren Theorie als Erweiterung der Theorie klassischer Jacobiformen betrachtet werden kann, spielen in unterschiedlichen Theorien, wie der Theorie der Modulformen auf orthogonalen Gruppen oder der Theorie der Vertexoperatoralgebren, eine bedeutende Rolle. Jede Jacobiform mit Gitterindex hat eine Theta-Entwicklung, die bei ungeradem Index eine Verbindung zu Modulformen von halbganzen Gewichten herstellt und daher via Shimura-Liftung eine Beziehung zu Modulformen ganzen Gewichts impliziert, und bei geradem Index eine direkte Verbindung zu Moldulformen ganzzahligen Gewicht suggeriert.

Das Ziel dieser Dissertation ist, eine Hecke-Theorie für Jacobiformen mit Gitterindex zu entwickeln, indem die Hecke-Theorie für die klassischen Jacobiformen erweitert wird, und zu untersuchen, wie sich die angedeuteten Beziehungen zu elliptischen Moldulformen unter Hecke-Operatoren verhalten.

Nachdem die Hecke-Operatoren als Doppelnebenklassen-Operatoren definiert werden, wird deren Wirkung auf die Fourier-Koeffizienten der Jacobiformen und die multiplikativen Relationen, welche von Hecke-Operatoren erfüllt werden, untersucht, indem z.B. die Struktur-Konstanten der Algebra berechnet werden, die von den Hecke-Operatoren erzeugt werden.

Daraufhin wird gezeigt, dass der Vektorraum der Jacobiformen mit Gitterindex über eine Basis von simultanen Eigenformen für die Hecke-Operatoren verfügt und es wird die präzise Beziehung zwischen der Hecke-Algebra und der Hecke-Algebra für Moldulformen ganzzahligen Gewichts aufgezeigt.

Letztgenannte Beziehung stützt die Erwartung, dass äquivariante Isomorphismen zwischen den Räumen der Jacobiformen mit Gitterindex und Räumen Moldulformen ganzzahligen Gewichts existieren. Es erfolgt eine Präzisierung und der Beweis, dass solche Liftungen in bestimmten Fällen

existieren. Zudem geben wir weitere Argumente für die Existenz solcher Liftungen im allgemeunen Fall durch das Studium numerischer Beispiele.

Acknowledgments

I would like to thank my supervisor, Prof. Dr. Nils-Peter Skoruppa, for his continued guidance and support throughout the duration of my Ph.D. He taught me a lot with his mathematical knowledge and skills, and gave me many valuable suggestions as well as kind help in the elucidation of difficulties. I also want to thank Prof. Dr. Lynne Walling, Prof. Dr. Volker Michel, and Prof. Dr. Gerd Mockenhaupt for agreeing to be in the jury.

This page is intentionally left blank.

To my wife, Karam.

This page is intentionally left blank.

Table of Contents

Introduction and Statement of Results

Jacobi forms of scalar index are a mixture of modular forms and elliptic functions. They have an arithmetic theory very analogous to the usual theory of modular forms; this began with Maass's proof of the Saito-Kurokawa conjecture and was developed systematically by Martin Eichler and Don Zagier in the monograph "The Theory of Jacobi Forms" [EZ85].

Because they have two variables, Jacobi forms have associated to them two characteristic integers: the weight, which describes the transformation properties of the form with respect to the modular group $\mathrm{SL}_2(\mathbb{Z})$, and the index, which describes the transformation properties in the elliptic variable. We shall use $J_{k,m}$ for the space of Jacobi forms on $\mathrm{SL}_2(\mathbb{Z})$ of weight k and index m. The basic features of the theory of Jacobi forms of scalar index are:

- $J_{k,m}$ is finite-dimensional.
- There exists a Hecke theory for Jacobi forms of scalar index, i.e., for each positive number ℓ relative prime to m, there exists a natural Hecke operator $T(\ell)$ on $J_{k,m}$, and the space $J_{k,m}$ has a basis consisting of simultaneous eigenforms with respect to all $T(\ell)$.
- There exists a natural notion of Jacobi Eisenstein series and Jacobi cusp forms.
- There exists a Petersson scalar product $< \phi, \psi >$ on the space of Jacobi cusp forms.

- A perfect correspondence with elliptic modular forms of integer weight: the result of the paper of Skoruppa and Zagier [SZ88] is a relation between Jacobi forms of weight k and index m on the one hand and ordinary elliptic modular forms of weight $2k-2$ and level m. Indeed, they proved that $J_{k,m}$ is isomorphic as module over the Hecke algebra to a natural subspace of the space $M_{2k-2}(m)$ of modular forms of weight $2k-2$ on $\Gamma_0(m)$. More precisely, the lifting from $J_{k,m}$ to $M_{2k-2}(m)$ is constructed as follows:

  Let $\phi \in J_{k,m}$, then ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{d\leq 0, r \\ d\equiv r^2 \bmod 4m}} C_\phi(d,r) q^{\frac{r^2-d}{4m}} \zeta^r.$$

  Let $D<0$ be a fundamental discriminant and r be an integer with $D \equiv r \bmod 4m$, then

$$\mathscr{S}_{D,r}(\tau) : \phi \to \sum_{\ell=0}^{\infty} \Big(\text{coefficient of } q^{\frac{r^2-D}{4m}} \zeta^r \text{ in } \phi \mid T(\ell)\Big) e^{2\pi i \ell \tau} \qquad (1)$$

  maps $J_{k,m}$ to a certain subspace of $M_{2k-2}(m)$, and the maps $\mathscr{S}_{D,r}$ commute with the action of the Hecke operators. (This is not quite true, since one would need here to define $T(\ell)$ for $\gcd(\ell,m) > 1$ which is not a part of the common theory.)

Jacobi forms of lattice index, which can be viewed as a generalization of the classical ones if one consider $2m$ as a Gram matrix of the lattice $(\mathbb{Z},(x,y) \to 2mxy)$, have been studied in [Zie89], [BK93], [CG11], [Sko07], [Boy11], [Bri04], and other authors. Recall that an integral lattice over $\mathbb{Z}$ is a pair $\underline{L} = (L,\beta)$, where L is a free $\mathbb{Z}$-module of a finite rank, and $\beta : L \times L \to \mathbb{Z}$ is a non-degenerate symmetric and $\mathbb{Z}$-bilinear. We shall use $L^\#$, $\mathrm{lev}(\underline{L})$, $\mathrm{rk}(\underline{L})$ and $\det(\underline{L})$ to denote the dual, the level, the rank, and the determinant of the lattice, respectively. Also, for each $r \in L^\#$, we shall use N_r for the smallest positive integer such that $N_r \cdot r \in L$.

Suppose that $\underline{L}$ is even and positive definite, i.e., such that $\beta(x,x)$ is even and strictly positive unless $x=0$. A Jacobi form $\phi(\tau,z)$ of weight $k \in \mathbb{N}$ and index $\underline{L} = (L,\beta)$ is a holomorphic function of a variable τ in the complex upper half plane $\mathfrak{H}$ and a variable $z \in L \otimes_{\mathbb{Z}} \mathbb{C}$ that satisfies certain transformation laws (see Definition 2.4.1) and has a Fourier expansion of the form:

$$\phi(\tau,z) = \sum_{\substack{D\leq 0, r\in L^{\#} \\ D\equiv\beta(r) \bmod \mathbb{Z}}} C_{\phi}(D,r)\,\mathfrak{e}\left((\beta(r)-D)\tau + \beta(r,z)\right),$$

The starting point for the current thesis is the expectation that there should be lifting for Jacobi forms of lattice index to elliptic modular forms. However, there is currently not even a Hecke theory for these forms.

In this thesis, I shall focus on Hecke operators acting on the vector space of Jacobi forms of lattice index. I shall use these operators and the arithmetic properties of Jacobi forms to give examples of correspondences with elliptic modular forms. More precisely, we will discuss correspondences supporting the mentioned expectation of the following sense:

Jacobi forms of weight k and index $\underline{L}$	$\xleftrightarrow[\text{if } \mathrm{rk}(\underline{L})\equiv 1 \bmod 2]{\text{correspondence}}$	Elliptic modular forms of weight $k_1 := 2k-1-\mathrm{rk}(\underline{L})$
Jacobi forms of weight k and index $\underline{L}$	$\xleftrightarrow[\text{if } \mathrm{rk}(\underline{L})\equiv 0 \bmod 2]{\text{correspondence}}$	Elliptic modular forms of weight $k_2 := k - \frac{\mathrm{rk}(\underline{L})}{2}$

The main results of this thesis can be subdivided as follows:

1. A Hecke theory for Jacobi forms of lattice index (chapter 2 and chapter 3): In this part we develop a systematic Hecke theory for Jacobi forms of lattice index along the lines of Hecke's theory of modular forms.

1.a. Explicit description of the action of Hecke operators (section 2.5, section 2.6 and section 2.7: Let $\underline{L} = (L,\beta)$ be a positive definite even lattice over $\mathbb{Z}$. We set

$$\Delta(\underline{L}) = \begin{cases} (-1)^{\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor} 2\det(\underline{L}) & \text{if } \mathrm{rk}(\underline{L}) \text{ is odd,} \\ (-1)^{\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor} \det(\underline{L}) & \text{if } \mathrm{rk}(\underline{L}) \text{ is even.} \end{cases}$$

For $a \in \mathbb{N}$ and $n \in \mathbb{Q}$ such that $n \cdot \Delta(\underline{L}) \in \mathbb{Z}$, we set

$$\chi_{\underline{L}}(n,a) := \left(\frac{n\cdot\Delta(\underline{L})}{a}\right), \quad \chi_{\underline{L}}(a) := \chi_{\underline{L}}(1,a) = \left(\frac{\Delta(\underline{L})}{a}\right).$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the usual Kronecker symbol (see chapter 1).

Let $J_{\underline{L}}(\mathbb{Z})$ be the Jacobi group of $\underline{L}$ (to be defined in section 2.2), and $k \in \mathbb{N}$ be a positive integer. For each $\ell \in \mathbb{N}$ with $\gcd(\ell, \operatorname{lev}(\underline{L})) = 1$ we define a double coset Hecke operator $T_0(\ell)$ on the vector space of Jacobi forms of weight k and index $\underline{L}$ in the following way:

$$\phi \mapsto T_0(\ell)\phi := \ell^{k-2-\operatorname{rk}(\underline{L})} \sum_A \phi|_{k,\underline{L}} A,$$

where A runs over a complete set of representatives for $J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})$. Then, we define the Hecke operator $T(\ell)$ on $J_{k,\underline{L}}$ as follows:

1. If $\operatorname{rk}(\underline{L})$ is odd, we set $k_1 := 2k - \operatorname{rk}(\underline{L}) - 1$ and

$$T(\ell) := \sum_{d^2|\ell,\ d>0} d^{k_1-2}\ T_0\left(\frac{\ell}{d^2}\right).$$

2. If $\operatorname{rk}(\underline{L})$ is even, we set $k_2 := k - \frac{\operatorname{rk}(\underline{L})}{2}$ and

$$T(\ell) := \sum_{\substack{d,s>0 \\ sd^2|\ell,\ s \text{ square-free}}} \chi_{\underline{L}}(s)(sd^2)^{k_2-2}\ T_0\left(\frac{\ell}{sd^2}\right).$$

The operators $T_0(\ell)$ and $T(\ell)$ are well-defined and map Jacobi forms of weight k and index $\underline{L}$ to Jacobi forms of the same weight and same rank. For $\operatorname{rk}(\underline{L}) = 1$ the operator $T(\ell)$ equals the Hecke operator for classical Jacobi forms as e.g. in [SZ88] and [EZ85].

Next, we describe the action of these operators on Jacobi forms in terms of Fourier coefficients and give their commutation relations.

Theorem (see Theorem 2.6.1). *Let $\underline{L} = (L, \beta)$ be a positive definite even lattice over $\mathbb{Z}$ of odd rank. Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L, \beta)$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{D \le 0, r \in L^\# \\ D \equiv \beta(r) \bmod \mathbb{Z}}} C_\phi(D, r)\, \mathfrak{e}\big((\beta(r) - D)\tau + \beta(r, z)\big).$$

Let $\ell \in \mathbb{N}$ with $\gcd(\ell, \operatorname{lev}(\underline{L})) = 1$, and let

$$\big(T(\ell)\phi\big)(\tau, z) = \sum_{\substack{D \le 0, r \in L^\# \\ D \equiv \beta(r) \bmod \mathbb{Z}}} C_{T(\ell)\phi}(D, r)\, \mathfrak{e}\big((\beta(r) - D)\tau + \beta(r, z)\big).$$

Then one has

$$C_{T(\ell)\phi}(D,r) = \sum_a a^{k_1/2-1}\varrho(D,a)C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' r\big).$$

The sum is over those $a \mid \ell^2$ *such that* $a^2 \mid \ell^2 \mathrm{lev}(\underline{L})D$*, and* a' *is an integer such that* $aa' \equiv 1 \pmod{\mathrm{lev}(\underline{L})}$*. Moreover,* $\varrho(D,a)$ *equals* $f \cdot \chi_{\underline{L}}(D/f^2, a/f^2)$ *if* $\gcd(\mathrm{lev}(\underline{L})D, a) = f^2$ *with* $f \in \mathbb{N}$*, and it equals* 0 *if* $\gcd(\mathrm{lev}(\underline{L})D, a)$ *is not a perfect square.*

Theorem (see Theorem 2.6.3)**.** *In the same notations as in the preceding theorem assume the rank of* $\underline{L}$ *is even. Then one has*

$$C_{T(\ell)\phi}(D,r) = \sum_{a \mid \ell^2, \mathrm{lev}(\underline{L})D} a^{k_2-1}\chi_{\underline{L}}(a)C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' r\big).$$

Using this explicit description of the action of the operator $T(\ell)$ on Jacobi forms of lattice index in terms of Fourier expansion, we can determine their multiplicative properties.

Theorem (see Theorem 2.7.11 and Theorem 2.7.4)**.** *Let* $\ell_1, \ell_2 \in \mathbb{N}$ *such that* ℓ_1, ℓ_2 *coprime to* $\mathrm{lev}(\underline{L})$*. We have the following multiplicative relation:*

$$T(\ell_1)\cdot T(\ell_2) = \begin{cases} \sum\limits_{d \mid \ell_1, \ell_2} d^{k_1-1} T\left(\frac{\ell_1\ell_2}{d^2}\right) & \text{If } \mathrm{rk}(\underline{L}) \text{ is odd,} \\ \sum\limits_{d \mid {\ell_1}^2, {\ell_2}^2} \chi_{\underline{L}}(d) d^{k_2-1} T\left(\frac{\ell_1\ell_2}{d}\right) & \text{If } \mathrm{rk}(\underline{L}) \text{ is even.} \end{cases}$$

As an easy consequence of these theorems we obtain an important insight into the arithmetic proprieties of Jacobi eigenforms of lattice index. More precisely, let $\underline{L} = (L, \beta)$ again be an even positive definite lattice over $\mathbb{Z}$, and consider a Jacobi form ϕ of weight k and index $\underline{L} = (L, \beta)$ with Fourier coefficients $C_\phi(D,r)$, which is an eigenfunction of all $T(\ell)$ with $\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$, say, $T(\ell)\phi = \lambda(\ell)\phi$.

Theorem (see Theorem 2.7.17 and Theorem 2.7.9)**.** *Let* $r \in L^\#$ *and* $\mathrm{lev}(r)$ *is the smallest positive integer such that* $\mathrm{lev}(r)\beta(r) \in \mathbb{Z}$*, and* $D \le 0$ *such that* $D \equiv \beta(r) \bmod \mathbb{Z}$ *and* $\mathrm{lev}(r)D$ *is a square-free integer. Then*

1. *if* $\mathrm{rk}(\underline{L})$ *is odd, one has*

$$\prod_{p\nmid \mathrm{lev}(\underline{L})}\left(1-\chi_{\underline{L}}(D,p)p^{k_1/2-1-s}\right)^{-1}\sum_{\substack{\ell\in\mathbb{N}\\ \gcd(\ell,\mathrm{lev}(\underline{L}))=1}} C_\phi(\ell^2D,\ell r)\ell^{-s}$$
$$=C_\phi(D,r)\prod_{p\nmid \mathrm{lev}(\underline{L})}\left(1-\lambda(p)p^{-s}+p^{k_1-1-2s}\right)^{-1},$$

2. *if* $\mathrm{rk}(\underline{L})$ *is even, one has*

$$\Big(\sum_{\substack{\ell\mid \mathrm{lev}(\underline{L})D\\ \gcd(\ell,\mathrm{lev}(\underline{L}))=1}}\chi_{\underline{L}}(\ell)\ell^{k_2-1-s}\Big)\sum_{\substack{\ell\geq 1\\ \gcd(\ell,\mathrm{lev}(\underline{L}))=1}} C_\phi\left(\ell^2D,\ell r\right)\ell^{-s}$$
$$=C_\phi(D,r)\prod_{p\nmid \mathrm{lev}(\underline{L})}\frac{1+\chi_{\underline{L}}(p)p^{k_2-1-s}}{1-\left(\lambda(p)-p^{k_2-1}\chi_{\underline{L}}(p)\right)p^{-s}+p^{2(k_2-1-s)}}.$$

The products are over all primes p not dividing $\mathrm{lev}(\underline{L})$*.*

Note that the right-hand sides of these identities are nothing else than $C_\phi(D,r)\,L(s,\phi)$, where we set $L(s,\phi)=\sum_\ell\lambda(\ell)\ell^{-s}$ (all sums are taken over ℓ coprime to $\mathrm{lev}(\underline{L})$). If the ϕ in the first identity lifts to an elliptic modular form f of weight $k_1=2k-1-\mathrm{rk}(\underline{L})$ then $L(s,\phi)$ should be (up to a finite number of Euler factors) the L-series of f. We observe that the right-hand side of the first identity has indeed the right shape.

The right-hand of the second identity looks, at the first glance, slightly more complicated. However, if we think of an elliptic modular form of weight $k_2=k-\frac{\mathrm{rk}(\underline{L})}{2}$ with nebentypus, say, $\chi_{\underline{L}}\xi$, and with Hecke eigenvalues $\gamma(\ell)$, then $\sum_\ell\overline{\xi(\ell)}\gamma(\ell^2)\ell^{-s}$ (again taken over all ℓ coprime to $\mathrm{lev}(\underline{L})$) equals $L(s,\phi)$ if we replace $\lambda(p)$ with $\overline{\xi(p)}\gamma(p^2)$. This suggests, for each ξ and suitable levels m, the existence of maps from $M_{k_2}(m,\chi_{\underline{L}}\xi)$ to $J_{k,\underline{L}}$ such that $T(\ell^2)$ on the left corresponds to $\xi(\ell)T(\ell)$ on the Jacobi form side. We shall construct in this thesis examples for such maps.

1.b. Basis of simultaneous Hecke eigenforms (chapter 3): It is easy to see (for weights greater than $\frac{\mathrm{rk}(\underline{L})}{2}+2$) that the space of Jacobi forms is a direct sum of the subspaces spanned by Eisenstein series and cusp forms, which will be introduced in section 3.3. We shall prove that our Hecke operators $T(\ell)$ (ℓ coprime to $\mathrm{lev}(\underline{L})$) leave these subspaces invariant, and are Hermitian with respect to a suitably defined Petersson scalar product on the subspace of cusp forms. Also, we shall prove that the orthogonal group $O(D_{\underline{L}})$ of the discriminant module $D_{\underline{L}}$ of $\underline{L}$ acts naturally on the spaces of Jacobi forms with index $\underline{L}$, and that the corresponding operators $W(\alpha)$ ($\alpha \in O(D_{\underline{L}})$) are Hermitian too and commute with Hecke operators. By the spectral theory of such operators, one has the following theorem.

Theorem (see Theorem 3.2.13)**.** *The space of Jacobi cusp forms $S_{k,\underline{L}}$ has a basis of simultaneous eigenforms for all operators $T(\ell)$ ($\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$) and for all operators $W(\alpha)(\alpha \in O(D_{\underline{L}}))$.*

Let $D_{\underline{L}} = (L^\#/L, \beta)$ be the associated discriminant module with the lattice $\underline{L}$ (see Definition 1.2.10). We set $\mathrm{Iso}(D_{\underline{L}}) := \{x \in L^\#/L \mid \beta(x) \in \mathbb{Z}\}$. Let k be a positive integer with $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$. For each $r \in \mathrm{Iso}(D_{\underline{L}})$, we define a Jacobi Eisenstein series of weight k and index $\underline{L}$, in terms of Jacobi theta series $\vartheta_{\underline{L},r}$ which are naturally associated to $\underline{L}$ (see section 2.3), as follows:

$$E_{k,\underline{L},r} := \frac{1}{2} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \vartheta_{\underline{L},r+L}\big|_{k,\underline{L}} A.$$

We shall use $J^{Eis}_{k,\underline{L}}$ for the subspace in $J_{k,\underline{L}}$ that spanned by all Eisenstein series $E_{k,\underline{L},r}$.

Theorem (see Theorem 3.3.18)**.** *The series*

$$E_{k,\underline{L},x,\xi} := \sum_{d \in \mathbb{Z}^\times_{N_x}} \xi(d) E_{k,\underline{L},dx},$$

where x runs through a set of representatives for the orbits in the orbit space $\mathrm{Iso}(D_{\underline{L}})/\mathbb{Z}^\times_{\mathrm{lev}(\underline{L})}$, and ξ runs through all primitive Dirichlet characters mod F *with $F \mid N_x$ such that $\xi(-1) = (-1)^k$, form a basis of Hecke eigenforms of $J^{Eis}_{k,\underline{L}}$. More precisely:*

$$T(\ell)E_{k,\underline{L},x,\xi} = \sigma^{\xi,\overline{\xi}}_{k_1-1}(\ell) E_{k,\underline{L},x,\xi} \qquad \textit{if } \mathrm{rk}(\underline{L}) \textit{ is odd},$$

$$T(\ell)E_{k,\underline{L},x,\xi} = \overline{\xi(\ell)}\, \sigma^{\xi,\chi_{\underline{L}}}_{k_2-1}(\ell^2) E_{k,\underline{L},x,\xi} \qquad \textit{if } \mathrm{rk}(\underline{L}) \textit{ is even},$$

for all $\ell \in \mathbb{N}$ *such that* $\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$. *Here,* $k_1 = 2k - 1 - \mathrm{rk}(\underline{L})$, $k_2 = k - \frac{\mathrm{rk}(\underline{L})}{2}$, *and for any two Dirichlet characters* ξ *and* χ *we use*

$$\sigma_k^{\xi,\chi}(\ell) := \sum_{d|\ell} \xi\left(\tfrac{\ell}{d}\right)\chi(d)d^k.$$

Note that this theorem supports the interpretations of the last theorem in 1.a, which we proposed at the end of that section. See Remark 3.3.19 for a detailed discussion.

2. Lifting to Elliptic Modular Forms (chapter 4): A natural question in view of the preceding theorems is whether we can construct in general a relation between Jacobi forms of lattice index of odd rank and elliptic modular forms which extends the work of Skoruppa and Zagier [SZ88] for the case of scalar index (see Equation (1)). For obtaining at least partially such lifts we apply two methods:

2.a. Lifting via Shimura correspondence for half integral weight (section 4.1): We know that every Jacobi form of lattice index has a theta expansion which implies, for odd rank index, a connection to half integral weight modular forms. We can try to use the Shimura correspondence for half integral weight forms to map Jacobi forms of lattice index to modular forms of integral weight.

For this let $\underline{L}$ be of odd rank, and k be a positive integer such that $2k - \mathrm{rk}(\underline{L}) - 1 \geq 2$. Let $x \in L^\#$ and $D \in \mathbb{Q}$ such that $D \equiv \beta(x) \bmod \mathbb{Z}$. Assuming that ${N_x}^2 D$ is a square free negative integer, where N_x is the smallest positive integer such that $N_x x \in L$. For a Jacobi cusp form $\phi \in S_{k,\underline{L}}$, set

$$\mathscr{S}_{D,x}(\phi) = \sum_{\ell=1}^{\infty}\Big(\sum_{a|\ell} a^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\chi_{\underline{L}}(D,a)C_\phi\big(\tfrac{\ell^2}{a^2}D, \tfrac{\ell}{a}x\big)\Big)\mathfrak{e}(\ell\tau),$$

$$\mathscr{S}^{\xi}_{D,x}(\phi) = \sum_{\substack{s \bmod N_x \\ D \equiv \beta(sx) \bmod \mathbb{Z}}} \xi(s)\big(\mathscr{S}_{D,sx}(\phi) \otimes \xi\big),$$

where $\xi(\cdot) = \left(\frac{(-1)^k {N_x}^2}{\cdot}\right)$, and $\mathscr{S}_{D,sx}(\phi) \otimes \xi$ denotes the function obtained from $\mathscr{S}_{D,sx}(\phi)$ by multiplying its n-th Fourier coefficient by $\xi(n)$.

Theorem (See Theorem 4.1.4). *$\mathscr{S}^{\xi}_{D,x}(\phi)$ is an elliptic modular form of weight $2k-1-\mathrm{rk}(\underline{L})$ on $\Gamma_0(\mathrm{lev}(\underline{L})N_x^2/2)$, and, in fact, a cusp form if $2k-1-\mathrm{rk}(\underline{L})>2$. Moreover, one has*

$$T(p)\mathscr{S}^{\xi}_{D,x}(\phi)=\xi(p)\mathscr{S}^{\xi}_{D,x}(T(p)\phi) \tag{2}$$

for all primes p with $\gcd(p,\mathrm{lev}(\underline{L}))=1$

Indeed, this theorem supports the interpretations of the last theorem in 1.a. Namely, if $\mathscr{S}_{D,x}$ takes $J_{k,\underline{L}}$ to elliptic modular forms on $\Gamma_0(\mathrm{lev}(\underline{L})/2)$, then the twisted version $\mathscr{S}^{\xi}_{D,x}$ takes $J_{k,\underline{L}}$ to elliptic modular forms on $\Gamma_0({N_x}^2\,\mathrm{lev}(\underline{L})/2)$, which we proved indeed. If in addition, $\mathscr{S}_{D,x}$ commutes with Hecke operators, we deduce for $\mathscr{S}^{\xi}_{D,x}$ the relation (2), which is again what we proved.

2.b. Lifting via stable isomorphisms between lattices (section 4.2): Two even lattices $\underline{L_1}=(L_1,\beta_1)$ and $\underline{L_2}=(L_2,\beta_2)$, are said to be *stably isomorphic* if and only if there exists even unimodular lattices U_1,U_2 such that $\underline{L_1}\oplus U_1\cong\underline{L_2}\oplus U_2$. If $\underline{L_1},\underline{L_2}$ are stably isomorphic, then one can show that there is an isomorphism

$$I_{\underline{L_2},\underline{L_1}}:J_{k+\lceil\mathrm{rk}(\underline{L_2})/2\rceil,\underline{L_2}}\longrightarrow J_{k+\lceil\mathrm{rk}(\underline{L_1})/2\rceil,\underline{L_1}}$$

(see Theorem 4.2.2 and Theorem 4.2.4). Note that stably isomorphic lattices have the same level and determinant. We shall show that this isomorphism commutes with the Hecke operators $T(\ell)$ (see Theorem 4.2.4). Also, we obtain the following important result:

Theorem (see Theorem 4.2.5). *If the lattice $\underline{L}=(L,\beta)$ is stably isomorphic to the lattice $\big(\mathbb{Z},(x,y)\mapsto\det(\underline{L})xy\big)$, then there is a Hecke-equivariant isomorphism*

$$J_{k,\underline{L}}\xrightarrow{\cong}\mathfrak{M}_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)^{-},$$

where $\mathfrak{M}_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)$ is the Certain Space *inside $M_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)$ which was introduced in [SZ88, 3], and where the "$-$" sign denotes the subspace of all $f\in\mathfrak{M}_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)$ such that $f\,|\,W_{\mathrm{lev}(\underline{L})/4}=-(-1)^{k/2}f$.*

3. Lifting from elliptic modular forms (section 5.1): We saw above that, for even rank index, we expect *liftings* from spaces $M_{k_2}(m, \chi_{\underline{L}}\xi)$ to $J_{k,\underline{L}}$ in the sense that $T(\ell^2)$ corresponds to $\xi(\ell)T(\ell)$. We shall study this in the case that the determinant of the lattice $\underline{L} = (L, \beta)$ is an odd prime. In this case we shall explicitly construct a lifting

$$\mathscr{S} : M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\left(\mathrm{lev}(\underline{L}), \chi_{\underline{L}}\right) \to J_{k,\underline{L}}$$

(see Theorem 5.1.2). The map $\mathscr{S}$ will turn out to be surjective. However, in general, it will not be injective, but we shall see that we can restrict $\mathscr{S}$ to a natural subspace of $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\left(\mathrm{lev}(\underline{L}), \chi_{\underline{L}}\right)$, invariant under all $T(\ell^2)$, so that we obtain an isomorphism.

To complete the picture, we will discuss briefly the relation between the Hecke operators on Jacobi forms and the Hecke operators on the vector-valued components (see section 5.2). We also construct some numerical examples using the method of theta blocks (see section 6.1).

Chapter 1

Preliminaries

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ the set of all strictly positive natural numbers, the ring of rational integers, the rational number field, the real number field, and the complex number field. Also we put $S_1 = \{z \in \mathbb{C}, |z| = 1\}$. For a prime number p, $\mathbf{Q}_p$ denotes the field of p-adic numbers, while $\mathbf{Z}_p$ denotes the subring of p-adic integers. This should not to be confused with the ring of integers modulo p which we denote by $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$.

For complex valued function f, $\overline{f}$ denotes its complex conjugate. We set $i = \sqrt{-1}$. We shall always take the branch of the square root having argument in $(-\pi/2, \pi/2]$. Thus for $z \in \mathbb{C}\backslash\{0\}$, we define $\sqrt{z} = z^{\frac{1}{2}}$ so that $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$, and we set $\sqrt{0} = 0$. The function $\sqrt{z}$ on the complex plane takes positive reals to positive reals, complex numbers in the upper half-plane to the first quadrant, and complex numbers in the lower half-plane to the fourth quadrant. Its restriction to $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ is holomorphic. Further, we put $z^{k/2} = (\sqrt{z})^k$ for every $k \in \mathbb{Z}$.

Let n be an integer, with prime factorization $u \cdot {p_1}^{e_1} \cdots {p_k}^{e_k}$, where $u = \pm 1$ and the p_i are primes. Let a be an integer. The Kronecker symbol $\left(\frac{a}{n}\right)$ is defined to be $\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right) \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{e_i}$. For odd p_i, the number $\left(\frac{a}{p_i}\right)$ is simply the usual Legendre symbol. This leaves the case when $p_i = 2$. We define $\left(\frac{a}{2}\right)$ to be 0 if a is even, 1 if $a \equiv \pm 1 \pmod 8$, and -1 if $a \equiv \pm 3 \pmod 8$. The quantity

$\left(\frac{a}{u}\right)$ is 1 when $u = 1$. When $u = -1$, we define it by

$$\left(\frac{a}{-1}\right) = \begin{cases} -1 & \text{if } a < 0 \\ 1 & \text{if } a > 0 \end{cases}.$$

These definitions extend the Legendre symbol for all pair of integers a, n.

Let p be a prime number. The p-adic order or p-adic valuation for $\mathbb{Z}$ is defined as $\mathrm{ord}_p : \mathbb{Z} \to \mathbb{N}$ by:

$$\mathrm{ord}_p(n) = \begin{cases} \max\{v \in \mathbb{N},\ p^v | n\} & \text{if } n \neq 0 \\ \infty & \text{if } n = 0 \end{cases}$$

It follows that one can write $n = p^{\mathrm{ord}_p(n)}.n_p$ where $n_p \in \mathbb{Z}$ with $\gcd(n_p, p) = 1$.

Let H be a subgroup of finite index in the group G, and f is a function on G which is H-right invariant. We use $\sum_{g \in G/H} f(g)$ (by slight abuse of language) as notation for $\sum_{g \in R} f(g)$, where R is a complete set of representatives for G/H.

Given two integers $b \neq 0, a$. By $b \,||\, a$ we mean that $b \mid a$ and $\gcd(a, \frac{a}{b}) = 1$. In sums of the forms $\sum_{b|a}$ or $\sum_{ab=\ell}$ it is understood that the summation is over positive divisors only. By $\lfloor x \rfloor$ we mean the function $\max\{n \in \mathbb{Z} \mid n \leq x\}$ and similarly $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$. By $\boldsymbol{\delta}$ we mean the logical Kronecker delta, that is the Boolean function which has as argument a logical expression with

$$\boldsymbol{\delta}(\text{logical expression}) := \begin{cases} 1 & \text{if the logical expression is a true expression,} \\ 0 & \text{otherwise.} \end{cases}$$

For example, given two integers c, d. $\boldsymbol{\delta}(c \mid d)$ is equal to 1 if c divides d and is equal to 0 otherwise. For two integers a, b, we will use the symbol $(a, b) := \gcd(a, b)$ to denote the greatest common divisor of a and b. We shall use $\mathfrak{e}_r(x)$ for $e^{2\pi i x/r}$. We set $\mathfrak{e}(x) := \mathfrak{e}_1(x)$, and $\Gamma := \mathrm{SL}_2(\mathbb{Z})$.

The symbol ":=" means that the expression on the right is the definition of what is on the left.

Let R denote a commutative ring with identity element $1 = 1_R$. We set

$$R[G] = \{\varphi : G \to R \mid \varphi(g) \neq 0 \text{ only for finitely many } g \in G\}.$$

The elements of $\varphi \in R[G]$ have finite support, i.e.,

$$\text{support}(\varphi) := \{g \in G \mid \varphi(g) \neq 0\}$$

is a finite set. Given $\alpha \in R$ and $\varphi, \psi \in R[G]$ we define

$$\begin{aligned}
(\alpha\varphi)(g) &:= \alpha\varphi(g), \\
(\varphi + \psi)(g) &:= \varphi(g) + \psi(g), \\
(\varphi.\psi)(g) &:= \sum_{h \in G} \varphi(h)\psi(h^{-1}g) = \sum_{h_1,h_2 \in G, h_1h_2 = g} \varphi(h_1)\psi(h_2).
\end{aligned} \tag{1.1}$$

Since φ and ψ have finite support, the sums in Equation (1.1) are finite. Given $g \in G$ we define the Kronecker-delta $\delta_g \in R[G]$ by

$$\delta_g(h) := \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{if } h \neq g. \end{cases}$$

Theorem 1.0.1 ([Kri, Theorem 1.1]). *Let R be a commutative unitary ring and G a group with identity element e. Then $R[G]$ is an associative R-algebra with δ_e as its identity element. The mappings $\delta_g, g \in G$, form a basis of the R-module $R[G]$ and satisfy*

$$\delta_g \cdot \delta_h = \delta_{gh} \text{ for } g, h \in G.$$

The algebra $R[G]$ is commutative if and only if G is abelian.

1.1 Elliptic Modular Forms

In this section we recall notations from the theory of modular forms that are going to be used in the rest of the thesis.

Let $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ denote the upper half plane (the Poincaré half plane). We shall use $\text{G} := \text{GL}_2^+(\mathbb{Q})$ for the group of rational 2 by 2 matrices with positive determinant. We have a natural action of $\text{GL}_2^+(\mathbb{Q})$ on the upper half plane. It is given by

$$(A, \tau) \mapsto A\tau = \frac{a\tau + b}{c\tau + d} \qquad (A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau \in \mathfrak{H}). \tag{1.2}$$

Definition 1.1.1. We write $\widetilde{\mathrm{GL}_2^+(\mathbb{Q})}$ for the metaplectic cover of $\mathrm{GL}_2^+(\mathbb{Q})$, that is the elements $(A, w(\tau))$, where $A \in \mathrm{GL}_2^+(\mathbb{Q})$ and $w : \mathfrak{H} \to \mathbb{C}$ is a holomorphic function on $\mathfrak{H}$ satisfying

$$w(\tau)^2 = \det(A)^{-\frac{1}{2}}(c\tau + d)$$

with the following group law

$$\big((A, w(\tau)), (B, v(\tau))\big) \mapsto (A, w(\tau))(B, v(\tau)) = (AB, w(B\tau)v(\tau)). \tag{1.3}$$

The application $(A, w(\tau)) \mapsto A$ defines a homomorphism $P : \widetilde{\mathrm{GL}_2^+(\mathbb{Q})} \to \mathrm{GL}_2^+(\mathbb{Q})$. For a subgroup Γ' of $\mathrm{GL}_2^+(\mathbb{Q})$, we let $\widetilde{\Gamma}' = P^{-1}(\Gamma')$ be the inverse image of Γ' under P. It is known that $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ is generated by

$$\widetilde{T} = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right), \qquad \widetilde{S} = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right).$$

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$, we let $\widetilde{A} = \left(A, \sqrt{\det(A)^{-\frac{1}{2}}(c\tau + d)}\right) \in \widetilde{\mathrm{GL}_2^+(\mathbb{Q})}$. Note that $A \mapsto \widetilde{A}$ is not a group homomorphism.

Definition 1.1.2. A congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is any subgroup that contains

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}), \gamma \mapsto \gamma \bmod N)$$

for some N. The smallest such N is the level of Γ. For example,

$$\Gamma_1(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{N}\right\}$$

and

$$\Gamma_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{N}\right\}$$

are congruence subgroups of level N. One has $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$.

Definition 1.1.3. Let N be a positive integer with $4 \mid N$. We set

$$\Gamma_0(N)^* = \{(A, j(A, \tau)) \mid A \in \Gamma_0(N)\},$$

where for each $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, the automorphic factor $j(A, \tau)$ is defined by

$$j(A, \tau) := \epsilon_d^{-1} \left(\frac{c}{d}\right) \sqrt{c\tau + d} \qquad (\tau \in \mathfrak{H}).$$

Lemma 1.1.4. $\Gamma_0(N)^*$ *is a subgroup of* $\widetilde{\mathrm{GL}_2^+(\mathbb{Q})}$.

Definition 1.1.5 (Petersson Slash Operator)**.** Let $k \in \frac{1}{2}\mathbb{Z}$. We define the weight k right action of $\widetilde{\mathrm{GL}_2^+(\mathbb{Q})}$ on the set of functions $f : \mathfrak{H} \to \mathbb{C}$ as follows: For $\tau \in \mathfrak{H}$ and $\widetilde{A} = (A, w(\tau)) \in \widetilde{\mathrm{GL}_2^+(\mathbb{Q})}$, we set

$$\left(f\big|_k \widetilde{A}\right)(\tau) := w(\tau)^{-2k} f(A\tau). \tag{1.4}$$

In particular,

$$f\big|_k \gamma_1\gamma_2 = f\big|_k \gamma_1\big|_k \gamma_2 \quad \text{for all } \gamma_1, \gamma_2 \in \widetilde{\mathrm{GL}_2^+(\mathbb{Q})}.$$

Note that for integral k this action factors into an action of $\mathrm{GL}_2^+(\mathbb{Q})$, which is nothing else than the usual $\big|_k$-action of $\mathrm{GL}_2^+(\mathbb{Q})$ given by

$$\left(f\big|_k A\right)(\tau) := \det(A)^{\frac{k}{2}} (c\tau + d)^{-k} f(A\tau),$$

with c, d, again, the lower row of A.

Definition 1.1.6 (Dirichlet Characters)**.** Let N be a positive integer , and $\widetilde{\chi}$ a character of $(\mathbb{Z}/N\mathbb{Z})^\times$. For any integer n, we put

$$\chi(n) = \begin{cases} \widetilde{\chi}(n \bmod N) & \text{if} \quad (n, N) = 1, \\ 0 & \text{if} \quad (n, N) \neq 1, \end{cases}$$

then χ is mapping of $\mathbb{Z}$ into $\mathbb{C}$ satisfying

(i) $\chi(mn) = \chi(m)\chi(n)$

(ii) $\chi(m) = \chi(n) \quad if \quad m \equiv n \bmod N$

(iii) $\chi(n) \neq 0 \quad$ if and only if $\quad (n, N) = 1$.

We called such a mapping χ of $\mathbb{Z}$ into $\mathbb{C}$ a Dirichlet character $\bmod N$, and we call N the modulus of χ. We call the Dirichlet character corresponding to the trivial character of $(\mathbb{Z}/N\mathbb{Z})^\times$ the trivial character $\bmod N$. For a Dirichlet character $\chi \bmod N$, we define the complex conjugate $\bar{\chi}$ by $\bar{\chi}(n) = \overline{\chi(n)} \quad (n \in \mathbb{Z})$.

If χ is a Dirichlet character $\bmod N$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we set $\chi(A) := \chi(d)$. The map $A \mapsto \chi(A)$ defines a character of $\Gamma_0(N)$ which we denote by the same symbol χ.

Definition 1.1.7. Given a Dirichlet character χ, we define the Dirichlet L-series of χ by

$$L(\chi, s) := \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_{p \text{ prime}} (1 - \chi(p) p^{-s})^{-1}.$$

Let $N \in \mathbb{N}$. We set

$$L_N(\chi, s) := \sum_{\substack{n=1 \\ \gcd(n,N)=1}}^{\infty} \chi(n) n^{-s} = L(\chi, s) \prod_{\substack{p \text{ prime} \\ p \mid N}} (1 - \chi(p) p^{-s}).$$

Definition 1.1.8 (Elliptic Modular Forms)**.** Let $k \in \frac{1}{2}\mathbb{Z}$, N a positive integer. If $k \in \frac{1}{2} + \mathbb{Z}$, we assume that $4 \mid N$. Let Γ be a congruence group of level N, and χ be a character of Γ of finite order with $\Gamma(N) \subset \ker(\chi)$. An elliptic modular form of weight k for Γ with character χ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that the following holds true:

1. For all $A \in \Gamma$ one has
   $f\big|_k A = \chi(A) f$ if $k \in \mathbb{Z}$, and $f\big|_k (A, j(A, \tau)) = \chi(A) f$ if $k \in \frac{1}{2} + \mathbb{Z}$.

2. The function f is holomorphic at all cusps.

The condition (2) means that $f|_k \tilde{\alpha}$ for any $\tilde{\alpha} \in \widetilde{\mathrm{GL}_2^+(\mathbb{Q})}$ has a Fourier expansion

$$\big(f|_k \tilde{\alpha}\big)(\tau) = \sum_{n=0}^{\infty} a_{f|_k \tilde{\alpha}}(n) \mathfrak{e}(2\pi i n \tau / h) \qquad \text{for some } h \in \mathbb{N}.$$

If $a_{f|_k \tilde{\alpha}}(0) = 0$ for all $\tilde{\alpha} \in \widetilde{\mathrm{GL}_2^+(\mathbb{Q})}$, then f is called a cusp form. The $\mathbb{C}$-vector space of elliptic modular forms of weight $k \in \mathbb{Z}$ (resp. $k \in \frac{1}{2} + \mathbb{Z}$) and character χ for Γ is denoted by $M_k(\Gamma, \chi)$ (resp. $M_k(N, \chi)$). The subspace of cusp forms will be denoted by $S_k(\Gamma, \chi)$ (resp. $S_k(N, \chi)$).

Remarks 1.1.9.

1. If $k \in \frac{1}{2} + \mathbb{Z}$, then it is obvious that $M_k(N,\chi) = 0$ if χ is an odd character, that is $\chi(-1) = -1$. Henceforth we will be assuming χ to be an even character.

2. If χ is trivial character, we write $M_k(\Gamma,\chi)$ (resp. $M_k(N,\chi)$) simply by $M_k(\Gamma)$ (resp. $M_k(N)$). Similarly, we write $S_k(\Gamma,\chi)$ (resp. $S_k(N,\chi)$) simply by $S_k(\Gamma)$ (resp. $S_k(N)$).

3. Since $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \Gamma_0(N)$, the condition (2) in Definition 1.1.8 means that every modular form f of weight $k \in \mathbb{Z}$ on $\Gamma_0(N)$ has a Fourier expansion of the form
$$f(\tau) = \sum_{n \geq 0} a_f(n)\mathfrak{e}(n\tau). \tag{1.5}$$

1.1.1 Operators on the Space of Elliptic Modular Forms

At the beginning we introduce the Hecke operators and state some of their properties. For $\ell \in \mathbb{N}$ we define the Hecke operator

$$T(\ell) : M_k(\Gamma_0(N),\chi) \to M_k(\Gamma_0(N),\chi)$$

by

$$f \mapsto T(\ell)f := \ell^{\frac{k}{2}-1} \sum_{ad|\ell} \sum_{b=0}^{d-1} \chi(a) f|_k \left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right). \tag{1.6}$$

If $f \in M_k(\Gamma_0(N),\chi)$ has a Fourier expansion

$$f(\tau) = \sum_{n \geq 0} a_f(n)\mathfrak{e}(n\tau),$$

then

$$(T(\ell)f)(\tau) = \sum_{n \geq 0} a_{T(\ell)f}(n)\mathfrak{e}(n\tau),$$

where

$$a_{T(\ell)f}(n) = \sum_{d|\ell,n} \chi(d) d^{k-1} a_f(\ell n/d^2). \tag{1.7}$$

In particular, $T(\ell)f$ is again a modular form, and is a cusp form if f is one.

Definition 1.1.10. A modular form $f \in M_k(\Gamma_0(N),\chi)$ is called a Hecke eigenfunction for the operator $T(\ell)$ if there exists $\lambda(\ell,f) \in \mathbb{C}$ with $T(\ell)f = \lambda(\ell,f)f$.

Proposition 1.1.11. *Suppose that $f(\tau) = \sum_{n\geq 0} a_f(n)\mathfrak{e}(n\tau) \in M_k(\Gamma_0(N),\chi)$ is an eigenfunction for all Hecke operators $T(\ell)$ with eigenvalues $\lambda(\ell,f)$. Then,*

1. *If $f(\tau)$ is non constant, then $a_f(1) \neq 0$.*

2. *If $f(z)$ is a normalized, i.e., $a_f(1) = 1$, then $\lambda(\ell,f) = a_f(\ell)$.*

3. *If $a_f(0) \neq 0$, then $\lambda(\ell,f) = \sum_{a|\ell} \chi(a)a^{k-1}$.*

Proof. See e.g. [Kob93, Proposition 40]. □

Proposition 1.1.12. *The operators $T(\ell)$ on $M_k(\Gamma_0(N),\chi)$ satisfy the formal power series*

$$\sum_{n=1}^{\infty} T(\ell)\ell^{-s} = \prod_{p\,\text{prime}} \left(1 - T(p)p^{-s} + \chi(p)p^{k-1-2s}\right). \tag{1.8}$$

Proof. See e.g. [Kob93, Proposition 36]. □

Theorem 1.1.13 ([Shi73, Theorem 1.7])**.** *Let $k \in \mathbb{Z}$ be an odd integer. Let*

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n)\mathfrak{e}(n\tau) \in M_{k/2}(N,\chi).$$

For each prime number p there is a Hecke operator $T(p^2)$ acting on $M_{k/2}(N,\chi)$. The action of $T(p^2)$ in terms of Fourier expansion is given as follows:

$$(T(p^2)f)(\tau) = \sum_{n=0}^{\infty} a_{T(p^2)f}(n)\mathfrak{e}(n\tau),$$

where

$$a_{T(p^2)f}(n) = a_f(p^2n) + \chi(p)\left(\frac{-1}{p}\right)^{\lambda}\left(\frac{n}{p}\right)p^{\lambda-1}a_f(n) + \chi(p^2)p^{k-2}a_f(n/p^2), \tag{1.9}$$

and $\lambda = (k-1)/2$ and $a_f(n/p^2) = 0$ whenever $p^2 \nmid n$.

There are other important operators on the space of elliptic modular forms of integral weight. Let $f(\tau)=\sum_{n\geq 0} a_f(n)\mathfrak{e}(n\tau)\in M_k(\Gamma_0(N),\chi)$, $d\in\mathbb{N}$, and ψ is a Dirichlet character modulo a positive integer M. We set

$$(f|U_d)(\tau) := \sum_{n\geq 0} a_f(dn)\mathfrak{e}(n\tau), \tag{1.10}$$

$$(f|V_d)(\tau) := \sum_{n\geq 0} a_f(n)\mathfrak{e}(nd\tau) \tag{1.11}$$

$$(f\otimes\psi)(\tau) := \sum_{n\geq 0} \psi(n)a_f(n)\mathfrak{e}(n\tau). \tag{1.12}$$

Theorem 1.1.14. *One has*

1. *$f|V_d$ and $f|U_d$ are elements of $S_k(\Gamma_0(dN),\chi)$. If $d|N$, then $f|U_d$ is an element of $S_k(\Gamma_0(N),\chi)$.*

2. *$f\otimes\psi$ is an element of $M_k(\Gamma_0(NM^2),\chi\psi^2)$.*

Proof. Assertion (1) can be found in [DS06, 5.6]. For (2) we refer the reader to [Kob93, Proposition 17 (P.127)]. □

1.1.2 Atkin-Lehner Theory of Newforms

Now, we recall some facts of the Atkin-Lehner theory of newforms from [AL70]. Let N be a posiitve integer, and $Q||N$. We define the Atkin-Lehner operator $W_Q^N : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(N))$ by the $|_k$-action of any matrix

$$W_Q^N = \begin{pmatrix} Qa & b \\ Nc & Qd \end{pmatrix}, \quad a,b,c,d\in\mathbb{Z}, \text{ and } \det(W_Q^N)=Q.$$

Note that the different choice of a,b,c,d do not effect the $|_k$-action of W_Q^N on $M_k(\Gamma_0(N))$. Also, we define the operator $W_N : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(N))$ ("Fricke involution") by the $|_k$-action of the matrix

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

We define the subspace $S_k^{\mathrm{old}}(\Gamma_0(N))$ in $S_k(\Gamma_0(N))$ by:

$$S_k^{\mathrm{old}}(\Gamma_0(N)) = \bigoplus_{\substack{M|N \\ M\neq N}} \bigoplus_{d|\frac{M}{N}} S_k(\Gamma_0(N))\big|V_d.$$

The subspace $S_k^{\text{new}}(\Gamma_0(N))$ in $S_k(\Gamma_0(N))$ is defined to be the orthogonal complement of $S_k^{\text{old}}(\Gamma_0(N))$ with respect to the Petersson inner product. Note that these subspaces are preserved under $T(\ell)$ for $(\ell, N) = 1$.

An element f of $S_k^{\text{new}}(\Gamma_0(N))$ is called a newform if it is a normalized eigenfunction under all Hecke operators $T(\ell)$ and under all operators $W_{p^v}^N$ for $p^v || N$ and W_N.

Theorem 1.1.15. *Let $f(\tau) = \sum_{n\geq 0} a_f(n)\mathfrak{e}(n\tau) \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform. Then,*

1. *$T(\ell)f = a_f(\ell)f$ for all $\ell \in \mathbb{N}$.*
2. *If $p||N$, then $a_f(p) = -w_p p^{k/2-1}$, where $w_p \in \{\pm 1\}$ is such that $f \mid W_p^N = w_p f$.*

Proof. See e.g. [AL70, Theorem 3]. □

1.2 Lattices and Finite Quadratic Modules

In this section we will recall basics of the theory of finite quadratic modules, and lattices. The basic references in this section are [Sko14],[Ser93], and [Nik80].

Let R be a commutative ring with unity 1, and M, N be R-modules where M is free of a finite rank. A map $\beta : M \times M \to N$ is called a symmetric R-bilinear form if

$$\beta(x,y) = \beta(y,x), \quad \beta(x+y,z) = \beta(x,z) + \beta(y,z), \quad \beta(\lambda x,z) = \lambda\beta(x,z),$$

for all $x, y, z \in M$, $\lambda \in R$. If $N = R$ we say that β is integral. Moreover, it is called non-degenerate if $\beta(x,y) = 0$ for all y in M is only possible for $x = 0$.

Let $\{e_1, e_2, \cdots, e_m\}$ be an R-basis for M, i.e., $M = Re_1 \otimes \cdots \otimes Re_n$. Let $x, y \in M$ with $x = \sum_i x_i e_i$, $y = \sum_i y_i e_i$ for $x_i, y_i \in R$. The values of β are determined by its values $\beta(e_i, e_j)$ on all pairs of basis elements, since $\beta(x,y) = \sum_{i,j} x_i y_j \beta(e_i, e_j)$. The matrix $F = \left(\beta(e_i, e_j)\right)_{i,j} \in \text{Mat}_m(N)$ is called the Gram matrix of β with respect to the basis $\{e_1, e_2, \cdots, e_m\}$. Thus the bilinear form can be written using matrix multiplication as $\beta(x,y) = \widetilde{x}^T F \widetilde{y}$, where, for any $x = \sum_i x_i e_i \in M$ we use $\widetilde{x} = (x_1, x_2, \cdots, x_m)^T$. Let F' be another matrix representing for β with respect to a basis $\{e'_1, e'_2, \cdots, e'_m\}$, then $F' = X^T F X$, where X denotes the matrix of the basis change.

1.2.1 Finite Quadratic Modules over $\mathbb{Z}$

Definition 1.2.1. A finite quadratic module over $\mathbb{Z}$ is a pair (M,Q), where M is a finite abelian group (i.e. a finite $\mathbb{Z}$-module), and where $Q: M \to \mathbb{Q}/\mathbb{Z}$ is a non-degenerate quadratic form of M, i.e.,

1. $Q(ax) = a^2 Q(x)$ for all $a \in \mathbb{Z}$ and for all $x \in M$.

2. The symmetric function $\beta : M \times M \to \mathbb{Q}/\mathbb{Z}$ given by $(x,y) \mapsto Q(x+y) - Q(x) - Q(y)$ is $\mathbb{Z}$-bilinear and non-degenerate.

The map β is called the bilinear form associated to Q.

Definition 1.2.2. Let (M,Q) and $(M\prime,Q\prime)$ be two finite quadratic modules. A $\mathbb{Z}$-linear map $\sigma : M \to M'$ is called isometry if σ is injective and

$$Q'(\sigma(x)) = Q(x)$$

for all $x \in M$. If such isometric $\sigma : M \to M'$ exist, we say that M and M' are isometric, and Q is R-equivalent to Q' (we write $Q \sim_R Q'$).

Theorem 1.2.3. *Any finite quadratic module (M,Q) is isomorphic to a direct sum of finite quadratic modules of the following forms:*

1. $A^a_{p^r} = (\mathbb{Z}_{p^r}, \frac{ax^2}{p^r})$ *for some odd prime p and $a \in \mathbb{Z}$ such that* $\gcd(a,p) = 1$.

2. $A^a_{2^r} = (\mathbb{Z}_{2^r}, \frac{ax^2}{2^{r+1}})$ *for some $a \in \mathbb{Z}$ such that* $\gcd(a,2) = 1$.

3. $B_{2^r} = (\mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}, \frac{x^2+xy+y^2}{2^r})$.

4. $C_{2^r} = (\mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}, \frac{xy}{2^r})$.

Proof. A proof can be found in [Sko14, Chapter 1]. □

1.2.2 Integral Lattices

Definition 1.2.4 (Lattice)**.** Let R be a commutative ring with unity 1, and L,N be R-modules where L is free of a finite rank. Let $\beta : L \times L \to N$ be a symmetric non-degenerate bilinear form. The pair $\underline{L} = (L,\beta)$ is called a lattice over R. The lattice $\underline{L}$ is called integral if the associated bilinear form is integral, i.e, $\beta(L,L) \subseteq R$.

Remark 1.2.5. Throughout this thesis we will consider only integral lattices over $R = \mathbb{Z}$. Thus for sake of simplicity we shall refer to them simply as integral lattices.

Definition 1.2.6 (Positive Definite Lattice)**.** An integral lattice $\underline{L} = (L,\beta)$ is called positive definite if and only if $\beta(x,x) > 0$ for all $x \in L$ such that $x \neq 0$.

Definition 1.2.7 (Even Lattice)**.** An integral lattice $\underline{L} = (L,\beta)$ is called even if $\beta(x,x)$ is even for all $x \in L$, otherwise $\underline{L}$ is called odd.

Notation Let (L,β) be an even lattice. For every $x \in L$ we set

$$\beta(x) := \tfrac{1}{2}\beta(x,x).$$

Definition 1.2.8. Let $\underline{L} = (L,\beta)$ be an even positive definite lattice. If S is a ring extension of $\mathbb{Z}$ we consider β via linear continuation as a bilinear form on $L \otimes S$, which we shall denote by the same letter. In particular, we shall use the notation $\underline{L}_S = (L \otimes_{\mathbb{Z}} S, \beta)$ for the "S-version" of $\underline{L}$, which is a lattice over S.

Definition 1.2.9 (Dual Lattice)**.** Let $\underline{L} = (L,\beta)$ be an integral Lattice. We define its dual $\underline{L}^\# := (L^\#,\beta)$, where

$$L^\# = \left\{ y \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(y,x) \in \mathbb{Z} \text{ for all } x \in L \right\}.$$

It is well-known that $L^\#$ is again a free $\mathbb{Z}-$module of the same rank as L. If $L^\# = L$, then $\underline{L}$ is called unimodular .

Definition 1.2.10 (The Discriminant Form)**.** Let $\underline{L} = (L, \beta)$ be an even lattice. L contained in $L^\#$ and $L^\#/L$ is a finite abelian group. Since $\underline{L}$ is integral and even, the reduction of β modulo $\mathbb{Z}$ induces a bilinear form on $L^\#/L$. We set

$$D_{\underline{L}} := (L^\#/L, x + L \mapsto \beta(x) + \mathbb{Z})$$

and we call it the discriminant form associated with the lattice $\underline{L}$.

Note that the lattice $\underline{L}$, which shall be considered in this thesis, is non-degenerate (by definition), thus its Gram matrix F (with respect to given basis) is symmetric and invertible. Moreover, F can be written in the form

$$M^T F M = \mathrm{diag}(\underbrace{1, \cdots, 1}_{n_+}, \underbrace{-1, \cdots, -1}_{n_-})$$

with suitable M in $\mathrm{GL}_{\mathrm{rk}(\underline{L})}(\mathbb{R})$. By Sylvesterís law of inertia the numbers n_+ and n_- of $+1$ and -1's on the diagonal do not depend on the particular choice of M. We define the signature of $\underline{L}$ by

$$\mathrm{sign}(\underline{L}) := n_+ - n_-.$$

Lemma 1.2.11 (Milgram's formula)**.** *The bilinear form on $L^\#/L$ determines* $\mathrm{sign}(\underline{L})$ *mod* 8 *by Milgram's formula:*

$$\sum_{x \in L^\#/L} \mathfrak{e}\left(\beta(x)\right) = \sqrt{\det(\underline{L})}\,\mathfrak{e}\left(\mathrm{sign}(\underline{L})/8\right). \tag{1.13}$$

Remarks 1.2.12.

1. According to [Wal63, Theorem 6], any finite quadratic module can be obtained as the discriminant form of an even lattice.

2. Let F be a Gram matrix of the even positive definite lattice $\underline{L}$. In many cases, it is useful to identify L with $\mathbb{Z}^{\mathrm{rk}(\underline{L})}$, and $L^\#$ with $F^{-1}\mathbb{Z}^{\mathrm{rk}(\underline{L})}$. Thus we may write
$$D_{\underline{L}} = (F^{-1}\mathbb{Z}^{\mathrm{rk}(\underline{L})}/\mathbb{Z}^{\mathrm{rk}(\underline{L})}, x \mapsto \tfrac{1}{2}x^T F x).$$

Definition 1.2.13. Let F be a Gram matrix corresponding to a basis of L. We set:

$$\det(\underline{L}) := \left|L^\#/L\right| = \left|\det(F)\right|.$$

Note that if β is a positive definite then $\det(F) > 0$ and $\operatorname{sign}(\underline{L}) = \operatorname{rk}(\underline{L})$.

Definition 1.2.14 (Level of a Lattice)**.** Let $\underline{L} = (L, \beta)$ be an integral lattice. We define the level of the lattice $\underline{L}$ to be the smallest positive integer $\operatorname{lev}(\underline{L})$ such that $\operatorname{lev}(\underline{L}) \cdot \beta(x) \in \mathbb{Z}$ for all $x \in L^{\#}$.

For given $x \in L^{\#}$, we use $\operatorname{lev}(x)$ for the smallest positive integer such that $\operatorname{lev}(x) \cdot \beta(x) \in \mathbb{Z}$. It is obvious that $\operatorname{lev}(\underline{L}) = \mathrm{l.c.m}\left(\mathrm{lev(x)}\right)_{x \in L^{\#}/L}$.

Definition 1.2.15. We set

$$\Delta(\underline{L}) = \begin{cases} (-1)^{\lfloor \frac{\operatorname{rk}(\underline{L})}{2} \rfloor} \det(\underline{L}) & \text{if } \operatorname{rk}(\underline{L}) \equiv 0 \bmod 2, \\ (-1)^{\lfloor \frac{\operatorname{rk}(\underline{L})}{2} \rfloor} 2\det(\underline{L}) & \text{if } \operatorname{rk}(\underline{L}) \equiv 1 \bmod 2. \end{cases} \tag{1.14}$$

Lemma 1.2.16. *One has* $\Delta(\underline{L}) \equiv 0, 1 \bmod 4$*, and if* $\operatorname{rk}(\underline{L})$ *is odd, then* $\Delta(\underline{L})$ *and* $\operatorname{lev}(\underline{L})$ *are both divisible by* 4*. In particular the application* $a \mapsto \left(\frac{\Delta(\underline{L})}{a}\right)$ *define Dirichlet character modulo* $|\Delta(\underline{L})|$.

Proof. Follows from the Jordan decomposition of $D_{\underline{L}_{\mathbb{Z}_2}}$. □

Notation Let $\underline{L} = (L, \beta)$ be an integral lattice. We shall use $\mathbb{N}_{\underline{L}}$ to denote the set of all positive integer $\ell \in \mathbb{N}$ with $(\ell, \operatorname{lev}(\underline{L})) = 1$.

Chapter 2

Hecke Theory of Jacobi Forms of Lattice Index

Jacobi forms are interesting for their arithmetic properties, which are reflected by the theory of Hecke operators which we shall develop in this chapter. First, we shall state and prove some lemmas on lattices and Gauss sums, then we shall recall the definition of the Jacobi group, and the Jacobi forms of lattice index, then we construct a collection of extremely important and fundamental linear operators acting on the vector space of Jacobi forms of lattice index, called the Hecke operators. These operators extend the classical theory of Hecke operators for the scalar index Jacobi forms that developed in [EZ85].

In this chapter we will use the notation $\underline{L} = (L, \beta)$ to denote a positive definite even lattice over $\mathbb{Z}$. We shall use $\mathrm{lev}(\underline{L})$, $\mathrm{rk}(\underline{L})$, and $\det(\underline{L})$ to denote the level, the rank, and the determinant of the lattice $\underline{L}$, as defined in section 1.2 respectively.

2.1 Some Lemmas on Lattices and Gauss Sums

Here, we shall state and prove some lemmas which we will need later in section 2.6 and section 2.7 when we discuss the action of Hecke operators on

Jacobi forms of lattice index in terms of Fourier coefficients. First, we state the main results of this section:

2.1.1 The Main Results of this Section

Definition 2.1.1. For $a \in \mathbb{N}$ and $n \in \mathbb{Q}$ such that $n \cdot \Delta(\underline{L}) \in \mathbb{Z}$, we set

$$\chi_{\underline{L}}(n,a) := \left(\frac{n\cdot\Delta(\underline{L})}{a}\right), \quad \chi_{\underline{L}}(a) := \chi_{\underline{L}}(1,a) = \left(\frac{\Delta(\underline{L})}{a}\right).$$

Definition 2.1.2. For integers $a, c \in \mathbb{Z}$ where a is positive, we set

$$\mathscr{W}(c,a) := \sum_{t|a} \mu(a/t)\, t^{1-\mathrm{rk}(\underline{L})} \# \left\{ x \in L \,/\, tL \mid c \equiv \beta(x) \bmod t \right\}. \tag{2.1}$$

$$\mathscr{W}_{\mathrm{I}}(c,a) := \sum_{\substack{b|a \\ a/b=\text{perfect square}}} b^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil -1} \mathscr{W}(c,b). \tag{2.2}$$

$$\mathscr{W}_{\mathrm{II}}(c,a) := \sum_{b|a} \chi_{\underline{L}}(b)\, b^{-\frac{\mathrm{rk}(\underline{L})}{2}} \mathscr{W}\left(c, \frac{a}{b}\right). \tag{2.3}$$

Here, μ is the Möbius function.

Proposition 2.1.3. *Assuming that* $\mathrm{rk}(\underline{L})$ *is odd.*

1. *For $a \in \mathbb{N}_{\underline{L}}$ and $c \in \mathbb{Z}$, one has*

$$\mathscr{W}_{\mathrm{I}}(c,a) = \chi_{\underline{L}}(a) \left(\frac{c/f^2}{a/f^2}\right) f \; \boldsymbol{\delta}\left(\gcd(c,a) = f^2,\ f \in \mathbb{N}\right). \tag{2.4}$$

2. *If $x \in \mathbb{N}_{\underline{L}}$ such that x^2 divides c and a, then*

$$x \cdot \mathscr{W}_{\mathrm{I}}(c/x^2, a/x^2) = \mathscr{W}_{\mathrm{I}}(c,a). \tag{2.5}$$

Proposition 2.1.4. *Assuming that* $\mathrm{rk}(\underline{L})$ *is even. For integers $a \in \mathbb{N}_{\underline{L}}$ and $c \in \mathbb{Z}$ one has*

$$\mathscr{W}_{\mathrm{II}}(c,a) = \chi_{\underline{L}}(a)\, a^{-\frac{\mathrm{rk}(\underline{L})}{2}+1} \boldsymbol{\delta}(a \mid c). \tag{2.6}$$

The rest of this section is devoted to the proof of the preceding propositions.

2.1.2 Proofs

To prove Proposition 2.1.3 and Proposition 2.1.4 we need the following series of definitions and lemmas.

Definition 2.1.5. Let $a, b \in \mathbb{Z}$, $c \in \mathbb{N}$. The generalized Gauss sum $G(a,b,c)$ is defined by:

$$G(a,b,c) = \sum_{n=0}^{c-1} \mathfrak{e}\left(\frac{an^2+bn}{c}\right). \tag{2.7}$$

Definition 2.1.6. For any odd integer a, we set: $\epsilon(a) = 1$ if $a \equiv 1 \pmod 4$ and $\epsilon(a) = i$ if $a \equiv 3 \pmod 4$.

Lemma 2.1.7 ([BEW98]). *Let p be an odd prime integer, $v \in \mathbb{N}$ be a positive integer, and $a \in \mathbb{Z}$. One has*

$$G(a,0,p^v) = \begin{cases} \epsilon(p^{v-\mathrm{ord}_p(a)}) p^{\frac{v+\mathrm{ord}_p(a)}{2}} \left(\frac{a/p^{\mathrm{ord}_p(a)}}{p^{v-\mathrm{ord}_p(a)}}\right) & \text{if } v > \mathrm{ord}_p(a), \\ p^v & \text{if } v \le \mathrm{ord}_p(a). \end{cases} \tag{2.8}$$

Lemma 2.1.8. *Let $t = 2^v$ for some positive integer v, a be an odd integer, and $b \in \mathbb{Z}$. The sum $G(a,b,t)$ has the value*

$$t^{\frac{1}{2}}(1+i)\left(\frac{-t}{a}\right)\epsilon(a)\mathfrak{e}_t\left(-\frac{\bar{a}b^2}{4}\right)$$

if $t > 2$ and $b \equiv 0 \bmod 2$ where $\bar{a} \in \mathbb{Z}$ such that $\bar{a}a \equiv 1 \bmod 4t$, has the value t if $t = 2$ and $b \not\equiv 0 \bmod 2$, and is 0 otherwise.

Proof. Follows from [BEW98, Theorem 1.2.2]. □

Lemma 2.1.9. *Let $t = 2^v$ for some positive integer v, and d be an odd integer. One has*

$$\sum_{x,y \bmod t} \mathfrak{e}_t\left(d(x^2+xy+y^2)\right) = -t\left(\frac{-t}{3}\right). \tag{2.9}$$

Proof. One has

$$\begin{aligned} \sum_{x,y \bmod t} \mathfrak{e}_t\left(d(x^2+xy+y^2)\right) &= \sum_{x \bmod t} \mathfrak{e}_t\left(dx^2\right) \sum_{y \bmod t} \mathfrak{e}_t\left(d(xy+y^2)\right) \\ &= \sum_{x \bmod t} \mathfrak{e}_t\left(dx^2\right) \cdot G(d, dx, t). \end{aligned}$$

First, we assmue that $t \neq 2$. By Lemma 2.1.8, the right-hand side (RHS) of the above equation equals

$$\text{RHS} = \sum_{x \bmod t} \mathfrak{e}_t\left(dx^2\right) \times \begin{cases} t^{\frac{1}{2}}(1+i)\left(\frac{-t}{d}\right)\epsilon(d)\mathfrak{e}_t\left(\frac{-d^2\bar{d}x^2}{4}\right) & \text{if } x \equiv 0 \bmod 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{d}$ is an integer such that $\bar{d}d \equiv 1 \bmod 4t$. Thus

$$\begin{aligned} \sum_{x,y \bmod t} \mathfrak{e}_t\left(d(x^2+xy+y^2)\right) &= \sum_{\substack{x \bmod t \\ x\equiv 0 \bmod 2}} \mathfrak{e}_t\left(dx^2\right) t^{\frac{1}{2}}(1+i)\left(\frac{-t}{d}\right)\epsilon(d)\mathfrak{e}_t\left(\frac{-d^2\bar{d}x^2}{4}\right) \\ &= t^{\frac{1}{2}}(1+i)\left(\frac{-t}{d}\right)\epsilon(d) \sum_{\substack{x \bmod t \\ x\equiv 0 \bmod 2}} \mathfrak{e}_t\left(\tfrac{3}{4}dx^2\right). \end{aligned}$$

By replacing $\frac{x}{2}$ with y we see, using Lemma 2.1.8, that

$$\sum_{\substack{x \bmod t \\ x\equiv 0 \bmod 2}} \mathfrak{e}_t\left(\tfrac{3}{4}dx^2\right) = \frac{1}{2}\sum_{y \bmod t} \mathfrak{e}_t\left(3dy^2\right) = \frac{1}{2}t^{\frac{1}{2}}(1+i)\left(\frac{-t}{3d}\right)\epsilon(3d).$$

Inserting this into the last formula for $\sum_{x,y \bmod t}\mathfrak{e}_t\left(d(x^2+xy+y^2)\right)$, we obtain the claimed result. Now we assume that $t=2$. One has

$$\sum_{x,y \bmod 2} \mathfrak{e}_2\left(d(x^2+xy+y^2)\right) = \sum_{x \bmod 2} \mathfrak{e}_2\left(dx^2\right)\cdot G(d,dx,2) = 2\sum_{\substack{x \bmod 2 \\ x \text{ odd}}} \mathfrak{e}_2\left(dx^2\right) = -2\left(\frac{-2}{3}\right).$$

The proof is complete. □

Lemma 2.1.10. *Let $t = 2^v$ for some positive integer v, and d be an odd integer. One has*

$$\sum_{x,y \bmod t} \mathfrak{e}_t(dxy) = t. \tag{2.10}$$

Proof. One has

$$\begin{aligned} \sum_{x,y \bmod t} \mathfrak{e}_t(dxy) &= \sum_{\substack{x,y \bmod t \\ x\equiv 0 \bmod t}} \mathfrak{e}_t(dxy) + \sum_{\substack{x,y \bmod t \\ x\not\equiv 0 \bmod t}} \mathfrak{e}_t(dxy) \\ &= \sum_{\substack{x(t) \\ x\equiv 0 \bmod t}} t + 0 = t \end{aligned}$$

as stated in the lemma. □

Definition 2.1.11. For integers $\ell, c \in \mathbb{Z}$, the Ramanujan sum $G_\ell(c)$ is defined as follows:

$$G_\ell(c) = \sum_{\substack{d \bmod \ell \\ (d,\ell)=1}} \mathfrak{e}_\ell(cd). \tag{2.11}$$

It is known that the Ramanujan sum $G_\ell(c)$ is multiplicative when considered as a function of ℓ for a fixed value of c.

Definition 2.1.12. For integers $\ell, c \in \mathbb{Z}$ where ℓ is positive and odd, we set

$$F(c,\ell) = \epsilon(\ell) \sum_{d \bmod \ell} \left(\frac{d}{\ell}\right) \mathfrak{e}_\ell(cd). \tag{2.12}$$

Lemma 2.1.13. *The function $F(c,\ell)$ for fixed c is multiplicative in ℓ, i.e.,*

$$F(c,\ell_1\ell_2) = F(c,\ell_1)F(c,\ell_2) \tag{2.13}$$

for all $c \in \mathbb{Z}$ and odd positive integers ℓ_1, ℓ_2 such that $(\ell_1,\ell_2) = 1$.

Proof. Let ℓ_1, ℓ_2 be two relatively prime integers. Any integer $d \bmod \ell_1\ell_2$ can be written with the Chinese Remainder Theorem as $d = b_2\ell_1 + b_1\ell_2$, where b_1 runs through integers $\bmod\ \ell_1$ and b_2 runs $\bmod\ \ell_2$. Then

$$\begin{aligned}
F(c,\ell_1\ell_2) &= \epsilon(\ell_1\ell_2) \sum_{d \bmod \ell_1\ell_2} \left(\frac{d}{\ell_1\ell_2}\right) \mathfrak{e}_{\ell_1\ell_2}(cd) \\
&= \epsilon(\ell_1\ell_2) \sum_{b_1 \bmod \ell_1} \sum_{b_2 \bmod \ell_2} \left(\frac{b_2\ell_1+b_1\ell_2}{\ell_1\ell_2}\right) \mathfrak{e}_{\ell_1\ell_2}(c(b_2\ell_1+b_1\ell_2)) \\
&= \epsilon(\ell_1\ell_2) \sum_{b_1 \bmod \ell_1} \left(\frac{b_2\ell_1+b_1\ell_2}{\ell_1}\right) \mathfrak{e}_{\ell_1\ell_2}(cb_1\ell_2) \sum_{b_2 \bmod \ell_2} \left(\frac{b_2\ell_1+b_1\ell_2}{\ell_2}\right) \mathfrak{e}_{\ell_1\ell_2}(cb_2\ell_1) \\
&= \epsilon(\ell_1\ell_2) \sum_{b_1 \bmod \ell_1} \left(\frac{b_1\ell_2}{\ell_1}\right) \mathfrak{e}_{\ell_1}(cb_1) \sum_{b_2 \bmod \ell_2} \left(\frac{b_2\ell_1}{\ell_2}\right) \mathfrak{e}_{\ell_2}(cb_2) \\
&= \epsilon(\ell_1\ell_2) \left(\frac{\ell_2}{\ell_1}\right)\left(\frac{\ell_1}{\ell_2}\right) \sum_{b_1 \bmod \ell_1} \left(\frac{b_1}{\ell_1}\right) \mathfrak{e}_{\ell_1}(cb_1) \sum_{b_2 \bmod \ell_2} \left(\frac{b_2}{\ell_2}\right) \mathfrak{e}_{\ell_2}(cb_2) \\
&= \epsilon(\ell_1\ell_2)\epsilon(\ell_1)^{-1}\epsilon(\ell_2)^{-1} \left(\frac{\ell_2}{\ell_1}\right)\left(\frac{\ell_1}{\ell_2}\right) F(c,\ell_1)F(c,\ell_2).
\end{aligned}$$

Using the quadratic reciprocity, we see that $\epsilon(\ell_1\ell_2)\epsilon(\ell_1)^{-1}\epsilon(\ell_2)^{-1}\left(\frac{\ell_2}{\ell_1}\right)\left(\frac{\ell_1}{\ell_2}\right) = 1$. Namely, $\epsilon(\ell_1\ell_2)\epsilon(\ell_1)^{-1}\epsilon(\ell_2)^{-1}$ equals $+1$ if $\ell_1 \equiv \ell_2 \bmod 4$ and -1 otherwise, where by quadratic reciprocity equals $\left(\frac{\ell_2}{\ell_1}\right)\left(\frac{\ell_1}{\ell_2}\right)$. It follows $F(c,\ell_1\ell_2) = F(c,\ell_1)F(c,\ell_2)$. □

Lemma 2.1.14 ([BEW98],[And87])**.** *Let p be an odd prime, $v \in \mathbb{N}$ be an odd positive integer, and $c \in \mathbb{Z}$. One has*

$$F(c,p^v) = p^{v-\frac{1}{2}}\left(\frac{-c/p^{v-1}}{p}\right)\delta(p^{v-1}\|c). \tag{2.14}$$

Recall that $\underline{L} = (L,\beta)$ denotes a positive definite even lattice over $\mathbb{Z}$.

Definition 2.1.15. For integers $\ell, s \in \mathbb{Z}$ where ℓ is positive, we set

$$\mathfrak{S}_{\underline{L}}(s,\ell) := \sum_{x \in L/\ell L} \mathfrak{e}_\ell\left(s\beta(x)\right). \tag{2.15}$$

Lemma 2.1.16. *Let $s \in \mathbb{Z}$, $\ell_1, \ell_2 \in \mathbb{N}_{\underline{L}}$ such that* $\gcd(\ell_1,\ell_2) = 1$. *One has*

$$\mathfrak{S}_{\underline{L}}(s,\ell_1\ell_2) = \mathfrak{S}_{\underline{L}}(s\ell_2,\ell_1)\mathfrak{S}_{\underline{L}}(s\ell_1,\ell_2). \tag{2.16}$$

Proof. For $\ell_1, \ell_2 \in \mathbb{N}_{\underline{L}}$ with $\gcd(\ell_1,\ell_2) = 1$ we have a $\mathbb{Z}$-module homomorphism

$$\varphi : L/\ell_2 L \oplus L/\ell_1 L \to L/\ell_1\ell_2 L$$

given by

$$\varphi(x + \ell_2 L, y + \ell_1 L) = \ell_1 x + \ell_2 y + \ell_1\ell_2 L.$$

φ is surjective: Let $z \in L$. Since $\gcd(\ell_1,\ell_2) = 1$, then there are $h,k \in \mathbb{Z}$ such that $1 = \ell_1 h + \ell_2 k$, so $z = \ell_1 hz + \ell_2 kz$. Thus $\varphi(hz + \ell_1\ell_2 L, kz + \ell_1\ell_2 L) = z + \ell_1\ell_2 L$.
φ is injective: $\varphi(x + \ell_2 L, y + \ell_1 L) = 0 + \ell_1\ell_2 L$ if and only if $\ell_1 x + \ell_2 y \in \ell_1\ell_2 L$, that is $\ell_1 x + \ell_2 y = \ell_1\ell_2 z$ for some $z \in L$, which implies $\ell_2 y = \ell_1(x - \ell_2 z)$. Write $x = \sum_i a_i v_i$, $y = \sum_i b_i v_i$ and $z = \sum_i c_i v_i$, where $\{v_i \mid 1 \le i \le \mathrm{rk}(\underline{L})\}$ is a basis for L. Then

$$\ell_2 b_i = \ell_1(a_i - \ell_2 c_i) \qquad (1 \le i \le \mathrm{rk}(\underline{L})).$$

Since $\gcd(\ell_1,\ell_2) = 1$, we have $\ell_1 | b_i$. Thus $y = \ell_1 g$ for some $g \in L$, so $\ell_2 y = \ell_1\ell_2 g \in \ell_1\ell_2 L$. In a similar way we find that $\ell_1 x \in \ell_1\ell_2 L$. Thus

$$\begin{aligned}
\mathfrak{S}_{\underline{L}}(s,\ell_1\ell_2) &= \sum_{x \in L/\ell_1\ell_2 L} \mathfrak{e}_{\ell_1\ell_2}\left(s\beta(x)\right) \\
&= \sum_{x_1 \in L/\ell_2 L} \sum_{x_2 \in L/\ell_1 L} \mathfrak{e}_{\ell_1\ell_2}\left(s\beta(\ell_1 x_1 + \ell_2 x_2)\right) \\
&= \sum_{x_1 \in L/\ell_2 L} \mathfrak{e}_{\ell_2}\left(s\ell_1\beta(x_1)\right) \sum_{x_2 \in L/\ell_1 L} \mathfrak{e}_{\ell_1}\left(s\ell_2\beta(x_2)\right) \\
&= \mathfrak{S}_{\underline{L}}(s\ell_1,\ell_2)\mathfrak{S}_{\underline{L}}(s\ell_2,\ell_1)
\end{aligned}$$

as stated in the lemma. □

Lemma 2.1.17. *Let $p \in \mathbb{N}_{\underline{L}}$ be a prime number, $v \in \mathbb{N}$, and $s \in \mathbb{Z}$. Assume that $p \nmid s$. One has*

$$\mathfrak{S}_{\underline{L}}(s,p^v) = \begin{cases} \epsilon(p^v)^{\mathrm{rk}(\underline{L})}\left(\frac{(2s)^{\mathrm{rk}(\underline{L})}\det(\underline{L})}{p^v}\right)p^{v\frac{\mathrm{rk}(\underline{L})}{2}} & \text{if } p \neq 2, \\ \left(\frac{2^v}{\det(\underline{L})}\right)2^{v\frac{\mathrm{rk}(\underline{L})}{2}} & \text{if } p = 2. \end{cases}$$

Proof. Since there is one-to-one correspondence between the cosets of L/p^vL and the cosets of L_p/p^vL_p, where $L_p = L \otimes \mathbf{Z}_p$, we have in particular that $\mathfrak{S}_{\underline{L}}(s,p^v) = \mathfrak{S}_{\underline{L}_{\mathbf{Z}_p}}(s,p^v)$. First we assume that p is odd. Then every lattice over $\mathbf{Z}_p$ is isomorphic to a direct sum of lattices of rank 1 (See e.g. [Sko14]), so we can find $u_1, u_2, \cdots, u_{\mathrm{rk}(\underline{L})}$ p-adic units such that $\beta(\sum_i x_i e_i) \sim u_1x_1^2 + u_2x_2^2 + \cdots + u_{\mathrm{rk}(\underline{L})}x_{\mathrm{rk}(\underline{L})}^2$, which is integrally equivalent using [Cas08, Lemma 3.4] to

$$\beta(\sum_i x_i e_i) \sim x_1^2 + x_2^2 + \cdots + \prod_{i=1}^{\mathrm{rk}(\underline{L})} u_i x_{\mathrm{rk}(\underline{L})}^2.$$

When $x_1, x_2, \cdots, x_{\mathrm{rk}(\underline{L})}$ run through $\mathbb{Z}_{p^v}$, $\sum_{i=1}^{\mathrm{rk}(\underline{L})} x_i e_i$ runs through L_p/p^vL_p. Thus one can write

$$\mathfrak{S}_{\underline{L}}(s,p^v) = \Big(\sum_{x \bmod p^v} \mathfrak{e}_{p^v}\left(sx^2\right)\Big)^{\mathrm{rk}(\underline{L})-1}\Big(\sum_{x \bmod p^v} \mathfrak{e}_{p^v}\Big(s\prod_{i=1}^{\mathrm{rk}(\underline{L})} u_i x^2\Big)\Big).$$

By using Lemma 2.1.7, which gives a closed formula for the above Gauss sums, we obtain

$$\mathfrak{S}_{\underline{L}}(s,p^v) = \left(\epsilon(p^v)\right)^{\mathrm{rk}(\underline{L})}\left(p^v\right)^{\frac{\mathrm{rk}(\underline{L})}{2}}\left(\frac{\prod_{i=1}^{\mathrm{rk}(\underline{L})} u_i}{p^v}\right)\left(\frac{s}{p^v}\right)^{\mathrm{rk}(\underline{L})}.$$

The identity $\left(\frac{\prod_{i=1}^{\mathrm{rk}(\underline{L})} u_i}{p^v}\right) = \left(\frac{2^{\mathrm{rk}(\underline{L})}\det(\underline{L})}{p^v}\right)$ completes the proof for odd p. Now, we assmue that $p = 2$. Since $p = 2 \in \mathbb{N}_{\underline{L}}$ ($\det(\underline{L})$ is odd and $\mathrm{rk}(\underline{L})$ is even), then by [Cas08, Lemma 4.1], one has

$$\beta(\sum_i x_i e_i) \sim \left(a_1{x_2}^2 + x_2x_1 + b_1{x_1}^2\right) + \cdots + \left(a_{\frac{\mathrm{rk}(\underline{L})}{2}}{x_{\mathrm{rk}(\underline{L})}}^2 + x_{\mathrm{rk}(\underline{L})}x_{\mathrm{rk}(\underline{L})-1} + b_{\frac{\mathrm{rk}(\underline{L})}{2}}{x_{\mathrm{rk}(\underline{L})-1}}^2\right),$$

where for each i, the numbers $a_i = b_i \in \{0,1\}$. Thus

$$\mathfrak{S}_{\underline{L}}(s,2^v) = \Big(\sum_{x,y \bmod 2^v} \mathfrak{e}_{2^v}(sxy)\Big)^{r_1}\Big(\sum_{x,y \bmod 2^v} \mathfrak{e}_{2^v}\left(s(x^2+xy+y^2)\right)\Big)^{r_2},$$

where $r_1, r_2 \in \mathbb{N}$ with $\mathrm{rk}(\underline{L}) = 2r_1 + 2r_2$. By Lemma 2.1.8 we can write

$$\mathfrak{S}_{\underline{L}}(s, 2^v) = \left(-\left(\frac{-2^v}{-1}\right)2^v\right)^{r_1}\left(-\left(\frac{-2^v}{3}\right)2^v\right)^{r_2},$$

which is equal to

$$\mathfrak{S}_{\underline{L}}(s, 2^v) = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}}\left(2^v\right)^{\frac{\mathrm{rk}(\underline{L})}{2}}\left(\frac{-2^v}{\det(\underline{L})}\right) = \left(2^v\right)^{\frac{\mathrm{rk}(\underline{L})}{2}}\left(\frac{2^v}{\det(\underline{L})}\right),$$

where for deducing the second formula form the first we used $(-1)^{\frac{\mathrm{rk}(\underline{L})}{2}}\det(\underline{L}) \equiv 1 \bmod 4$. Now, the proof is complete. □

Remark 2.1.18. Obviously, for a prime $p \in \mathbb{N}_{\underline{L}}$, $v \in \mathbb{N}$, and arbitrary $s \in \mathbb{Z}$, one has

$$\mathfrak{S}_{\underline{L}}(s, p^v) = p^{t\cdot \mathrm{rk}(\underline{L})}\mathfrak{S}_{\underline{L}}(s/p^t, p^{v-t}),$$

where $t := \min(v, \mathrm{ord}_p(s))$. More precisely

1. If $p \neq 2$:

$$\mathfrak{S}_{\underline{L}}(s, p^v) = \left(\epsilon\left(p^{v-t}\right)\right)^{\mathrm{rk}(\underline{L})}\left(\frac{2^{\mathrm{rk}(\underline{L})}\det(\underline{L})}{p^{v-t}}\right)\left(p^{v+t}\right)^{\frac{\mathrm{rk}(\underline{L})}{2}}\left(\frac{s/p^t}{p^{v-t}}\right)^{\mathrm{rk}(\underline{L})}.$$

2. If $p = 2$:

$$\mathfrak{S}_{\underline{L}}(s, 2^v) = \left(2^{v+t}\right)^{\frac{\mathrm{rk}(\underline{L})}{2}}\left(\frac{2^{v+t}}{\det(\underline{L})}\right).$$

Definition 2.1.19. For $t \in \mathbb{N}$ and $c \in \mathbb{Z}$ we set:

$$B_{\underline{L}}(c, t) = t^{1-\mathrm{rk}(\underline{L})}\#\left\{x \in L \,/\, tL \mid c \equiv \beta(x) \bmod t\right\}. \tag{2.17}$$

Lemma 2.1.20. *For integers $a \in \mathbb{N}_{\underline{L}}$ and $c \in \mathbb{Z}$, we have*

$$\mathscr{W}(c, a) = a^{-\mathrm{rk}(\underline{L})}\sum_{\substack{d \bmod a\\(d,a)=1}}\mathfrak{e}_a(-cd)\mathfrak{S}_{\underline{L}}(d, a). \tag{2.18}$$

Proof. We have

$$\begin{aligned}\mathscr{W}(c, a) &= \sum_{t|a}\mu(a/t)B_{\underline{L}}(c, t)\\ &= \sum_{t|a}\mu(a/t)\,t^{-\mathrm{rk}(\underline{L})}\sum_{x\in L/tL}\sum_{d \bmod t}\mathfrak{e}_t\left(d(\beta(x) - c)\right)\end{aligned}$$

$$=\sum_{t|a}\mu(a/t)t^{-\mathrm{rk}(\underline{L})}\sum_{s|t}\sum_{\substack{d \bmod \frac{t}{s}\\(d,\frac{t}{s})=1}}\mathfrak{e}_{\frac{t}{s}}(-cd)\sum_{x\in L/tL}\mathfrak{e}_t\big(sd\beta(x)\big).$$

Here, for deducing the third formula from the second identity, we replaced the sum over d mod t by the sum $\sum_{s|t}\sum_{d \bmod t,(d,t)=s}$ and then replace d by ds. According to Remark 2.1.18, $\sum_{x\in L/tL}\mathfrak{e}_t\big(sd\beta(x)\big)=s^{\mathrm{rk}(\underline{L})}\mathfrak{S}_{\underline{L}}(d,t/s)$. This means that

$$\mathscr{W}(c,a)=\sum_{t|a}\mu(a/t)\sum_{g|t}g^{-\mathrm{rk}(\underline{L})}\sum_{\substack{d \bmod g\\(d,g)=1}}\mathfrak{e}_g(-cd)\mathfrak{S}_{\underline{L}}(d,g).$$

Thus using

$$\sum_{t|a}\mu(\frac{a}{t})\sum_{g|t}f(g)=f(a)$$

for any arithmetic function f, one obtains

$$\mathscr{W}(c,a)=a^{-\mathrm{rk}(\underline{L})}\sum_{\substack{d \bmod a\\(d,a)=1}}\mathfrak{e}_a(-cd)\mathfrak{S}_{\underline{L}}(d,a)$$

as stated in the proposition. □

Lemma 2.1.21. *For all $c\in\mathbb{Z}$ and $a_1,a_2\in\mathbb{N}_{\underline{L}}$ such that $(a_1,a_2)=1$ one has*

$$\mathscr{W}(c,a_1a_2)=\mathscr{W}(c,a_1)\mathscr{W}(c,a_2). \tag{2.19}$$

Proof. By Lemma 2.1.20 one has

$$\mathscr{W}(c,a_1a_2)=(a_1a_2)^{-\mathrm{rk}(\underline{L})}\sum_{\substack{d \bmod a_1a_2\\(d,a_1a_2)=1}}\mathfrak{e}_{a_1a_2}(-cd)\mathfrak{S}_{\underline{L}}(d,a_1a_2).$$

By Chinese Remainder Theorem $\mathbb{Z}^\times_{a_1a_2}\cong\mathbb{Z}^\times_{a_1}\times\mathbb{Z}^\times_{a_2}$. Thus, when b_1 runs through a complete system of representatives for the primitive residue classes modulo a_1, and b_2 runs through a complete system of representatives for the primitive residue classes modulo a_2, then $d=b_2a_1+b_1a_2$ runs through a complete system of representatives for the primitive residue classes modulo a_1a_2. It follows

$$\mathfrak{S}_{\underline{L}}(d,a_1a_2)=\mathfrak{S}_{\underline{L}}(da_1,a_2)\mathfrak{S}_{\underline{L}}(da_2,a_1)=\mathfrak{S}_{\underline{L}}({a_1}^2b_2,a_2)\mathfrak{S}_{\underline{L}}({a_2}^2b_1,a_1)$$

$$=\mathfrak{S}_{\underline{L}}(b_2,a_2)\mathfrak{S}_{\underline{L}}(b_1,a_1),$$

where the first identity follows from Lemma 2.1.16 and the last one from Lemma 2.1.17. This gives

$$\begin{aligned}\mathscr{W}(c,a_1a_2) &= (a_1a_2)^{-\mathrm{rk}(\underline{L})} \sum_{\substack{d \bmod a_1a_2\\(d,a_1a_2)=1}} \mathfrak{e}_{a_1a_2}(-cd)\mathfrak{S}_{\underline{L}}(d,a_1a_2)\\ &= a_1{}^{-\mathrm{rk}(\underline{L})} \sum_{\substack{b_1 \bmod a_1\\(b_1,a_1)=1}} \mathfrak{e}_{a_1}(-cb_1)\mathfrak{S}_{\underline{L}}(b_1,a_1)\; a_2{}^{-\mathrm{rk}(\underline{L})} \sum_{\substack{b_2 \bmod a_2\\(b_2,a_2)=1}} \mathfrak{e}_{a_2}(-cb_2)\mathfrak{S}_{\underline{L}}(b_2,a_2)\\ &= \mathscr{W}(c,a_1)\mathscr{W}(c,a_2)\end{aligned}$$

as stated in the lemma. □

Lemma 2.1.22. *Let $a \in \mathbb{N}_{\underline{L}}$, and $c \in \mathbb{Z}$.*

1. *We have $\mathscr{W}(c,a) = \prod_{p^v\|a}\mathscr{W}(c,p^v)$. The p-components $\mathscr{W}(c,p^v)$ are given as follows:*

$$\mathscr{W}\left(c,p^v\right) = p^{\lfloor\frac{-v\cdot\mathrm{rk}(\underline{L})}{2}\rfloor}\chi_{\underline{L}}(p^v)\times\begin{cases} G_{p^v}(-c) & \text{if } v\cdot\mathrm{rk}(\underline{L}) \text{ is even},\\ p^v\left(\frac{c/p^{v-1}}{p^v}\right)\delta(p^{v-1}\|c) & \text{if } v\cdot\mathrm{rk}(\underline{L}) \text{ is odd}.\end{cases} \tag{2.20}$$

2. *Let $x,b \in \mathbb{N}$ such that $x^2 \mid \gcd(c,a)$ and $\gcd(b,a)=1$, then*

$$x^{2-\mathrm{rk}(\underline{L})}\cdot\mathscr{W}(bc/x^2,a/x^2) = \left(\frac{b}{a}\right)^{\mathrm{rk}(\underline{L})}\mathscr{W}(c,a). \tag{2.21}$$

Proof. According to Lemma 2.1.21 one has $\mathscr{W}(c,a) = \prod_{p^v\|a}\mathscr{W}(c,p^v)$. The components $\mathscr{W}(c,p^v)$ given by Lemma 2.1.20 as

$$\mathscr{W}\left(c,p^v\right) = (p^v)^{-\mathrm{rk}(\underline{L})}\sum_{\substack{d \bmod p^v\\(d,p^v)=1}}\mathfrak{e}_{p^v}(-cd)\mathfrak{S}_{\underline{L}}(d,p^v).$$

If p is odd, then, by Lemma 2.1.17 which gives a closed formula for $\mathfrak{S}_{\underline{L}}(d,p^v)$, one has

$$\mathscr{W}\left(c,p^v\right) = \epsilon(p^v)^{\mathrm{rk}(\underline{L})}\left(\frac{2^{\mathrm{rk}(\underline{L})}\det(\underline{L})}{p^v}\right)p^{-\frac{v\cdot\mathrm{rk}(\underline{L})}{2}}\sum_{\substack{d \bmod p^v\\(d,p^v)=1}}\mathfrak{e}_{p^v}(-cd)\left(\frac{d}{p^v}\right)^{\mathrm{rk}(\underline{L})}.$$

Now, Equation (2.20) follows from this by Lemma 2.1.14 if $\mathrm{rk}(\underline{L})$ is odd, and by Definition 2.1.11 if $\mathrm{rk}(\underline{L})$ is even.

Assuming that $p = 2$. One has $\mathrm{lev}(\underline{L})$ is odd, and $\mathrm{rk}(\underline{L})$ is even. By using Lemma 2.1.17, we obtain

$$\mathscr{W}\left(c, 2^v\right) = 2^{-\frac{v \cdot \mathrm{rk}(\underline{L})}{2}} \left(\frac{2^v}{\det(\underline{L})}\right) \sum_{d \bmod 2^v} \mathfrak{e}_{2^v}(-cd).$$

The identity $\chi_{\underline{L}}(2^v) = \left(\frac{2^v}{\det(\underline{L})}\right)$ and Definition 2.1.11 complete the proof of the Equation (2.20). The second statement is just a corollary of the first one. □

Lemma 2.1.23. *The function $\mathscr{W}_{\mathrm{I}}(c,a)$ for fixed c, is multiplicative in a, i.e.,*

$$\mathscr{W}_{\mathrm{I}}(c, a_1 a_2) = \mathscr{W}_{\mathrm{I}}(c, a_1) \mathscr{W}_{\mathrm{I}}(c, a_2) \tag{2.22}$$

for all $c \in \mathbb{Z}$ and odd positive integers a_1, a_2 such that $(a_1, a_2) = 1$.

Proof. By Equation (2.2) one has

$$\mathscr{W}_{\mathrm{I}}(c, a_1 a_2) = \sum_{\substack{b \mid a_1 a_2 \\ a_1 a_2 / b = \text{perfect square}}} b^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil - 1} \mathscr{W}(c, b).$$

Denote the set of divisors of $a \in \mathbb{N}$ by $\mathrm{Div}(a)$. Then since a_1 and a_2 are coprime there is a bijection $\mathrm{Div}(a_1) \times \mathrm{Div}(a_2) \to \mathrm{Div}(a_1 a_2)$ given by $(b_1, b_2) \mapsto b_1 b_2$. It follows that

$$\begin{aligned}
\mathscr{W}_{\mathrm{I}}(c, a_1 a_2) &= \sum_{\substack{b_1 \mid a_1 \\ a_1 / b_1 = \text{perfect square}}} \sum_{\substack{b_2 \mid a_2 \\ a_2 / b_2 = \text{perfect square}}} (b_1 b_2)^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil - 1} \mathscr{W}(c, b_1 b_2) \\
&= \sum_{\substack{b_1 \mid a_1 \\ a_1 / b_1 = \text{perfect square}}} b_1^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil - 1} \mathscr{W}(c, b_1) \sum_{\substack{b_2 \mid a_2 \\ a_2 / b_2 = \text{perfect square}}} b_2^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil - 1} \mathscr{W}(c, b_2) \\
&= \mathscr{W}_{\mathrm{I}}(c, a_1) \mathscr{W}_{\mathrm{I}}(c, a_2)
\end{aligned}$$

as stated in the lemma. □

Now, we are ready to prove Proposition 2.1.3 and Proposition 2.1.4.

Proof of Proposition 2.1.3. By Lemma 2.1.23 we have

$$\mathscr{W}_{\mathrm{I}}(c,a) = \prod_{p^v \| a} \mathscr{W}_{\mathrm{I}}(c,p^v),$$

where

$$\mathscr{W}_{\mathrm{I}}(c,p^v) = \sum_{\substack{\alpha | p^v \\ p^v/\alpha = \text{perfect square}}} \alpha^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil - 1} \mathscr{W}(c,\alpha).$$

By using Lemma 2.1.22, which gives an explicit formula for $\mathscr{W}(c,\alpha)$, one has

$$\mathscr{W}_{\mathrm{I}}(c,p^v) = \sum_{\substack{\alpha | p^v \\ p^v/\alpha = \text{perfect square}}} \chi_{\underline{L}}(\alpha) \times \begin{cases} \alpha^{-\frac{1}{2}} G_\alpha(-c) & \text{if } v \text{ is even,} \\ (\alpha/p)^{\frac{1}{2}} \left(\frac{pc/\alpha}{\alpha}\right) \boldsymbol{\delta}(\alpha \| pc) & \text{if } v \text{ is odd.} \end{cases}$$

First, we assume that v is even. It is well-known that

$$G_\alpha(-c) = \frac{\mu\left(\frac{\alpha}{(\alpha,c)}\right)}{\varphi\left(\frac{\alpha}{(\alpha,c)}\right)} \varphi(\alpha),$$

where μ is the Möbius function and φ is Euler's totient function. Inserting this into the last formula for $\mathscr{W}_{\mathrm{I}}(c,p^v)$ gives

$$\begin{aligned}
\mathscr{W}_{\mathrm{I}}(c,p^v) &= \sum_{\substack{\alpha | p^v \\ \alpha = \text{perfect square}}} \alpha^{-\frac{1}{2}} G_\alpha(-c) \\
&= \boldsymbol{\delta}\left(\mathrm{ord}_p(c) \equiv 0 \bmod 2\right) \sum_{\substack{\alpha | p^{\min(v,\mathrm{ord}_p(c))} \\ \alpha = \text{perfect square}}} \alpha^{-\frac{1}{2}} \varphi(\alpha) \\
&= \boldsymbol{\delta}\left(\mathrm{ord}_p(c) \equiv 0 \bmod 2\right) \left(1 + \varphi(p) \sum_{r=1}^{\min(v,\mathrm{ord}_p(c))/2} p^{r-1}\right) \\
&= \boldsymbol{\delta}\left(\mathrm{ord}_p(c) \equiv 0 \bmod 2\right) p^{\min(v,\mathrm{ord}_p(c))/2} \\
&= \left(\frac{c/f^2}{p^v/f^2}\right) f \boldsymbol{\delta}\left(\gcd(c,p^v) = f^2, f \in \mathbb{N}\right).
\end{aligned}$$

Now, we assume that v is odd. One has

$$\begin{aligned}
\mathscr{W}_{\mathrm{I}}(c,p^v) &= \chi_{\underline{L}}(p^v) \sum_{\substack{\alpha | p^v \\ \alpha \neq \text{perfect square}}} (\alpha/p)^{\frac{1}{2}} \left(\frac{pc/\alpha}{\alpha}\right) \boldsymbol{\delta}(\alpha \| pc) \\
&= \chi_{\underline{L}}(p^v) \sum_{\substack{x | p^{v-1}, c \\ x = \text{perfect square}}} \left(\frac{c/x}{p^v}\right) x^{\frac{1}{2}}.
\end{aligned}$$

The inner sum in the last equation is empty or contains exactly one term

$$\sum_{\substack{x|p^{v-1},c\\ x=\text{perfect square}}} \left(\frac{c/x}{p^v}\right) x^{\frac{1}{2}} = \left(\frac{c/p^{\mathrm{ord}_p(c)}}{p^v}\right) p^{\frac{\mathrm{ord}_p(c)}{2}} \boldsymbol{\delta}\left(\mathrm{ord}_p(c) < v \text{ and } \mathrm{ord}_p(c) \text{ is even}\right)$$

$$= \left(\frac{c/f^2}{p^v/f^2}\right) f \boldsymbol{\delta}\left(\gcd(c,p^v) = f^2, f \in \mathbb{N}\right).$$

Now, the proof is complete. □

Proof of Proposition 2.1.4. By Lemma 2.1.22, one has

$$\begin{aligned}
\mathscr{W}_{\mathrm{II}}(c,a) &= \sum_{b|a} \chi_{\underline{L}}(b) b^{-\frac{\mathrm{rk}(\underline{L})}{2}} \mathscr{W}(c, \frac{a}{b}) \\
&= \sum_{b|a} \chi_{\underline{L}}(b) b^{-\frac{\mathrm{rk}(\underline{L})}{2}} (a/b)^{-\frac{\mathrm{rk}(\underline{L})}{2}} \chi_{\underline{L}}(a/b) G_{\frac{a}{b}}(-c) \\
&= \chi_{\underline{L}}(a) a^{-\frac{\mathrm{rk}(\underline{L})}{2}} \sum_{b|a} G_{\frac{a}{b}}(-c) = \chi_{\underline{L}}(a) a^{-\frac{\mathrm{rk}(\underline{L})}{2}+1} \boldsymbol{\delta}(a \mid c).
\end{aligned}$$

Here in the last step we used the well-known property of the Ramanujan sum $\sum_{b|a} G_{\frac{a}{b}}(-c) = \sum_{d \pmod a} \mathfrak{e}_a(-dc) = a\boldsymbol{\delta}(a \mid c)$. □

2.2 Jacobi Groups of Integral Lattice

In this section we define the Jacobi group of an integral lattice ($\mathbb{Z}$-Lattice), and study its properties. To define the Jacobi group, we need first to define the Heisenberg group. Indeed, there are two constructions called the Heisenberg group. If K is a field of characteristic not equal to 2 and V is a $2n$-dimensional vector space over K with symplectic form $\beta : V \times V \to k$, one of these groups is $V \times K$ with the operation $(v,a)\cdot(w,b) = (v+w, a+b+\frac{1}{2}\beta(v,w))$ and is usually called the polarized Heisenberg group. The other group is the subgroup of $\mathrm{GL}_{n+2}(K)$ consisting of matrices with 1s along the diagonal and 0s elsewhere, except for the top row and rightmost column. This construction has the advantage of working over K of arbitrary characteristic, but uses coordinates. It is usually called the classical Heisenberg group. Here we define a Jacobi group using a Heisenberg group in the classical sense, but instead of using coordinates we will use a symmetric bilinear form.

As we mentioned in the beginning of this chapter, we shall use the notation $\underline{L} = (L, \beta)$ to denote a positive definite even lattice over $\mathbb{Z}$. Also, we shall use lev($\underline{L}$), rk($\underline{L}$), and det($\underline{L}$) to denote the level, the rank, and the determinant of the lattice $\underline{L}$, respectively.

Definition 2.2.1 (Heisenberg Group). The Heisenberg group associated with $\underline{L} = (L, \beta)$ is defined by

$$H_{\underline{L}}(\mathbb{Q}) := \left\{(x, y, \xi) : x, y \in L \otimes \mathbb{Q}, \xi \in S^1\right\}, \tag{2.23}$$

with the composition law

$$(x_1, y_1, \xi_1)(x_2, y_2, \xi_2) = \left(x_1 + x_2, y_1 + y_2, \xi_1\xi_2 \mathfrak{e}\left(\beta(x_1, y_2)\right)\right). \tag{2.24}$$

We consider the subgroup

$$H_{\underline{L}}(\mathbb{Z}) := \left\{(x, y, 1) : x, y \in L\right\}. \tag{2.25}$$

Note that $H_{\underline{L}}(\mathbb{Q})$ is defined in [Boy11] using the decomposition law

$$(x_1, y_1, \xi_1)(x_2, y_2, \xi_2) = \left(x_1 + x_2, y_1 + y_2, \xi_1\xi_2 \mathfrak{e}\left(\tfrac{1}{2}(\beta(x_1, y_2) - \beta(x_2, y_1))\right)\right).$$

In particular, it is easy to switch from one version to the other. The propositions in the rest of this section could be deduced from [Boy11, Section 3.4]. However, for the convenience of the reader, we give independent proofs.

Proposition 2.2.2. *The composition law 2.24 defines a group structure on the set $H_{\underline{L}}(\mathbb{Q})$.*

Proof. The neutral element is $(0, 0, 1)$. For an element $(x, y, \xi) \in H_{\underline{L}}(\mathbb{Q})$, the inverse element equals $(-x, -y, \xi^{-1}\mathfrak{e}\left(\beta(x, y)\right))$.
For $(x_1, y_1, \xi_1), (x_2, y_2, \xi_2), (x_3, y_3, \xi_3) \in H_{\underline{L}}(\mathbb{Q})$, the associativity follows from

$$\begin{aligned}
&((x_1, y_1, \xi_1)(x_2, y_2, \xi_2))(x_3, y_3, \xi_3) \\
&= \left(x_1 + x_2, y_1 + y_2, \xi_1\xi_2 \mathfrak{e}\left(\beta(x_1, y_2)\right)\right)(x_3, y_3, \xi_3) \\
&= \left(x_1 + x_2 + x_3, y_1 + y_2 + y_3, \xi_1\xi_2\xi_3 \mathfrak{e}\left(\beta(x_1, y_2)\right) \mathfrak{e}\left(\beta(x_1 + x_2, y_3)\right)\right) \\
&= \left(x_1 + x_2 + x_3, y_1 + y_2 + y_3, \xi_1\xi_2\xi_3 \mathfrak{e}\left(\beta(x_2, y_3)\right) \mathfrak{e}\left(\beta(x_1, y_2 + y_3)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&=(x_1,y_1,\xi_1)\left(x_2+x_3,y_2+y_3,\xi_2\xi_3\mathfrak{e}\left(\beta(x_2,y_3)\right)\right)\\
&=(x_1,y_1,\xi_1)((x_2,y_2,\xi_2)(x_3,y_3,\xi_3)).
\end{aligned}
$$

Now, the proof is complete. □

Proposition 2.2.3. *The group* $\mathrm{SL}_2(\mathbb{Q})$ *acts on the group* $H_{\underline{L}}(\mathbb{Q})$ *from right via*

$$
\left((x,y,\xi),A\right)\to(x,y,\xi)^A:=\left((x,y)A,\xi\mathfrak{e}\left(\tfrac{1}{2}\beta\big((x,y)A\big)-\tfrac{1}{2}\beta(x,y)\right)\right). \tag{2.26}
$$

Here, $(x,y)A$ *stands for the formal multiplication of the row vector* (x,y) *and* A. *i.e., if* $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$*, then* $(x,y)A:=(ax+cy,bx+dy)$.

Proof. We need to check the group axioms, and that, for given $A\in\mathrm{SL}_2(\mathbb{Q})$, the map $((x,y,\xi),A)\to(x,y,\xi)^A$ is a group homomorphism of $H_{\underline{L}}(\mathbb{Q})$. For fixed $h_i=(x,y,\xi)\in H_{\underline{L}}(\mathbb{Q})$, and $A_1=\left(\begin{smallmatrix}a_1&b_1\\c_1&d_1\end{smallmatrix}\right),A_2=\left(\begin{smallmatrix}a_2&b_2\\c_2&d_2\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Q})$ we clearly have $(x,y,\xi)^1=(x,y,\xi)$, and

$$
\begin{aligned}
\left((x,y,\xi)^{A_1}\right)^{A_2}&=\left((x,y)A_1,\xi\mathfrak{e}\left(\tfrac{1}{2}\beta((x,y)A_1)-\tfrac{1}{2}\beta(x,y)\right)\right)^{A_2}\\
&=\left((x,y)A_1A_2,\xi\mathfrak{e}\left(\tfrac{1}{2}\beta((x,y)A_1A_2)-\tfrac{1}{2}\beta((x,y))\right)\right)\\
&=(x,y,\xi)^{A_1A_2}.
\end{aligned}
$$

Similarly, for $h_1,h_2\in H_{\underline{L}}(\mathbb{Q})$, one has $h_1^{A_1}h_2^{A_1}=(h_1h_2)^{A_1}$, which can be verified easily. □

Definition 2.2.4 (Jacobi Group). We define the Jacobi group $J_{\underline{L}}(\mathbb{Q})$ as a semi-direct product of $\mathrm{SL}_2(\mathbb{Q})$ and $H_{\underline{L}}(\mathbb{Q})$

$$
J_{\underline{L}}(\mathbb{Q}):=\mathrm{SL}_2(\mathbb{Q})\ltimes H_{\underline{L}}(\mathbb{Q}).
$$

i.e.,

$$
J_{\underline{L}}(\mathbb{Q})=\left\{(A,h):A\in\mathrm{SL}_2(\mathbb{Q}),h\in H_{\underline{L}}(\mathbb{Q})\right\},
$$

with the group operation

$$
(A,h)(A',h')=(AA',h^{A'}h'). \tag{2.27}
$$

Also, we set

$$
J_{\underline{L}}(\mathbb{Z}):=\mathrm{SL}_2(\mathbb{Z})\ltimes H_{\underline{L}}(\mathbb{Z}).
$$

Remarks 2.2.5. There exist canonical monomorphisms $H_{\underline{L}}(\mathbb{Q}) \to J_{\underline{L}}(\mathbb{Q})$ and $\mathrm{SL}_2(\mathbb{Q}) \to J_{\underline{L}}(\mathbb{Q})$, given by

$$h \mapsto (\mathbf{1}_{\mathrm{SL}_2(\mathbb{Q})}, h), \qquad h \in H_{\underline{L}}(\mathbb{Q})$$
$$A \mapsto (A, \mathbf{1}_{H_{\underline{L}}(\mathbb{Q})}), \qquad A \in \mathrm{SL}_2(\mathbb{Q})$$

where $\mathbf{1}_{H_{\underline{L}}(\mathbb{Q})} = (0,0,1)$ (resp. $\mathbf{1}_{\mathrm{SL}_2(\mathbb{Q})} = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$) is the identity element of $H_{\underline{L}}(\mathbb{Q})$ (resp. $\mathrm{SL}_2(\mathbb{Q})$). These monomorphisms are so natural that we will treat $H_{\underline{L}}(\mathbb{Q})$ and $\mathrm{SL}_2(\mathbb{Q})$ as subgroups of $J_{\underline{L}}(\mathbb{Q})$ under these inclusions.

Using this identification, we have, in particular, for $h \in H_{\underline{L}}(\mathbb{Q})$ and $A \in \mathrm{SL}_2(\mathbb{Q})$ that

$$A^{-1}hA = h^A. \tag{2.28}$$

Definition 2.2.6. For $A \in \mathrm{SL}_2(\mathbb{Q})$ and $\tau \in \mathfrak{H}$ we set

$$\mathfrak{J}(A,\tau) = c\tau + d \qquad (A = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)).$$

It is easy to see that the map $\mathfrak{J} : \mathrm{SL}_2(\mathbb{Q}) \times \mathfrak{H} \to \mathfrak{H}$ satisfies the following multiplicative property:

$$\mathfrak{J}(A, B\tau)\mathfrak{J}(B,\tau) = \mathfrak{J}(AB,\tau).$$

Proposition 2.2.7. *The group $J_{\underline{L}}(\mathbb{Q})$ acts from the left on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ as follows:*

$$\big(\big(A,(\lambda,\mu,\xi)\big),(\tau,z)\big) \mapsto \big(A,(\lambda,\mu,\xi)\big)(\tau,z) := \big(A\tau, \tfrac{z+\lambda\tau+\mu}{\mathfrak{J}(A,\tau)}\big). \tag{2.29}$$

Proof. It is obvious that $\big(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),(0,0,1)\big)(\tau,z) = (\tau,z)$. Let $\big(A_1,(\lambda_1,\mu_1,\xi_1)\big)$, $\big(A_2,(\lambda_2,\mu_2,\xi_2)\big) \in J_{\underline{L}}(\mathbb{Q})$. One has

$$\begin{aligned}
&\big(A_1,(\lambda_1,\mu_1,\xi_1)\big)\big(\big(A_2,(\lambda_2,\mu_2,\xi_2)\big)(\tau,z)\big)\\
=&\big(A_1,(\lambda_1,\mu_1,\xi_1)\big)\big(A_2\tau, \big(z+\lambda_2\tau+\mu_2\big)\mathfrak{J}(A_2,\tau)^{-1}\big)\\
=&\Big((A_1A_2)\tau, \frac{z+\lambda_2\tau+\mu_2+\mathfrak{J}(A_2,\tau)\lambda_1A_2\tau+\mathfrak{J}(A_2,\tau)\mu_1}{\mathfrak{J}(A_2,\tau)\mathfrak{J}(A_1,A_2\tau)}\Big)\\
=&\Big((A_1A_2)\tau, \Big(z+\big((\lambda_1,\mu_1)A_2(\lambda_2,\mu_2)\big)(\tau,1)^t\Big)\mathfrak{J}(A_1A_2,\tau)^{-1}\Big)\\
=&\big(A_1A_2,(\lambda_1,\mu_1,\xi_1)^{A_2}(\lambda_2,\mu_2,\xi_2)\big)(\tau,z)\\
=&\big(\big(A_1,(\lambda_1,\mu_1,\xi_1)\big)\big(A_2,(\lambda_2,\mu_2,\xi_2)\big)\big)(\tau,z)
\end{aligned}$$

which completes the proof. □

Proposition 2.2.8 (Jacobi slash operators). *Let k be a positive integer. The group $J_{\underline{L}}(\mathbb{Q})$ acts from the right on* $\mathrm{Hol}(\mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C}))$ *via*

$$\big(\phi,(A,h)\big)\mapsto \phi\big|_{k,\underline{L}}(A,h):=(\phi\big|_{k,\underline{L}}A)\big|_{k,\underline{L}}h, \tag{2.30}$$

where

(i) For all $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Q})$

$$\big(\phi\big|_{k,\underline{L}}A\big)(\tau,z):=\phi\big(A\tau,\tfrac{z}{\mathfrak{J}(A,\tau)}\big)\mathfrak{J}(A,\tau)^{-k}\,\mathfrak{e}\left(\tfrac{-c\beta(z)}{\mathfrak{J}(A,\tau)}\right). \tag{2.31}$$

(ii) For all $(x,y,\xi)\in H_{\underline{L}}(\mathbb{Q})$

$$\big(\phi\big|_{k,\underline{L}}(x,y,\xi)\big)(\tau,z):=\xi\phi(\tau,z+x\tau+y)\,\mathfrak{e}\left(\tau\beta(x)+\beta(x,z)\right). \tag{2.32}$$

Proof. It is obvious that $\phi\big|_{k,\underline{L}}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)=\phi$. Let $A=\left(\begin{smallmatrix}*&*\\c_A&*\end{smallmatrix}\right), B=\left(\begin{smallmatrix}*&*\\c_B&*\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Q})$. According to Equation (2.31) one sees that:

$$\begin{aligned}
&\Big((\phi\big|_{k,\underline{L}}A)\big|_{k,\underline{L}}B\Big)(\tau,z)=\Big(\big(\phi\big|_{k,\underline{L}}A\big)\big(B\tau,\tfrac{z}{\mathfrak{J}(B,\tau)}\big)\Big)\mathfrak{J}(B,\tau)^{-k}\,\mathfrak{e}\left(\tfrac{-c_B\beta(z)}{\mathfrak{J}(B,\tau)}\right)\\
&=\phi\left(AB\tau,\tfrac{z}{\mathfrak{J}(A,B\tau)\mathfrak{J}(B,\tau)}\right)\mathfrak{J}(A,B\tau)^{-k}\,\mathfrak{e}\left(\tfrac{-c_A\beta(z/\mathfrak{J}(B,\tau))}{\mathfrak{J}(A,B\tau)}\right)\mathfrak{J}(B,\tau)^{-k}\,\mathfrak{e}\left(\tfrac{-c_B\beta(z)}{\mathfrak{J}(B,\tau)}\right)\\
&=\phi\left(AB\tau,\tfrac{z}{\mathfrak{J}(AB,\tau)}\right)\mathfrak{J}(AB,\tau)^{-k}\,\mathfrak{e}\left(-\big(\tfrac{c_A}{\mathfrak{J}(A,B\tau)\mathfrak{J}(B,\tau)^2}+\tfrac{c_B}{\mathfrak{J}(B,\tau)}\big)\beta(z)\right).
\end{aligned}$$

Using that $\frac{c_A}{\mathfrak{J}(A,B\tau)\mathfrak{J}(B,\tau)^2}+\frac{c_B}{\mathfrak{J}(B,\tau)}=\frac{c_{AB}}{\mathfrak{J}(AB,\tau)}$, where $AB=\left(\begin{smallmatrix}*&*\\c_{AB}&*\end{smallmatrix}\right)$, we obtain

$$\big(\phi\big|_{k,\underline{L}}A\big)\big|_{k,\underline{L}}B=\phi\big|_{k,\underline{L}}AB. \tag{2.33}$$

From Equation (2.32) it is clear that $\phi\big|_{k,\underline{L}}(0,0,1)=\phi$. For $h_1=(x_1,y_1,\xi_1)$ and $h_2=(x_2,y_2,\xi_2)\in H_{\underline{L}}(\mathbb{Q})$ one has

$$\begin{aligned}
\big((\phi\big|_{k,\underline{L}}h_1)\big|_{k,\underline{L}}h_2\big)(\tau,z)&=\xi_2\big((\phi\big|_{k,\underline{L}}h_1)(\tau,z+x_2\tau+y_2)\big)\mathfrak{e}\left(\tau\beta(x_2)+\beta(x_2,z)\right)\\
&=\xi_1\xi_2\phi(\tau,z+x_2\tau+y_2+x_1\tau+y_1)\\
&\quad\times\mathfrak{e}\left(\tau\beta(x_1)+\beta(x_1,z+x_2\tau+y_2)+\tau\beta(x_2)+\beta(x_2,z)\right)\\
&=\xi_1\xi_2\,\mathfrak{e}\left(\beta(x_1,y_2)\right)\phi(\tau,z+(x_1+x_2)\tau+(y_1+y_2))\\
&\quad\times\mathfrak{e}\left(\tau\beta(x_1+x_2)+\beta(x_1+x_2,z)\right)\\
&=\big(\phi\big|_{k,\underline{L}}h_1h_2\big)(\tau,z).
\end{aligned}$$

Namely, we proved that

$$\phi\big|_{k,\underline{L}}h_1\big|_{k,\underline{L}}h_2=\phi\big|_{k,\underline{L}}h_1h_2. \tag{2.34}$$

It is left to show the group axioms. Using Remarks 2.2.5, one has

$$\big(\phi|_{k,\underline{L}}h^{A^{-1}}\big)|_{k,\underline{L}}A = \big(\phi|_{k,\underline{L}}A\big)|_{k,\underline{L}}h. \tag{2.35}$$

Thus, for all $(A_1,h_1),(A_2,h_2) \in J_{\underline{L}}(\mathbb{Q})$, one has

$$\begin{aligned}
\phi|_{k,\underline{L}}\big((A_1,h_1)(A_2,h_2)\big) &= \phi|_{k,\underline{L}}\big(A_1A_2,h_1^{A_2}h_2\big) = \phi|_{k,\underline{L}}A_1A_2|_{k,\underline{L}}h_1^{A_2}h_2 \\
&= \phi|_{k,\underline{L}}A_1|_{k,\underline{L}}A_2|_{k,\underline{L}}h_1^{A_2}|_{k,\underline{L}}h_2 = \phi|_{k,\underline{L}}A_1|_{k,\underline{L}}h_1|_{k,\underline{L}}A_2|_{k,\underline{L}}h_2 \\
&= \phi|_{k,\underline{L}}(A_1,h_1)|_{k,\underline{L}}(A_2,h_2),
\end{aligned}$$

where the first identity follows from Definition 2.2.4, the third from Equation (2.33) and Equation (2.34), and the fourth from Equation (2.35). □

Remarks 2.2.9. We have the following remarks:

1. We can write out the action of the Jacobi slash operator on Jacobi forms more explicitly as

$$\big(\phi|_{k,\underline{L}}(A,h)\big)(\tau,z) = \xi\mathfrak{J}(A,\tau)^{-k}\mathfrak{e}\left(\tfrac{-c\beta(z+x\tau+y)}{\mathfrak{J}(A,\tau)} + \tau\beta(x)+\beta(x,z)\right)\phi((A,h)(\tau,z))$$

   for all $(A,h) \in J_{\underline{L}}(\mathbb{Q})$ with $h=(x,y,\xi)$ and $A = \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right)$.

2. We can extend the definition of the slash operator to the case of half integral weight $k \in \frac{1}{2}\mathbb{Z}$ by replacing $\mathrm{SL}_2(\mathbb{Q})$ with its metaplectic cover $\widetilde{\mathrm{SL}_2(\mathbb{Q})}$. The group $\widetilde{\mathrm{SL}_2(\mathbb{Q})}$ acts on the group $H_{\underline{L}}(\mathbb{Q})$ from right via

$$\big((x,y,\xi),(A,w)\big) \mapsto (x,y,\xi)^{(A,w)},$$

   where

$$(x,y,\xi)^{(A,w)} := \Big((x,y)A, \xi\mathfrak{e}\left(\tfrac{1}{2}\beta((x,y)A) - \tfrac{1}{2}\beta(x,y)\right)\Big)$$

   If we put $J_{\underline{L}}(\mathbb{Q}) := \widetilde{\mathrm{SL}_2(\mathbb{Q})} \ltimes H_{\underline{L}}(\mathbb{Q})$, then the group $J_{\underline{L}}(\mathbb{Q})$ acts from the right on $\mathrm{Hol}(\mathfrak{H}\times(L\otimes_{\mathbb{Z}}\mathbb{C}))$ via

$$\big(\phi,((A,w),h)\big) \mapsto \phi|_{k,\underline{L}}((A,w),h) = (\phi|_{k,\underline{L}}(A,w)|_{k,\underline{L}}h,$$

   where for all $\widetilde{A} = (A,w) \in \widetilde{\mathrm{SL}_2(\mathbb{Q})}$

$$\big(\phi|_{k,\underline{L}}\widetilde{A}\big)(\tau,z) := \phi\big(A\tau, \tfrac{z}{w(\tau)^2}\big)w(\tau)^{-2k}\mathfrak{e}\left(\tfrac{-c\beta(z)}{w(\tau)^2}\right). \tag{2.36}$$

2.3 Theta Series

Let $\underline{L}$ be an even positive definite lattice over $\mathbb{Z}$. Let $D_{\underline{L}} = (L^\#/L, \beta)$ be the associated discriminant form. Let $\{\delta_x\}_{x\in L^\#/L}$ denote the stranded basis of $\mathbb{C}[L^\#/L]$, so that $\delta_x\delta_y = \delta_{x+y}$. By $[\![\cdot|\cdot]\!]$ we denote the associated scalar product on $\mathbb{C}[L^\#/L]$ that satisfies

$$[\![\sum_{x\in L^\#/L} f_x\delta_x | \sum_{x\in L^\#/L} g_x\delta_x]\!] = \sum_{x\in L^\#/L} f_x\overline{g_x} \tag{2.37}$$

We denote by $\rho_{\underline{L}}$ the Weil representation $\rho_{\underline{L}} : \widetilde{\mathrm{SL}_2(\mathbb{Z})} \to GL(\mathbb{C}[L^\#/L])$ associated with $\underline{L}$. The action of the generators of the group $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ on the standard basis of $\mathbb{C}[L^\#/L]$ is given as follows:

$$\rho_{\underline{L}}(\widetilde{T})\delta_x = \mathfrak{e}\left(\beta(x)\right)\delta_x \tag{2.38}$$

$$\rho_{\underline{L}}(\widetilde{S})\delta_x = \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}} \sum_{y\in L^\#/L} \mathfrak{e}\left(-\beta(x,y)\right)\delta_y. \tag{2.39}$$

It is known that $\rho_{\underline{L}}$ factors through $\widetilde{\mathrm{SL}_2}(\mathbb{Z}_{\mathrm{lev}(\underline{L})})$ (for details see [Str11]).

Proposition 2.3.1. *For* $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(\mathrm{lev}(\underline{L}))$ *and* $\gamma \in L^\#/L$ *one has*

$$\rho_{\underline{L}}(A, \sqrt{c\tau+d})\delta_\gamma = \begin{cases} v_\theta(A)\mathfrak{e}\left(bd\beta(\gamma)\right)\chi_{\underline{L}}(d)\delta_{d\gamma} & \text{if } \mathrm{rk}(\underline{L}) \equiv 1 \bmod 2, \\ \mathfrak{e}\left(bd\beta(\gamma)\right)\chi_{\underline{L}}(d)\delta_{d\gamma} & \text{if } \mathrm{rk}(\underline{L}) \equiv 0 \bmod 2, \end{cases} \tag{2.40}$$

where $v_\theta(A) := \left(\frac{c}{d}\right)\epsilon(d)^{-1}$ *is the theta multiplier system (cf. [Shi73]).*

Proof. For the proof and the discussion about the character $\chi_{\underline{L}}$ see [McG03, Lemma 4.6], [Bor99, Theorem 5.4],[Ebe12, corollary 3.1],[Boy11, 3.35], and the discussion on [Ebe12, page 94]. □

Definition 2.3.2 (Jacobi theta series)**.** Let $x \in L^\#$. We define the Jacobi theta series $\vartheta_{\underline{L},x}(\tau,z)$ $\left((\tau,z) \in \mathfrak{H} \times L \otimes_{\mathbb{Z}} \mathbb{C}\right)$ by

$$\vartheta_{\underline{L},x}(\tau,z) := \sum_{\substack{r\in L^\# \\ r\equiv x \pmod{L}}} \mathfrak{e}\left(\tau\beta(r) + \beta(r,z)\right). \tag{2.41}$$

Theorem 2.3.3 ([Boy11, 3.34])**.** *The Jacobi theta series has the following transformation laws:*

(i) $\vartheta_{\underline{L},x}\big|_{\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\widetilde{T} = \mathfrak{e}\left(\beta(x)\right)\vartheta_{\underline{L},x}$.

(ii) $\vartheta_{\underline{L},x}\big|_{\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\widetilde{S} = \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}}\sum_{y\in L^\#/L}\mathfrak{e}\left(-\beta(x,y)\right)\vartheta_{\underline{L},y}$.

Theorem 2.3.4. *The vector-valued function*

$$\Theta_{\underline{L}}(\tau,z) = \sum_{x\in L^\#/L}\vartheta_{\underline{L},x}(\tau,z)\delta_x \tag{2.42}$$

has the following transformation formula

$$\Theta_{\underline{L}}\big|_{\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\widetilde{A} = \rho_{\underline{L}}(\widetilde{A})\Theta_{\underline{L}}, \tag{2.43}$$

for all $A \in \widetilde{\mathrm{SL}_2(\mathbb{Z})}$.

Proof. The transformation formula for $\Theta_{\underline{L}}$ under $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ follows from Theorem 2.3.3 and the formulas for $\rho_{\underline{L}}$ in Equation (2.38) and Equation (2.39). □

2.4 Jacobi Forms of Lattice Index

Jacobi forms whose indices are positive definite half-integral matrices have been studied in [BK93], [CG11],[Sko07],[Boy11], [Bri04], and by other authors. Here, we will restrict ourselves to Jacobi forms whose indices are positive definite integral matrices. For some facts that are important for us or cannot be found in the literature, we will work out a proof.

Definition 2.4.1 (Jacobi Form of Lattice Index)**.** For a positive integer k, and a positive definite even lattice $\underline{L} = (L,\beta)$, the space $J_{k,\underline{L}}$ of Jacobi forms of weight k and index $\underline{L}$ consists of all holomorphic functions $\phi(\tau,z)$ of variable τ in the complex upper half plane $\mathfrak{H}$ and a variable $z \in L\otimes_{\mathbb{Z}}\mathbb{C}$ which satisfies the following properties:

(i) For all $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, one has

$$\phi\big|_{k,\underline{L}}A = \phi. \tag{2.44}$$

(ii) For all $(x,y,1)\in H_{\underline{L}}(\mathbb{Z})$, one has

$$\phi\big|_{k,\underline{L}}(x,y,1) = \phi. \tag{2.45}$$

(iii) The function ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^\#\\ n\geq\beta(r)}} c_\phi(n,r)\mathfrak{e}\left(n\tau+\beta(r,z)\right). \tag{2.46}$$

Remarks 2.4.2.

1. Let G be a symmetric integral positive definite matrix of size n with an even diagonal. Set $\underline{L} = (\mathbb{Z}^n, (x,y) \mapsto x^t G y)$. If we identify $\mathbb{Z}^n \otimes \mathbb{C}$ with $\mathbb{C}^n$ (via the map $z \otimes c \mapsto zc$), then the space $J_{k,\underline{L}}$ is nothing else but the space of Jacobi forms of weight k and of matrix index G which have been studied in literature.

2. Let m be a positive integer, $\underline{L} = (2m\mathbb{Z}, (x,y) \mapsto \frac{xy}{2m})$. The space $J_{k,\underline{L}}$ is nothing else but the space of Jacobi forms of weight k and scalar index m which have been studied in [EZ85].

Proposition 2.4.3. *Let ϕ be a Jacobi form of weight k and index $\underline{L}$ with a Fourier development*

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^\#\\ n\geq\beta(r)}} c_\phi(n,r)\mathfrak{e}\left(n\tau+\beta(r,z)\right).$$

Then $c_\phi(n,r)$ depends only on $n-\beta(r)$ and on $r \bmod L$. Moreover, one has

$$c_\phi(n,r) = (-1)^k c_\phi(n,-r).$$

Proof. Let $(\lambda,0,1) \in H_{\underline{L}}(\mathbb{Z})$. By the definition of Jacobi forms, we have

$$\phi = \phi\big|_{k,\underline{L}}(\lambda,0,1).$$

Thus, using the Jacobi slash operator, we obtain

$$\phi(\tau,z) = \phi(\tau,z+\lambda\tau)\mathfrak{e}\left(\tau\beta(\lambda)+\beta(z,\lambda)\right) = \sum_{n,r} c_\phi(n,r)\mathfrak{e}\left(\left(n+\beta(\lambda,r)+\beta(\lambda)\right)\tau+\beta(z,\lambda)\right).$$

By replacing r with $r-\lambda$, and replacing n with $n-\beta(\lambda,r)+\beta(\lambda)$, we can write

$$\phi(\tau,z) = \sum_{n,r} c_\phi\left(n-\beta(\lambda,r)+\beta(\lambda), r-\lambda\right)\mathfrak{e}\left(n\tau+\beta(r,z)\right),$$

and hence $c_\phi(n,r) = c_\phi\big(n - \beta(\lambda,r) + \beta(\lambda), r - \lambda\big)$, i.e., $c_\phi(n,r) = c_\phi\big(n', r'\big)$ whenever $r' \equiv r \pmod{L}$ and $n - \beta(r) = n' - \beta(r')$ as stated in the proposition. The relation $c_\phi(n,r) = (-1)^k c_\phi(n,-r)$ follows directly by applying the transformation law of Jacobi forms to $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. □

Definition 2.4.4. Let $\phi = \sum_{n,r} c_\phi(n,r)\mathfrak{e}\big(n\tau + \beta(r,z)\big)$ be a Jacobi form of index $\underline{L} = (L,\beta)$. For $D \le 0$ and $r \in L^\#$ with $D \equiv \beta(r) \pmod{\mathbb{Z}}$ we set

$$C_\phi(D,r) := c_\phi\big(\beta(r) - D, r\big). \tag{2.47}$$

Remark 2.4.5. It is clear from the previous Proposition 2.4.3 that

$$C_\phi(D,r) = C_\phi(D, r \bmod L).$$

Thus every Jacobi form ϕ of weight k and index $\underline{L}$ has a Fourier expansion

$$\phi(\tau,z) = \sum_{(D,r)\in \mathrm{supp}(\underline{\mathrm{L}})} C_\phi(D,r)\mathfrak{e}\big((\beta(r) - D)\tau + \beta(r,z)\big),$$

where

$$\mathrm{supp}(\underline{\mathrm{L}}) := \Big\{(D,r) \;\Big|\; r \in L^\#, D \in \mathbb{Q}_{\le 0} \text{ such that } \beta(r) \equiv D \bmod \mathbb{Z}\Big\}.$$

Definition 2.4.6 (Jacobi cusp form)**.** A Jacobi form ϕ is called a cusp form if $C_\phi(0,r) = 0$ for all r such that $\beta(r) \in \mathbb{Z}$. By $S_{k,\underline{L}}$, we denote the subspace of Jacobi forms in $J_{k,\underline{L}}$ consisting of cusp forms.

Proposition 2.4.7. *Let $\phi \in J_{k,\underline{L}}$ be a Jacobi form of weight k and index $\underline{L} = (L,\beta)$. Then ϕ can be written as a sum*

$$\phi(\tau,z) = \sum_{x \in L^\#/L} h_x(\tau)\vartheta_{\underline{L},x}(\tau,z), \tag{2.48}$$

where

$$h_x(\tau) = \sum_{\substack{D\in\mathbb{Q} \\ (D,x)\in \mathrm{supp}(\underline{\mathrm{L}})}} C_\phi(D,x)\,\mathfrak{e}(-D\tau). \tag{2.49}$$

Proof. According to Remark 2.4.5 we can write

$$\phi(\tau,z) = \sum_{(D,r)\in \mathrm{supp}(\underline{\mathrm{L}})} C_\phi(D,r)\mathfrak{e}\big((\beta(r) - D)\tau + \beta(r,z)\big).$$

Since $C_\phi(D,r) = C_\phi(D, r \bmod L)$, we can split up the sum over $r \in L^\#$ into cosets

$$\phi(\tau,z) = \sum_{x \in L^\#/L} \sum_{\substack{r \in L^\# \\ r \equiv x \bmod L}} \sum_{(D,r)\in \mathrm{supp}(\underline{L})} C_\phi(D,r)\mathfrak{e}\left((\beta(r)-D)\tau + \beta(r,z)\right),$$

which can be reordered as follows:

$$\phi(\tau,z) = \sum_{x\in L^\#/L} \Bigg(\sum_{\substack{D \in \mathbb{Q}_{\le 0} \\ D \equiv \beta(x) \bmod \mathbb{Z}}} C_\phi(D,x)\,\mathfrak{e}(-D\tau)\Bigg)\Bigg(\sum_{\substack{r \in L^\# \\ r \equiv x \bmod L}} \mathfrak{e}\left(\tau\beta(r) + \beta(r,z)\right)\Bigg).$$

Now, the proof is complete. □

Remark 2.4.8. Combining Proposition 2.4.7 with the equation $\vartheta_{\underline{L},x}(\tau,-z) = \vartheta_{\underline{L},-x}(\tau,z)$ and $\phi(\tau,z)$=$(-1)^k$ $\phi(\tau,-z)$ we deduce the symmetry property:

$$h_x(\tau) = (-1)^k h_{-x}(\tau) \qquad (x \in L^\#/L). \tag{2.50}$$

Proposition 2.4.9. *Let $\phi = \sum_{x\in L^\#/L} h_x \vartheta_{\underline{L},x} \in J_{k,\underline{L}}$. For each $x \in L^\#/L$ the function h_x satisfies the following transformation laws:*

$$h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}} \widetilde{T} = \mathfrak{e}\left(-\beta(x)\right) h_x, \tag{2.51}$$

$$h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}} \widetilde{S} = \frac{i^{\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}} \sum_{y \in L^\#/L} \mathfrak{e}\left(\beta(x,y)\right) h_y. \tag{2.52}$$

Proof. By Proposition 2.4.7

$$\left(h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}} \widetilde{T}\right)(\tau) = h_x(\tau+1) = \sum_{(D,r)\in\mathrm{supp}(\underline{L})} C_\phi(D,x)\,\mathfrak{e}(-D\tau)\,\mathfrak{e}(-D).$$

Since $D \equiv \beta(x) \bmod \mathbb{Z}$, it follows that $\mathfrak{e}(-D) = \mathfrak{e}\left(-\beta(x)\right)$. Thus

$$\left(h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}} \widetilde{T}\right)(\tau) = \mathfrak{e}\left(-\beta(x)\right) h_x(\tau).$$

For the second we have, using Definition 2.4.1 and the definition of the slash operator, the following equation:

$$\phi(\tau,z) = \left(\phi\big|_{k,\underline{L}} S\right)(\tau,z) = \sum_{x\in L^\#/L} \left(\left(h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}} \widetilde{S}\right)(\tau)\right)\left(\left(\vartheta_{\underline{L},x}\big|_{\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}} \widetilde{S}\right)(\tau,z)\right).$$

Using the transformation laws of the theta functions from Theorem 2.3.3, we can write

$$\phi(\tau,z) = \sum_{y\in L^\#/L} \vartheta_{\underline{L},y}(\tau,z) \sum_{x\in L^\#/L} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}} \mathfrak{e}\left(-\beta(x,y)\right)\left(h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\widetilde{S}\right)(\tau).$$

For fixed τ, the functions $\vartheta_{\underline{L},y}(\tau,z)$ $(y \in L^\#/L)$ are linearly independent (see [Boy11], [Kri91, p. 609]), thus we have

$$h_y(\tau) = \sum_{x\in L^\#/L} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}} \mathfrak{e}\left(-\beta(x,y)\right)\left(h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\widetilde{S}\right)(\tau). \tag{2.53}$$

Now, if we apply $\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\widetilde{S}$ on both sides of Equation (2.53), we see, using $\widetilde{S}^2 = (-1,i)$, that

$$\begin{aligned}\left(h_y\big|_{k,\underline{L}}\widetilde{S}\right)(\tau) &= \sum_{x\in L^\#/L} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}} \mathfrak{e}\left(-\beta(x,y)\right)\left(h_x\big|_{k-\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}(-1,i)\right)(\tau) \\ &= (-1)^{-k+\frac{\mathrm{rk}(\underline{L})}{2}} \sum_{x\in L^\#/L} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}} \mathfrak{e}\left(\beta(-x,y)\right) h_x(\tau).\end{aligned}$$

Replace x by $-x$ and using $h_x = (-1)^k h_{-x}$ (see Remark 2.4.8) we obtain the claimed formula. □

2.5 Hecke Operators on the Space of Jacobi Forms

As can be seen in Skoruppa and Zagier's paper [SZ88], the main result of the theory of Jacobi forms is the relation between Jacobi forms of weight k and index m on the one hand, and ordinary elliptic modular forms of weight $2k-2$ and level m on the other. The lifting from $J_{k,m}$ to $M_{2k-2}(m)$ is constructed as follows: Let $\phi \in J_{k,m}$, then ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{d\le 0, r \\ d\equiv r^2 \bmod 4m}} C_\phi(d,r) q^{\frac{r^2-d}{4m}} \zeta^r.$$

Let $D < 0$ be a fundamental discriminant and r be an integer with $D \equiv r \bmod 4m$, then

$$\mathscr{S}_{m,D,r}(\tau) : \phi \to \sum_{\ell=0}^{\infty} \left(\text{coefficient of } q^{\frac{r^2-D}{4m}} \zeta^r \text{ in } \phi \mid T(\ell)\right) e^{2\pi i \ell \tau} \tag{2.54}$$

maps $J_{k,m}$ to a certain subspace of $M_{2k-2}(m)$. The maps $\mathscr{S}_{m,D,r}$ commute with the action of the Hecke operators $T(\ell)$. (Equation (2.54) is not quite true, since one would need here to define $T(\ell)$ for $\gcd(\ell,m)>1$, which is not a part of the common theory.)

The current thesis is based on the expectation that there should be lifting for Jacobi forms of lattice index to elliptic modular forms. However, a Hecke theory for these forms currently does not even exist.

Here, we shall construct such a theory and use our Hecke operators to give examples of correspondences with elliptic modular forms. More precisely, we will discuss correspondences supporting the mentioned expectation in the following sense:

Jacobi forms of weight k and index $\underline{L}$	$\xleftrightarrow[\text{if } \mathrm{rk}(\underline{L})\equiv 1 \bmod 2]{\text{correspondence}}$	elliptic modular forms of weight $2k-1-\mathrm{rk}(\underline{L})$
Jacobi forms of weight k and index $\underline{L}$	$\xleftrightarrow[\text{if } \mathrm{rk}(\underline{L})\equiv 0 \bmod 2]{\text{correspondence}}$	elliptic modular forms of weight $k-\frac{\mathrm{rk}(\underline{L})}{2}$

First, we state some basic information regarding the Hecke algebra associated with $\Gamma=\mathrm{SL}_2(\mathbb{Z})$. For this we follow essentially [Miy06, 4] and [Kri, 5].

Definition 2.5.1. For each $A\in\mathrm{Mat}(2,\mathbb{Z})$ we set

$$\gcd(A):=\text{g.c.d. of entries of A.}$$

We say that A is primitive if $\gcd(A)=1$.

Definition 2.5.2. For ℓ in $\mathbb{N}$ we set

$$\mathscr{M}(\ell):=\{A\in\mathrm{Mat}(2,\mathbb{Z})\mid \det A=\ell\},$$
$$\mathscr{M}^{\mathrm{pr}}(\ell):=\{A\in\mathscr{M}(\ell)\mid \text{A is primitive}\}.$$

It is clear that $\mathscr{M}(1)=\Gamma$. Also note that $\mathscr{M}:=\bigcup_{\ell\in\mathbb{N}}\mathscr{M}(\ell)$ is closed under matrix multiplication.

Theorem 2.5.3 (The Elementary Divisor Theorem for 2×2 Matrices)**.** *For given $A\in\mathscr{M}$, there exist $U,V\in\Gamma$, such that*

$$UAV=\begin{pmatrix}\gcd(A) & 0\\ 0 & \det(A)/\gcd(A)\end{pmatrix}.$$

Proof. See e.g. [Kri, Theorem 2.3] □

Theorem 2.5.4. *Let $\ell \in \mathbb{N}$. A complete set of coset representatives for the left cosets of Γ in $\mathcal{M}(\ell)$ is given by*

$$\Delta_\ell = \left\{ M = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a,b,d \in \mathbb{Z}, a,d \geq 0, ad = \ell \text{ and } 0 \leq b < d \right\}.$$

Moreover, a complete set of coset representatives for the left cosets of Γ in $\Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right)\Gamma$ is given by

$$\Delta_\ell^{\mathrm{pr}} = \left\{ M \in \Delta_\ell \mid \gcd(M) = 1 \right\}.$$

Proof. See e.g. [Miy06, Equation (4.5.24) and Equation (4.5.25)]. □

For developing our Hecke theory for lattice index Jacobi forms we extend ideas of [SZ87] which introduced there in the case of scalar index Jacobi forms. Let $\underline{L} = (L, \beta)$ be again a positive definite even lattice over $\mathbb{Z}$. Let $\ell \in \mathbb{N}$ such that $\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$. The set

$$J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z}) = \left\{ g_1 \left(\left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right), (0,0,1) \right) g_2 \;:\; g_1, g_2 \in J_{\underline{L}}(\mathbb{Z}) \right\} \tag{2.55}$$

is a double coset in $J_{\underline{L}}(\mathbb{Q})$. The Jacobi group $J_{\underline{L}}(\mathbb{Z})$ acts on the double coset $J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})$ by left multiplication, partitioning it into orbits. A typical orbit is $J_{\underline{L}}(\mathbb{Z}) g$ with a representative g, and $J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})$ is thus a disjoint union $\bigcup_i J_{\underline{L}}(\mathbb{Z}) g_i$ for some choice of representatives g_i.

Definition 2.5.5. Let ℓ be a positive integer with $\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$. We follow [SZ87] and define a double coset operator $T_0(\ell)$ on the vector space of Jacobi forms of weight k and index $\underline{L}$ as follows:

$$T_0(\ell)\phi := \ell^{k-2-\mathrm{rk}(\underline{L})} \sum_{g \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})} \phi\big|_{k,\underline{L}}\, g \qquad (\phi \in J_{k,\underline{L}}), \tag{2.56}$$

where the sum in Equation (2.56) runs over a complete set of representatives g for $J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})$.

Proposition 2.5.6. *The operator $T_0(\ell)$ given by Equation (2.56) is well defined, maps $J_{k,\underline{L}}$ to $J_{k,\underline{L}}$, and maps $S_{k,\underline{L}}$ to $S_{k,\underline{L}}$.*

Proof. Let $\phi \in J_{k,\underline{L}}$, $A \in J_{\underline{L}}(\mathbb{Z})$, and $B \in J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z})$. One has

$$\phi\big|_{k,\underline{L}}AB = \phi\big|_{k,\underline{L}}A\big|_{k,\underline{L}}B = \phi\big|_{k,\underline{L}}B.$$

Hence, each term in Equation (2.56) does not depend on the choice of the representative B, but only on the $J_{\underline{L}}(\mathbb{Z})$-orbit of B. Note that the sum in Equation (2.56) is finite (see Lemma 2.6.5). We want to show that the space of Jacobi forms of weight k and index $\underline{L}$ is invariant under the operator $T_0(\ell)$. Any $A \in J_{\underline{L}}(\mathbb{Z})$ permutes the orbit space $J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z})$ by right multiplication. That is, the map

$$\gamma_* : J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z}) \to J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z})$$

given by

$$J_{\underline{L}}(\mathbb{Z})B \mapsto J_{\underline{L}}(\mathbb{Z})BA$$

is well-defined and bijective. So if $\{B_i\}$ is a set of orbit representatives for the orbit space $J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z})$ then $\{B_iA\}$ is a set of orbit representatives for $J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z})$ as well. Thus

$$\left(T_0(\ell)\phi\right)\big|_{k,\underline{L}}A = \ell^{k-2-\mathrm{rk}(\underline{L})} \sum_{BA \in J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z})} \phi\big|_{k,\underline{L}}BA = T_0(\ell)\phi,$$

which means that $T_0(\ell)\phi$ transforms like a Jacobi form of weight k and index $\underline{L} = (L,\beta)$. It is left to show that

$$\left(T_0(\ell)\phi\right)\big|_{k,\underline{L}}A \qquad (A \in \mathrm{SL}_2(\mathbb{Z}))$$

has the correct Fourier development. Theorem 2.6.8 in the next pages will describe the Fourier development of this operator. Indeed, it shows also that this operator maps cusp forms to cusp forms. □

Definition 2.5.7 (The Operator $T(\ell)$). For all $\ell \in \mathbb{N}$ with $\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$, we define the Hecke operator $T(\ell) : J_{k,\underline{L}} \to J_{k,\underline{L}}$ in the following way:

1. If $\mathrm{rk}(\underline{L})$ is odd:

$$\phi \mapsto T(\ell)\phi := \sum_{d^2|\ell, d>0} d^{2k-\mathrm{rk}(\underline{L})-3}\, T_0\left(\tfrac{\ell}{d^2}\right)\phi. \tag{2.57}$$

2. If $\mathrm{rk}(\underline{L})$ is even:

$$\phi \mapsto T(\ell)\phi := \sum_{\substack{d,s>0 \\ sd^2|\ell,\ s \text{ square-free}}} \chi_{\underline{L}}(s)(sd^2)^{k-\frac{\mathrm{rk}(L)}{2}-2}\, T_0(\tfrac{\ell}{sd^2})\phi. \tag{2.58}$$

Our main goal now is to describe the action of these operators in terms of Fourier coefficients, and to give their commutation relations.

2.6 The Action of Hecke Operators on Fourier Coefficients

In this section we describe the action of the operators $T_0(\ell)$ and $T(\ell)$ on Jacobi forms in terms of Fourier development (see Theorem 2.6.8,Theorem 2.6.1, and Theorem 2.6.3). In the first subsection we state the main results of this section. The second subsection deals with the proof of the theorems.

2.6.1 The Main Results of this Section

Theorem 2.6.1. *Let $\underline{L} = (L,\beta)$ be a positive definite even lattice of odd rank. Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L,\beta)$ with a Fourier expansion*

$$\phi(\tau,z) = \sum_{\substack{D\le 0, r\in L^\# \\ D\equiv\beta(r) \bmod \mathbb{Z}}} C_\phi(D,r)\,\mathfrak{e}\big((\beta(r)-D)\tau + \beta(r,z)\big).$$

Let $\ell \in \mathbb{N}$ with $\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$*, and let*

$$\big(T(\ell)\phi\big)(\tau,z) = \sum_{\substack{D\le 0, r\in L^\# \\ D\equiv\beta(r) \bmod \mathbb{Z}}} C_{T(\ell)\phi}(D,r)\,\mathfrak{e}\big((\beta(r)-D)\tau + \beta(r,z)\big).$$

Then one has

$$C_{T(\ell)\phi}(D,r) = \sum_a a^{k-\lceil\frac{\mathrm{rk}(L)}{2}\rceil-1}\varrho(D,a)C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' r\big). \tag{2.59}$$

The sum in Equation (2.59) is over those $a \mid \ell^2$ such that $a^2 \mid \ell^2 \mathrm{lev}(\underline{L})D$, and a' is an integer such that $aa' \equiv 1 \pmod{\mathrm{lev}(\underline{L})}$. Moreover, $\varrho(D,a)$ equals $f \cdot \chi_{\underline{L}}(D/f^2, a/f^2)$ if $\gcd(\mathrm{lev}(\underline{L})D, a) = f^2$ with $f \in \mathbb{N}$, and it equals 0 if $\gcd(\mathrm{lev}(\underline{L})D, a)$ is not a perfect square.

Remark 2.6.2. Assume that $\mathrm{lev}(r)D$ is a square-free. Then Equation (2.59) simplifies to

$$C_{T(\ell)\phi}(D,r) = \sum_{a|\ell} a^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\chi_{\underline{L}}(D,a)C_\phi\left(\tfrac{\ell^2}{a^2}D, \tfrac{\ell}{a}r\right) \tag{2.60}$$

Theorem 2.6.3. *Let $\underline{L} = (L,\beta)$ be a positive definite even lattice of even rank. Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L,\beta)$ with a Fourier expansion*

$$\phi(\tau,z) = \sum_{\substack{D\le 0, r\in L^\# \\ D\equiv\beta(r) \bmod \mathbb{Z}}} C_\phi(D,r)\,\mathfrak{e}\left((\beta(r)-D)\tau + \beta(r,z)\right).$$

Let $\ell \in \mathbb{N}$ with $\gcd(\ell,\mathrm{lev}(\underline{L})) = 1$, and let

$$\left(T(\ell)\phi\right)(\tau,z) = \sum_{\substack{D\le 0, r\in L^\# \\ D\equiv\beta(r) \bmod \mathbb{Z}}} C_{T(\ell)\phi}(D,r)\,\mathfrak{e}\left((\beta(r)-D)\tau + \beta(r,z)\right).$$

Then one has

$$C_{T(\ell)\phi}(D,r) = \sum_{a|\ell^2, \mathrm{lev}(\underline{L})D} a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(a)C_\phi\left(\tfrac{\ell^2}{a^2}D, \ell a' r\right), \tag{2.61}$$

where a' is an integer such that $aa' \equiv 1 \pmod{\mathrm{lev}(\underline{L})}$.

Remark 2.6.4. Note that the conditions $a^2|\ell^2\mathrm{lev}(\underline{L})D$ and $\beta(r) \equiv D \bmod \mathbb{Z}$ imply that $\frac{\ell^2}{a^2}D \equiv \beta(\ell a' r) \bmod \mathbb{Z}$. Thus the coefficients $C_\phi\left(\frac{\ell^2}{a^2}D, \ell a' r\right)$ in Theorem 2.6.3 and Theorem 2.6.1 are well-defined.

2.6.2 Proofs

First we describe the action of the operator $T_0(\ell)$ on Jacobi forms in term of Fourier expansion, then we use this description to prove Theorem 2.6.1 and Theorem 2.6.3. To do this we need the following lemmas:

Lemma 2.6.5. *Let $\ell \in \mathbb{N}_{\underline{L}}$. The finite set of all elements (M,h), where M runs through a complete set of representatives for $\Gamma\backslash\Gamma\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)\Gamma$ and h runs through a complete set of representatives for $(H_{\underline{L}}(\mathbb{Z})\cap M^{-1}H_{\underline{L}}(\mathbb{Z})M)\backslash H_{\underline{L}}(\mathbb{Z})$, is a complete set of cosets representatives for $J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)J_{\underline{L}}(\mathbb{Z})$.*

Proof. Let $\gamma : J_{\underline{L}}(\mathbb{Z}))\left(\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right),1\right) J_{\underline{L}}(\mathbb{Z}) \to \Gamma\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)\Gamma$ be the canonical map $(M,h)\mapsto M$, and let $\gamma_* : J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})\left(\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right),1\right) J_{\underline{L}}(\mathbb{Z}) \to \Gamma\backslash\Gamma\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)\Gamma$ be the induced map. First we show that each coset in the fibre $\gamma_*^{-1}(\Gamma M)$ contains an element of the form (M,h), where $h\in H_{\underline{L}}(\mathbb{Z})$. Each class in the fibre $\gamma_*^{-1}(\Gamma M)$ has a representative $g=(A,h_1)\left(\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right),(0,0,1)\right)(B,h_2)$, where (A,h_1), $(B,h_2)\in J_{\underline{L}}(\mathbb{Z})$ with $M=A\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)B$. Write $A=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and $h_1=(\lambda_1,\mu_1,1)$. Since $(1_2,(-d\lambda_1,b\lambda_1,1))\in J_{\underline{L}}(\mathbb{Z})$, one has $\left(M,(0,\ell\mu_1,1)^B h_2\right)g^{-1}=\left(1_2,(-d\lambda_1,b\lambda_1,1)\right)\in J_{\underline{L}}(\mathbb{Z})$. The existence now is clear, since $(0,\ell\mu_1,1)^B h_2\in H_{\underline{L}}(\mathbb{Z})$. Let $M\in\Gamma\backslash\Gamma\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)\Gamma$ and $h_1,h_2\in H_{\underline{L}}(\mathbb{Z})$. The two classes $J_{\underline{L}}(\mathbb{Z})(M,h_1)$ and $J_{\underline{L}}(\mathbb{Z})(M,h_2)$ are equal if and only if $(M,h_1)(M,h_2)^{-1}\in J_{\underline{L}}(\mathbb{Z})$. This is equivalent, by the Jacobi group multiplication operation (see Definition 2.2.4), to $\left(h_1{h_2}^{-1}\right)^{M^{-1}}=Mh_1{h_2}^{-1}M^{-1}\in H_{\underline{L}}(\mathbb{Z})$. Since $h_1{h_2}^{-1}\in H_{\underline{L}}(\mathbb{Z})$, we obtain $h_1{h_2}^{-1}\in\left(M^{-1}H_{\underline{L}}(\mathbb{Z})M\right)\cap H_{\underline{L}}(\mathbb{Z})$. Thus the two classes $J_{\underline{L}}(\mathbb{Z})(M,h_1)$ and $J_{\underline{L}}(\mathbb{Z})(M,h_2)$ are equal if and only if

$$h_1\equiv h_2 \bmod \left(M^{-1}H_{\underline{L}}(\mathbb{Z})M\right)\cap H_{\underline{L}}(\mathbb{Z}).$$

We still need to show that this set of representatives is finite. In the view of the well-known fact that $\Gamma\backslash\Gamma\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)\Gamma$ is finite (see Theorem 2.5.4), it is enough to show that $(H_{\underline{L}}(\mathbb{Z})\cap M^{-1}H_{\underline{L}}(\mathbb{Z})M)\backslash H_{\underline{L}}(\mathbb{Z})$ is finite. But this is immediate, since

$$\left\{(\ell\lambda,\ell\mu,1)\;\middle|\;\lambda,\mu\in L\right\}\subseteq H_{\underline{L}}(\mathbb{Z})\cap M^{-1}H_{\underline{L}}(\mathbb{Z})M.$$

Now the proof is complete. □

Lemma 2.6.6. *Let ϕ be a Jacobi form of weight k and index $\underline{L}=(L,\beta)$. For each $\ell\in\mathbb{N}_{\underline{L}}$, the action of the operator $T_0(\ell)$ on ϕ can be written as follows:*

$$T_0(\ell)\phi=\ell^{k-2-2\,\mathrm{rk}(\underline{L})}\sum_{(\lambda,\mu)\in(L/\ell L)\times(L/\ell L)}\;\sum_{M\in\frac{1}{\ell}\Delta^{\mathrm{pr}}_{\ell^2}}\phi\big|_{k,\underline{L}}M\big|_{k,\underline{L}}(\lambda,\mu,1). \tag{2.62}$$

Proof. Note, first of all, that $\frac{1}{\ell}\Delta^{\mathrm{pr}}_{\ell^2}$ is a system of representatives for $\Gamma\backslash\Gamma\left(\begin{smallmatrix}\ell^{-1} & 0\\ 0 & \ell\end{smallmatrix}\right)\Gamma$. Let S be a subgroup of finite index in $H_{\underline{L}}(\mathbb{Z})$, which contained in the subgroup $H_{\underline{L}}(\mathbb{Z})\cap M^{-1}H_{\underline{L}}(\mathbb{Z})M$ $\;(M\in\frac{1}{\ell}\Delta^{\mathrm{pr}}_{\ell^2})$. Then, using Lemma 2.6.5, we can write

$$T_0(\ell)\phi=\ell^{k-2-\mathrm{rk}(\underline{L})}\sum_{M\in\frac{1}{\ell}\Delta^{\mathrm{pr}}_{\ell^2}}\;\sum_{h\in S\backslash H_{\underline{L}}(\mathbb{Z})}C(M)^{-1}\cdot\phi\big|_{k,\underline{L}}(M,h). \tag{2.63}$$

where

$$C(M) := [H_{\underline{L}}(\mathbb{Z}) \cap M^{-1}H_{\underline{L}}(\mathbb{Z})M : S].$$

We show that we can in fact choose

$$S = \{(\ell\lambda, \ell\mu, 1) \mid \lambda, \mu \in L\}.$$

Indeed, let $h = (\ell\lambda, \ell\mu, 1) \in S$. Clearly $h \in H_{\underline{L}}(\mathbb{Z})$. Moreover, for $M = \begin{pmatrix} \frac{a}{\ell} & \frac{b}{\ell} \\ 0 & \frac{d}{\ell} \end{pmatrix} \in \frac{1}{\ell}\Delta^{\mathrm{pr}}_{\ell^2}$, we have, using Proposition 2.2.3, that

$$h = \left((d\lambda, -b\lambda + a\mu, 1)\begin{pmatrix} \frac{a}{\ell} & \frac{b}{\ell} \\ 0 & \frac{d}{\ell} \end{pmatrix}, 1\right) = (d\lambda, -b\lambda + a\mu, 1)^M.$$

Thus $h \in H_{\underline{L}}(\mathbb{Z}) \cap M^{-1}H_{\underline{L}}(\mathbb{Z})M$. We still have to show that $C(M) = \ell^{\mathrm{rk}(\underline{L})}$. It is clear that for all $A \in \Gamma$ we have $C(AM) = C(M)$. We also have $C(MA) = C(M)$, since the function

$$\begin{aligned} H_{\underline{L}}(\mathbb{Z}) \cap M^{-1}H_{\underline{L}}(\mathbb{Z})M/S &\to H_{\underline{L}}(\mathbb{Z}) \cap A^{-1}M^{-1}H_{\underline{L}}(\mathbb{Z})MA/S \\ gS &\mapsto A^{-1}gAS \end{aligned}$$

is a bijection. Namely $C(M) = C(AMB)$ for any $A, B \in \Gamma$. By the elementary divisor theorem (see Theorem 2.5.3), we can find matrices $A, B \in \Gamma$ such that $AMB = \frac{1}{\ell}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell^2 \end{smallmatrix}\right) = \begin{pmatrix} \frac{1}{\ell} & 0 \\ 0 & \ell \end{pmatrix}$. Thus

$$C(M) = C(AMB) = C\left(\begin{pmatrix} \frac{1}{\ell} & 0 \\ 0 & \ell \end{pmatrix}\right) = [L \times \ell L : \ell L \times \ell L] = \ell^{\mathrm{rk}(\underline{L})}.$$

This proves the lemma. □

Definition 2.6.7 (notation)**.** Let $\underline{L} = (L, \beta)$ be a lattice over $\mathbb{Z}$, and let a be a positive integer such that $(a, \mathrm{lev}(\underline{L})) = 1$. We shall use a' to denote an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L})$.

Theorem 2.6.8. *Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L, \beta)$ with Fourier expansion*

$$\phi(\tau, z) = \sum_{(D,r) \in \mathrm{supp}(\underline{L})} C_\phi(D, r)\mathfrak{e}\big((\beta(r) - D)\tau + \beta(r, z)\big).$$

Let $\ell \in \mathbb{N}$ be a positive integer such that $\gcd(\ell, \operatorname{lev}(\underline{L})) = 1$, *and let*

$$\big(T_0(\ell)\phi\big)(\tau,z) = \sum_{(D,r)\in \operatorname{supp}(\underline{L})} C_{T_0(\ell)\phi}(D,r)\mathfrak{e}\big((\beta(r)-D)\tau + \beta(r,z)\big).$$

Then

$$C_{T_0(\ell)\phi}(D,r) = \sum_a a^{k-2}\mathscr{W}\big(\operatorname{lev}(\underline{L})^2 D, a\big) C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' r\big), \tag{2.64}$$

where a runs over the positive divisors of ℓ^2 with $a^2|\ell^2\operatorname{lev}(\underline{L})D$, a' is an integer such that $aa' \equiv 1 \bmod \operatorname{lev}(\underline{L})$, and the function $\mathscr{W}\big(\operatorname{lev}(\underline{L})^2 D, a\big)$ is given by Equation (2.1), i.e.,

$$\mathscr{W}\big(\operatorname{lev}(\underline{L})^2 D, a\big) = \sum_{t|a}\mu(\frac{a}{t})t^{1-\operatorname{rk}(\underline{L})}\#\left\{x\in L\,/\,tL \mid \operatorname{lev}(\underline{L})^2 D \equiv \beta(x) \bmod t\right\}.$$

Proof. By Lemma 2.6.6 we write the definition of $T_0(\ell)$ as

$$T_0(\ell)\phi = \ell^{k-2}\sum_M \phi\big|_{k,\underline{L}}\tfrac{1}{\ell}M\big|_{k,\underline{L}}\mathscr{A}_\ell,$$

where the sum is over all $M \in \Delta_{\ell^2}^{\mathrm{pr}}$ and $\big|_{k,\underline{L}}\mathscr{A}_\ell$ is the operator which acts on any function ψ that transforms like a Jacobi form by

$$\psi\big|_{k,\underline{L}}\mathscr{A}_\ell = \ell^{-2\operatorname{rk}(\underline{L})}\sum_{(\lambda,\mu)\in L/\ell L\times L/\ell L}\psi\big|_{k,\underline{L}}(\lambda,\mu,1).$$

First, we do the usual computation of the action of the upper triangular representatives for the left $\mathrm{SL}_2(\mathbb{Z})$-cosets. For that we set

$$\phi|T_0(\ell) := \phi_1|\mathscr{A}_\ell,$$

where

$$\phi_1 := \ell^{k-2}\sum_{ad|\ell^2}\sum_{\substack{b \bmod d\\ \gcd(a,b,c)=1}}\phi\big|_{k,\underline{L}}\tfrac{1}{\ell}\left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right). \tag{2.65}$$

To get rid of the condition $\gcd(a,b,d) = 1$ in Equation (2.65) we use the identity

$$\sum_{\delta|g}\mu(\delta) = \boldsymbol{\delta}(g=1),$$

where μ here is the Möbius function. This gives

$$\phi_1(\tau,z) = \ell^{-2}\sum_{ad=\ell^2}a^k\sum_{\delta|(a,d)}\mu(\delta)\sum_{\substack{b(d)\\ b\equiv 0(\delta)}}\sum_{n\in\mathbb{N},r\in L^\#}c_\phi(n,r)\mathfrak{e}\big(n\tfrac{a\tau+b}{d}+\beta(z,\tfrac{\ell r}{d})\big)$$

$$= \sum_{ad=\ell^2} a^{k-1} \sum_{\delta|(a,d)} \frac{\mu(\delta)}{\delta} \sum_{\substack{n\in\mathbb{Z}, r\in L^\#\\ \frac{d}{\delta}|n}} c_\phi(n,r)\mathfrak{e}\left(n\tfrac{a\tau}{d} + \beta(z, \tfrac{\ell r}{d})\right).$$

To simplify the previous formula, we set

$$\Lambda(\alpha,\beta) := \sum_{\delta|\alpha, \alpha\delta^{-1}|\beta} \mu(\delta)/\delta \quad (\alpha,\beta\in\mathbb{Z}).$$

This gives

$$\phi_1(\tau,z) = \sum_{ad=\ell^2} a^{k-1} \sum_{\substack{n\in\mathbb{Z}, r\in L^\#\\ \frac{d}{(a,d)}|n}} \Lambda\big((a,d), \tfrac{n}{d/(a,d)}\big) c_\phi(n,r)\mathfrak{e}\left(n\tfrac{a\tau}{d} + \beta(z, \tfrac{\ell r}{d})\right).$$

Replacing $\frac{na}{d}$ with n and $\frac{r\ell}{d}$ with r allowing us to write

$$\begin{aligned}\phi_1(\tau,z) &= \sum_{ad=\ell^2} a^{k-1} \sum_{\substack{r\in L^\#, r\in \frac{\ell}{d}L^\#\\ n\in\mathbb{Z}, \frac{a}{(a,d)}|n, a|dn}} \Lambda\big((a,d), \tfrac{n}{a/(a,d)}\big) c_\phi\left(\tfrac{dn}{a}, \tfrac{\ell}{a}r\right) \mathfrak{e}\left(n\tau + \beta(r,z)\right)\\ &= \sum_{ad=\ell^2} a^{k-1} \sum_{\substack{n\in\frac{a}{(a,d)}\mathbb{Z}\\ r\in\frac{a}{\ell}L^\#}} \Lambda\big((a,d), \tfrac{n}{a/(a,d)}\big) c_\phi\big(\tfrac{\ell^2}{a^2}n, \tfrac{\ell}{a}r\big)\mathfrak{e}\left(n\tau+\beta(r,z)\right).\end{aligned}$$

This gives the Fourier development of ϕ_1. We still must apply the averaging operator $\mathscr{A}_\ell$. We can factor $\mathscr{A}_\ell$ as $\ell^{-2\mathrm{rk}(\underline{L})}(\mathscr{A}_{1,\ell}\circ\mathscr{A}_{2,\ell})$, where for any function ψ that transforms like a Jacobi form we set

$$\psi\big|_{k,\underline{L}}\mathscr{A}_{1,\ell} := \sum_{\mu\in L/\ell L} \psi\big|_{k,\underline{L}}(0,\mu,1), \qquad \psi\big|_{k,\underline{L}}\mathscr{A}_{2,\ell} := \sum_{\lambda\in L/\ell L} \psi\big|_{k,\underline{L}}(\lambda,0,1).$$

Thus to compute the action of $T_0(\ell)$ we still need apply $\mathscr{A}_{1,\ell}$ and $\mathscr{A}_{2,\ell}$. For that we set

$$\phi_2(\tau,z) := \left(\phi_1\big|_{k,\underline{L}}\mathscr{A}_{1,\ell}\right)(\tau,z) = \sum_{\mu\in L/\ell L} \phi_1(\tau, z+\mu).$$

Using the last formula of the Fourier expansion of ϕ_1, one has

$$\phi_2(\tau,z) = \sum_{ad=\ell^2} a^{k-1} \sum_{\substack{dn\equiv 0(a)\\ r\in\frac{a}{\ell}L^\#}} \Lambda\big((a,d), \tfrac{n}{a/(a,d)}\big) c_\phi\big(\tfrac{\ell^2}{a^2}n, \tfrac{\ell}{a}r\big)\mathfrak{e}\left(n\tau+\beta(r,z)\right) \sum_{\mu\in L/\ell L} \mathfrak{e}\left(\beta(\mu,r)\right).$$

Note that the inner sum in the last formula equals $[L:\ell L]\boldsymbol{\delta}(r\in L^{\#})$. This gives

$$\phi_2(\tau,z)=\ell^{\mathrm{rk}(\underline{L})}\sum_{ad=\ell^2}a^{k-1}\sum_{\substack{n\in\mathbb{N},dn\equiv 0(a)\\ r\in\frac{a}{\ell}L^{\#}\cap L^{\#}}}\Lambda\big((a,d),\tfrac{n}{a/(a,d)}\big)c_\phi\big(\tfrac{\ell^2}{a^2}n,\tfrac{\ell}{a}r\big)\mathfrak{e}\big(n\tau+\beta(r,z)\big),$$

i.e., ϕ_2 is obtained from ϕ_1 by omitting all terms with $r\notin L^{\#}$. Now, we apply the operator $\mathscr{A}_{2,\ell}$.

$$\begin{aligned}\big(T_0(\ell)\phi\big)(\tau,z)&=\ell^{-2\mathrm{rk}(\underline{L})}\sum_{\lambda\in L/\ell L}\phi_2\big|_{k,\underline{L}}(\lambda,0,1)(\tau,z)\\&=\ell^{-2\mathrm{rk}(\underline{L})}\sum_{\lambda\in L/\ell L}\phi_2(\tau,z+\lambda\tau)\mathfrak{e}\big(\tau\beta(\lambda)+\beta(z,\lambda)\big).\end{aligned}$$

Inserting the Fourier expansion of $phi_2(\tau,z+\lambda\tau)$ in the last equation gives

$$\big(T_0(\ell)\phi\big)(\tau,z)=\sum_{ad=\ell^2}\frac{a^{k-1}}{\ell^{\mathrm{rk}(\underline{L})}}\sum_{\substack{\lambda\in L/\ell L\\ n\in\mathbb{N},dn\equiv 0(a),r\in\frac{a}{\ell}L^{\#}\cap L^{\#}}}\Big(\Lambda\big((a,d),\tfrac{n}{a/(a,d)}\big)c_\phi\big(\tfrac{\ell^2}{a^2}n,\tfrac{\ell}{a}r\big)$$
$$\mathfrak{e}\big((n+\beta(\lambda,r)+\beta(\lambda))\tau+\beta(z,\lambda+r)\big)\Big).$$

By replacing r with $r-\lambda$, and replacing n with $n-\beta(\lambda,r)+\beta(\lambda)$, we can rewrite the previous expansion of $T_0(\ell)\phi$ as follows:

$$\big(T_0(\ell)\phi\big)(\tau,z)=\sum_{(D,r)\in\mathrm{supp}(\underline{\mathrm{L}})}C_{T_0(\ell)\phi}(D,r)\mathfrak{e}\big((\beta(r)-D)\tau+\beta(r,z)\big),$$

with

$$C_{T_0(\ell)\phi}(D,r)=\sum_{a|\ell^2}\frac{a^{k-1}}{\ell^{\mathrm{rk}(\underline{L})}}\sum_{\lambda}\Lambda\big(\tfrac{(a,\ell)^2}{a},\tfrac{(a,\ell)^2}{a^2}(\beta(r-\lambda)-D)\big)C_\phi\big(\tfrac{\ell^2}{a^2}D,\tfrac{\ell}{a}(r-\lambda)\big),$$

where λ runs through $L/\ell L$ such that $\beta(r-\lambda)\equiv D \bmod \frac{a^2}{(a,\ell)^2}\mathbb{Z}$ and $r-\lambda\in\frac{a}{(a,\ell)}L^{\#}$. Let a' be an integer such that $a'a\equiv 1 \pmod{\mathrm{lev}(\underline{L})}$, and $s=\frac{(a,\ell)}{a}(r-\lambda)$. Since $\ell\in\mathbb{N}_{\underline{L}}$, the condition $\frac{a}{(a,\ell)}s\equiv r \bmod L$ is equivalent to the equation $s\equiv(\ell,a)a'r \bmod L$ and the terms $C_\phi\big(\frac{\ell^2}{a^2}D,\frac{\ell}{(a,\ell)}s\big)$ equal $C_\phi\big(\frac{\ell^2}{a^2}D,\ell a'r\big)$. Thus

$$C_{T_0(\ell)\phi}(D,r)=\sum_{a|\ell^2}a^{k-1}\ell^{-\mathrm{rk}(\underline{L})}\psi(\ell,a,r,D)C_\phi\big(\tfrac{\ell^2}{a^2}D,\ell a'r\big),$$

where

$$\psi(\ell,a,r,D) := \sum_{\substack{\lambda\in L/\ell L,\ r-\lambda\in\frac{a}{(a,\ell)}L^\#\\ \beta(r-\lambda)\equiv D \bmod \frac{a^2}{(a,\ell)^2}\mathbb{Z}}} \Lambda\Big(\frac{(a,\ell)^2}{a}, \frac{(a,\ell)^2}{a^2}\big(\beta(r-\lambda)-D\big)\Big). \tag{2.66}$$

It remains to simplify Equation (2.66). Recall that

$$\Lambda(\alpha,\beta) = \sum_{\delta|\alpha,\alpha\delta^{-1}|\beta} \mu(\delta)/\delta = \frac{1}{\alpha}\sum_{\delta|(\alpha,\beta)} \mu(\tfrac{\alpha}{\delta})\delta.$$

One has

$$\psi(\ell,a,r,D) = \sum_{\substack{\lambda\in L/\ell L,\ r-\lambda\in\frac{a}{(a,\ell)}L^\#\\ \beta(r-\lambda)\equiv D \bmod \frac{a^2}{(a,\ell)^2}\mathbb{Z}}} \frac{a}{(a,\ell)^2} \sum_{\delta|(\frac{(a,\ell)^2}{a},\frac{(a,\ell)^2}{a^2}\beta(r-\lambda)-\frac{(a,\ell)^2}{a^2}D)} \mu\Big(\frac{(a,\ell)^2}{a\delta}\Big)\delta.$$

After reordering the summations in the above formula we see that

$$\psi(\ell,a,r,D) = \frac{a}{(a,\ell)^2}\sum_{\delta|\frac{(a,\ell)^2}{a}} \mu\big(\tfrac{(a,\ell)^2}{a\delta}\big)\delta\big(\tfrac{\ell(a,\ell)}{a\delta}\big)^{\mathrm{rk}(\underline{L})} \sum_{\substack{s\in L^\#/\delta L,\ \frac{a}{(a,\ell)}s\equiv r \bmod L\\ \frac{(a,\ell)^2}{a^2}D\equiv\beta(s) \bmod \delta}} 1.$$

Inserting this into the last formula for $C_{T_0(\ell)\phi}(D,r)$ gives

$$C_{T_0(\ell)\phi}(D,r) = \sum_{a|\ell^2} a^{k-2}C_\phi\big(\tfrac{\ell^2}{a^2}D,\ell a'r\big)\big(\tfrac{a}{(a,\ell)}\big)^{2-\mathrm{rk}(\underline{L})} \sum_{t|\frac{(a,\ell)^2}{a}} \mu\big(\tfrac{(a,\ell)^2}{at}\big)B_{\underline{L}}\big(\tfrac{(a,\ell)^2}{a^2}D,r,t\big), \tag{2.67}$$

where, $C_\phi\big(\frac{\ell^2}{a^2}D,\ell a'r\big)=0$ unless $\frac{\ell^2}{a^2}D-\beta(\ell a'r)$ is an integer, and

$$B_{\underline{L}}\big(\tfrac{(a,\ell)^2}{a^2}D,r,t\big) = t^{1-\mathrm{rk}(\underline{L})}\#\Big\{v\in L\,/\,tL \mid \tfrac{(a,\ell)^2}{a^2}D \equiv \beta\big((a,\ell)a'r+v\big) \bmod t\Big\}.$$

Since $\ell\in\mathbb{N}_{\underline{L}}$, $\frac{\ell^2}{a^2}D-\beta(\ell a'r)$ is an integer if and only if $a^2 \mid \ell^2\mathrm{lev}(\underline{L})D$. To simplify $B_{\underline{L}}\big(\frac{(a,\ell)^2}{a^2}D,r,t\big)$ we can therefore assume that the last condition is fulfilled. Let b be an integer such that $1=aa'+\mathrm{lev}(\underline{L})b$. For each t such that $t\mid\frac{(a,\ell)^2}{a}$, one has $\beta(v)-\mathrm{lev}(\underline{L})b\beta\big((a,\ell)a'r+v\big) = aa'\beta(v)+\mathrm{lev}(\underline{L})b\beta((a,\ell)a'r)+\mathrm{lev}(\underline{L})b\beta((a,\ell)a'r,v)\equiv 0 \bmod t$. Thus we obtain

$$B_{\underline{L}}\big(\tfrac{(a,\ell)^2}{a^2}D,r,t\big) = t^{1-\mathrm{rk}(\underline{L})}\#\Big\{v\in L\,/\,tL \mid \tfrac{Nb(a,\ell)^2}{a^2}D \equiv \beta(v) \bmod t\Big\}.$$

Inserting this in Equation (2.67) gives

$$C_{T_0(\ell)\phi}(D,r) = \sum_{\substack{a|\ell^2 \\ a^2|\ell^2 \mathrm{lev}(\underline{L})D}} a^{k-2} C_\phi\left(\tfrac{\ell^2}{a^2}D, \ell a' r\right)\left(\tfrac{a}{(a,\ell)}\right)^{2-\mathrm{rk}(\underline{L})} \mathscr{W}\left(\tfrac{(a,\ell)^2}{a^2}\mathrm{lev}(\underline{L})bD, \tfrac{(a,\ell)^2}{a}\right),$$

where the function $\mathscr{W}$ is given by Equation (2.1). Note that the equation $aa' + \mathrm{lev}(\underline{L})b = 1$ implies that $\left(\frac{\mathrm{lev}(\underline{L})b}{a}\right)^{\mathrm{rk}(\underline{L})} = \left(\frac{\mathrm{lev}(\underline{L})^2}{a}\right)^{\mathrm{rk}(\underline{L})}$. Thus by Equation (2.21) we have

$$\left(\tfrac{a}{(a,\ell)}\right)^{2-\mathrm{rk}(\underline{L})} \mathscr{W}\left(\tfrac{(a,\ell)^2}{a^2}\mathrm{lev}(\underline{L})bD, \tfrac{(a,\ell)^2}{a}\right) = \mathscr{W}(\mathrm{lev}(\underline{L})^2 D, a).$$

Inserting this into the last formula of $C_{T_0(\ell)\phi}(D,r)$, gives the claimed formula and completes the proof. □

Proof of Theorem 2.6.1. Recall that for odd rank lattice $\underline{L} = (L,\beta)$, the operator $T(\ell)$ and $T_0(\ell)$ are related by (see Definition 2.5.7)

$$T(\ell) = \sum_{d^2|\ell,\ d>0} d^{2k-\mathrm{rk}(\underline{L})-3} T_0(\ell/d^2). \tag{2.68}$$

Thus by Equation (2.68), one has

$$C_{T(\ell)\phi}(D,r) = \sum_{\substack{s|\ell \\ \ell/s=\text{perfect square}}} (\ell/s)^{k-2-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor} C_{T_0(s)\phi}(D,r).$$

By using Theorem 2.6.8, which describes the action of the operators $T_0(\ell)$ on Fourier coefficients, we can write

$$C_{T(\ell)\phi}(D,r) = \sum_{\substack{s|\ell \\ \frac{\ell}{s}=\text{perfect square}}} (\ell/s)^{k-2-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor} \sum_{\substack{g|s^2 \\ g^2|s^2\mathrm{lev}(\underline{L})D}} g^{k-2}\mathscr{W}\big(\mathrm{lev}(\underline{L})^2 D, g\big) C_\phi\left(\tfrac{s^2}{g^2}D, sg'r\right). \tag{2.69}$$

We want to simplify Equation (2.69). For that, we set $a := g\frac{\ell}{s}$ and $b := \frac{a}{g} = \frac{\ell}{s}$. When s runs over all positive divisors of ℓ such that $\frac{\ell}{s}$ is a perfect-square and g runs over the positive divisors of s^2 such that $g^2 \mid s^2\mathrm{lev}(\underline{L})D$, then a runs over all positive divisors of ℓ^2 and b runs over all positive divisors of $\gcd(a, \frac{\ell^2}{a})$. Moreover, if b' is an integer such that $bb' \equiv 1 \bmod \mathrm{lev}(\underline{L})$, we set $a' := b'g'$. It is obvious that $aa' \equiv 1 \bmod \operatorname{lev}(\underline{L})$ and $C_\phi\big(\frac{\ell^2}{a^2}D, \ell a' r\big) = C_\phi\big(\frac{s^2}{g^2}D, sg'r\big)$. Thus we can reorder the summation in Equation (2.69) as follows:

$$C_{T(\ell)\phi}(D,r) = \sum_{\substack{a|\ell^2 \\ a^2|\ell^2 \operatorname{lev}(\underline{L})D}} C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' r\big) a^{k-2} \sum_{\substack{b|(a,\frac{\ell^2}{a}) \\ b=\text{perfect square}}} b^{-\lfloor \frac{\operatorname{rk}(\underline{L})}{2} \rfloor} \mathcal{W}\big(\operatorname{lev}(\underline{L})^2 D, \tfrac{a}{b}\big).$$

To complete the proof, we need to simplify the inner sum in the last equation. Setting $x = \frac{a}{(a,\ell)}$, then $(a, \frac{\ell^2}{a}) = a/x^2$. The condition $a|\ell^2$ and $a^2|\ell^2 \operatorname{lev}(\underline{L})D$ implies that $\frac{\operatorname{lev}(\underline{L})^2 D}{x^2} \in \mathbb{Z}$. By using Lemma 2.1.22, we obtain

$$\mathcal{W}\big(\operatorname{lev}(\underline{L})^2 D, \tfrac{a}{b}\big) = x^{2-\operatorname{rk}(\underline{L})} \mathcal{W}\big(\operatorname{lev}(\underline{L})^2 D/x^2, \tfrac{a}{bx^2}\big).$$

Thus

$$\begin{aligned}\sum_{b|(a,\frac{\ell^2}{a}), b=\text{perfect square}} b^{-\lfloor \frac{\operatorname{rk}(\underline{L})}{2} \rfloor} \mathcal{W}\big(\operatorname{lev}(\underline{L})^2 D, \tfrac{a}{b}\big) &= a^{1-\lceil \frac{\operatorname{rk}(\underline{L})}{2} \rceil}\, x \cdot \mathcal{W}_{\mathrm{I}}\big(\operatorname{lev}(\underline{L})^2 D/x^2, a/x^2\big) \\ &= a^{1-\lceil \frac{\operatorname{rk}(\underline{L})}{2} \rceil}\, \mathcal{W}_{\mathrm{I}}\big(\operatorname{lev}(\underline{L})^2 D, a\big),\end{aligned}$$

where in the last step we used Proposition 2.1.3 (see Equation (2.5)). Inserting this into the last formula of $C_{T(\ell)\phi}(D,r)$ gives

$$C_{T(\ell)\phi}(D,r) = \sum_{\substack{a|\ell^2 \\ a^2|\ell^2 \operatorname{lev}(\underline{L})D}} a^{k-1-\lceil \frac{\operatorname{rk}(\underline{L})}{2} \rceil} \mathcal{W}_{\mathrm{I}}\big(\operatorname{lev}(\underline{L})^2 D, a\big) C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' r\big),$$

Again, by using Proposition 2.1.3 and $\gcd(\operatorname{lev}(\underline{L}), a) = 1$, we obtain the claimed formula. □

Proof of Theorem 2.6.3. Recall that for even $\operatorname{rk}(\underline{L})$ the operators $T(\ell)$ and $T_0(\ell)$ are related as follows (see Definition 2.5.7):

$$T(\ell) := \sum_{\substack{d,s>0 \\ sd^2|\ell,\ s \text{ square-free}}} \chi_{\underline{L}}(s)(sd^2)^{k-\frac{\operatorname{rk}(\underline{L})}{2}-2}\, T_0\big(\tfrac{\ell}{sd^2}\big). \tag{2.70}$$

Using Equation (2.70) and Theorem 2.6.8, which gives a closed formula for the action of $T_0(\ell)$ in terms of Fourier coefficients, we can write

$$C_{T(\ell)\phi}(D,r) = \ell^{k-\frac{\operatorname{rk}(\underline{L})}{2}-2} \sum_{\substack{\ell_1|\ell \\ \ell/\ell_1=\text{ perfect square}}} \sum_{\substack{s|\ell_1 \\ s \text{ is square free}}}$$

$$\times \sum_{\substack{d|(\ell_1/s)^2 \\ s^2d^2|\ell_1{}^2\operatorname{lev}(\underline{L})D}} \chi_{\underline{L}}(s)(\ell_1/s)^{\frac{\operatorname{rk}(\underline{L})}{2}+2-k} d^{k-2}\mathscr{W}(\operatorname{lev}(\underline{L})^2D,d)C_\phi\big(\tfrac{\ell_1{}^2}{s^2d^2}D,\tfrac{\ell_1}{s}d'r\big).$$

We want to simplify the above equation. Similarly as in the proof of the preceding theorem (see proof of Theorem 2.6.1), we can reorder the summation as follows (the substitutions here are $a := \frac{s\ell}{\ell_1}d$, $b := a/d = s\ell/\ell_1$):

$$C_{T(\ell)\phi}(D,r) = \sum_{\substack{a|\ell^2 \\ a^2|\ell^2\operatorname{lev}(\underline{L})D}} a^{k-2}C_\phi\big(\tfrac{\ell^2}{a^2}D,\ell a'r\big) \sum_{b|(a,\frac{\ell^2}{a})} b^{-\frac{\operatorname{rk}(\underline{L})}{2}}\mathscr{W}\big(\operatorname{lev}(\underline{L})^2D,\tfrac{a}{b}\big) \sum_{\substack{s|b \\ s \text{ square-free} \\ sb= \text{ perfect square}}} \chi_{\underline{L}}(s).$$

The inner sum $\sum_s \chi_{\underline{L}}(s)$ contains exactly one term. In fact, it equals $\chi_{\underline{L}}(b)$ (since $\chi_{\underline{L}}(s) = \chi_{\underline{L}}(b)$). This gives

$$C_{T(\ell)\phi}(D,r) = \sum_{\substack{a|\ell^2 \\ a^2|\ell^2\operatorname{lev}(\underline{L})D}} a^{k-2}C_\phi\big(\tfrac{\ell^2}{a^2}D,\ell a'r\big) \sum_{b|(a,\frac{\ell^2}{a})} \chi_{\underline{L}}(b)b^{-\frac{\operatorname{rk}(\underline{L})}{2}}\mathscr{W}\big(\operatorname{lev}(\underline{L})^2D,\tfrac{a}{b}\big).$$

Setting $x := \frac{a}{(a,\ell)}$, one has $(a,\frac{\ell^2}{a}) = \frac{a}{x^2}$. The conditions $(\ell,\operatorname{lev}(\underline{L}))=1$, $a \mid \ell^2$, and $a^2 \mid \ell^2\operatorname{lev}(\underline{L})D$ imply that $\frac{\operatorname{lev}(\underline{L})^2D}{x^2} \in \mathbb{Z}$. Thus

$$\begin{aligned} C_{T(\ell)\phi}(D,r) &= \sum_{\substack{a|\ell^2 \\ a^2|\ell^2\operatorname{lev}(\underline{L})D}} a^{k-2}C_\phi\big(\tfrac{\ell^2}{a^2}D,\ell a'r\big)x^{2-\operatorname{rk}(\underline{L})}\mathscr{W}_{\mathrm{II}}\big(\tfrac{\operatorname{lev}(\underline{L})^2D}{x^2},\tfrac{a}{x^2}\big) \\ &= \sum_{a|\gcd(\ell^2,\operatorname{lev}(\underline{L})D)} a^{k-\frac{\operatorname{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(a)C_\phi\big(\tfrac{\ell^2}{a^2}D,\ell a'r\big), \end{aligned}$$

where the first identity follows from Equation (2.21) and Equation (2.5), and the second using Proposition 2.1.4. Now, the proof is complete. □

2.7 Hecke Operators and Euler Products

The eigenfunctions for the Hecke operators $T(\ell)$ ($\ell \in \mathbb{N}_{\underline{L}}$) correspond naturally to Dirichlet series having Euler product expansions. These Dirichlet series, the L-functions of eigenfunctions, will express the connection between Jacobi forms and elliptic modular forms.

Definition 2.7.1. A Jacobi form $\phi \in J_{k,\underline{L}}$ is called a Hecke eigenfunction for the operator $T(\ell)$ if there exists $\lambda(\ell,\phi) \in \mathbb{C}$ with

$$T(\ell)\phi = \lambda(\ell,\phi)\phi.$$

The number $\lambda(\ell) = \lambda(\ell,\phi)$ known as the eigenvalue of $T(\ell)$ with corresponding eigenfunction ϕ.

Definition 2.7.2. Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L,\beta)$ which is an eigenfunction of the Hecke operators $T(\ell)$ for all $\ell \in \mathbb{N}_{\underline{L}}$. The formal L-function of ϕ in s is defined as follows:

$$L(s,\phi) = \sum_{\ell \in \mathbb{N}_{\underline{L}}} \lambda(\ell)\ell^{-s}. \tag{2.71}$$

Definition 2.7.3. Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L,\beta)$ which is an eigenfunction of the Hecke operator $T(\ell)$ for $\ell \in \mathbb{N}_{\underline{L}}$. For a pair $(D,r) \in \mathrm{Supp}(\underline{L})$ such that $\mathrm{lev}(r)D$ is a square free integer, we set

$$F(s,D,r,\phi) := \sum_{\ell \in \mathbb{N}_{\underline{L}}} C_\phi(\ell^2 D, \ell r)\ell^{-s}. \tag{2.72}$$

2.7.1 Even Rank Lattices Case

In this subsection we assume that the rank of the lattice $\underline{L} = (L,\beta)$ is even. First, we will study the multiplicative properties of the operator $T(\ell)$, then we shall use these multiplicative properties to determine the Euler product of the L-functions.

According to Theorem 2.6.3, the operator $T(\ell)$ acts on each $\phi \in J_{k,\underline{L}}$ in terms of Fourier of coefficients as follows:

$$C_{T(\ell)\phi}(D,r) = \sum_{a \mid \ell^2, \mathrm{lev}(\underline{L})D} a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(a) C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' r\big), \tag{2.73}$$

Recall that for any positive integer a we use a' to denote an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L})$.

Theorem 2.7.4. *The Hecke operators $T(\ell)$ on $J_{k,\underline{L}}$ satisfy the following multiplicative relation:*

$$T(m)\cdot T(n) = \sum_{d|m^2,n^2} d^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(d)T\left(\tfrac{mn}{d}\right) \tag{2.74}$$

for every $m,n \in \mathbb{N}_{\underline{L}}$.

To prove Theorem 2.7.4 we need the following lemmas.

Lemma 2.7.5. *Let $m,n \in \mathbb{N}_{\underline{L}}$ be coprime positive integers. Then one has*

$$T(m)T(n) = T(mn).$$

Proof. Let $\phi \in J_{k,\underline{L}}$. By Equation (2.73) one has

$$C_{T(m)(T(n)\phi)}(D,r) = \sum_{a|(m^2,\Delta)} \sum_{b|(n^2,m^2\Delta/a^2)} (ab)^{k_2-1}\chi_{\underline{L}}(ab)C_\phi\left(\tfrac{n^2m^2}{a^2b^2}D, nma'b'r\right),$$

where $k_2 = k - \frac{\mathrm{rk}(\underline{L})}{2}$, and $\Delta = \mathrm{lev}(\underline{L})D$. First we will prove that

$$(n^2,\Delta) = (n^2, m^2\Delta/a^2) \tag{2.75}$$

for all $a|(m^2,\Delta)$. Clearly if $d|(n^2,m^2\Delta/a^2)$, then $d|m^2\Delta$, and as $d|n^2$, we have $(d,m^2) = 1$. It follows that $d \mid (n^2,\Delta)$, i.e., $(n^2,m^2\Delta/a^2) \mid (n^2,\Delta)$. Conversely, suppose that $d|(n^2,\Delta)$. Since $a \mid m^2$ we have $(d,a) = 1$. Now $d|m^2\Delta = a^2\frac{m^2\Delta}{a^2}$, and thus $d|\frac{m^2\Delta}{a^2}$. This shows that $(n^2,\Delta)|(n^2,m^2\Delta/a^2)$. Thus Equation (2.75) holds true and $(m^2n^2,\Delta) = (m^2,\Delta)(n^2,\Delta) = (m^2,\Delta)(n^2,m^2\Delta/a^2)$, where the two factors (m^2,Δ) and $(n^2,m^2\Delta/a^2)$ are coprime. Therefore when a runs over positive divisors of (m^2,Δ) and b runs over positive divisors of $(n^2,m^2\Delta/a^2)$, then ab runs over all positive divisors of (m^2n^2,Δ). This gives

$$C_{T(m)(T(n)\phi)}(D,r) = \sum_{g|(m^2n^2,\Delta)} g^{k_2-1}\chi_{\underline{L}}(g)C_\phi\left(\tfrac{n^2m^2}{g^2}D, nmg'r\right) = C_{T(mn)\phi}(D,r).$$

Now, the proof is complete. □

Lemma 2.7.6. *Let r be a positive integer. For a prime $p \in \mathbb{N}_{\underline{L}}$ one has*

$$T(p^r)\cdot T(p) = T(p^{r+1}) + p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p)T(p^r) + p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1)}T(p^{r-1}). \tag{2.76}$$

Proof. Let $\phi \in J_{k,\underline{L}}$. Then, by Equation (2.73), the action of the left-hand side of Equation (2.76) on ϕ in terms of Fourier coefficients is given by

$$\begin{aligned} C_{T(p^r)(T(p)\phi)}(D,x) &= \sum_{a|(p^{2r},\Delta)} a^{k_2-1}\chi_{\underline{L}}(a)C_{T(p)\phi}(\tfrac{p^{2r}}{a^2}\Delta, p^r a'x) \\ &= \sum_{a|(p^{2r},\Delta)} \sum_{b|(p^2,\frac{p^{2r}}{a^2}\Delta)} (ab)^{k_2-1}\chi_{\underline{L}}(ab)C_\phi(\tfrac{p^{2r+2}}{a^2b^2}\Delta, p^{r+1}a'b'x) \\ &= \sum_e N(e)e^{k_2-1}\chi_{\underline{L}}(e)C_\phi(\tfrac{p^{2r+2}}{e^2}\Delta, p^{r+1}e'x), \end{aligned}$$

where $k_2 = k - \frac{\mathrm{rk}(\underline{L})}{2}$, $\Delta = \mathrm{lev}(\underline{L})D$, and $N(e)$ is the number of ways of writing e as ab in the preceding sum. If such a decomposition exists then $e \mid p^2 a$ and hence $a = \frac{e}{(e,p^2)}\delta$ for some integer δ; writing down the conditions on a and $b = e/a$ we find formula

$$N(e) = \text{number of divisors } \delta \text{ of } \left(\Delta, p^2, e, \Delta\tfrac{p^2}{e}, \tfrac{p^{2r+2}}{e}\right),$$

where $N(e) = 0$ unless $e|(\Delta p^2, p^{2r+2})$, $e^2|\Delta p^{2r+2}$. The action of the right-hand side of Equation (2.76) on ϕ is given by

$$\begin{aligned} &\sum_{d|p^{2r},p^2} d^{k_2-1}\chi_{\underline{L}}(d)C_{T(p^{r+1}/d)\phi}(D,x) \\ &= \sum_{d|p^{2r},p^2} \sum_{a|(p^{2r+2}/d^2,\Delta)} (ad)^{k_2-1}\chi_{\underline{L}}(ad)C_\phi(\tfrac{p^{2r+2}}{a^2d^2}\Delta, p^{r+1}a'd'x) \\ &= \sum_e N''(e)e^{k_2-1}\chi_{\underline{L}}(e)C_\phi(\tfrac{p^{2r+2}}{e^2}\Delta, p^{r+1}e'x), \end{aligned}$$

where now $N''(e)$ counts the decomposition of e as ad satisfying the conditions in the sum; from $e|\Delta d$ we find that $\frac{e}{(\Delta,e)}$ divides d, and writing d as $\frac{e}{(\Delta,e)}\delta$ we obtain for $N''(e)$ the same formula as for $N(e)$. Thus the action of both sides of Equation (2.76) are equal and the proof is complete. □

Lemma 2.7.7. *Let r,s be positive integers. For each prime $p \in \mathbb{N}_{\underline{L}}$ one has*

$$\begin{aligned} T(p^r)T(p^s) &= \sum_{d|p^{2r},p^{2s}} d^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(d)T(p^{r+s}/d) \\ &= \sum_{v=0}^{2\min(r,s)} p^{v(k-\frac{\mathrm{rk}(\underline{L})}{2}-1)}\chi_{\underline{L}}(p^v)T(p^{r+s-v}). \end{aligned} \tag{2.77}$$

Proof. We fix r and prove Equation (2.77) inductively for prime powers p^s. It is obvious that Equation (2.77) holds true for $s=0$ since $T(p^r)T(p^0)=T(p^r)$. For $s=1$ we refer to Lemma 2.7.6. Assume that $s \geq 2$ and Equation (2.77) has been proven for $1, p, \cdots, p^s$. We want to prove it for p^{s+1}. By definition of $T(p^{s+1})$ and Lemma 2.7.6

$$\begin{aligned} T(p^r)T(p^{s+1}) =& T(p^r)T(p^s)T(p) - p^{k_2-1}\chi_{\underline{L}}(p)T(p^r)T(p^s) \\ & - p^{2(k_2-1)}T(p^r)T(p^{s-1}), \end{aligned}$$

where $k_2 = k - \frac{\mathrm{rk}(L)}{2}$. By inductive hypothesis we obtain

$$\begin{aligned} T(p^r)T(p^{s+1}) = & \sum_{v=0}^{\min(2r,2s)} p^{v(k_2-1)}\chi_{\underline{L}}(p^v)T(p^{r+s-v})T(p) \\ & - p^{k_2-1}\chi_{\underline{L}}(p) \sum_{v=0}^{\min(2r,2s)} p^{v(k_2-1)}\chi_{\underline{L}}(p^v)T(p^{r+s-v}) \\ & - p^{2(k_2-1)} \sum_{v=0}^{\min(2r,2s-2)} p^{v(k_2-1)}\chi_{\underline{L}}(p^v)T(p^{r+s-1-v}). \end{aligned}$$

Again, using Lemma 2.7.6, we obtain

$$\begin{aligned} T(p^r)T(p^{s+1}) = & \sum_{v=0}^{\min(2r,2s)} p^{v(k_2-1)}\chi_{\underline{L}}(p^v)T(p^{r+s+1-v}) \\ & + p^{2(k_2-1)} \sum_{v>\min(2r,2s-2)}^{\min(2r,2s)} p^{v(k_2-1)}\chi_{\underline{L}}(p^v)T(p^{r+s-1-v}). \end{aligned}$$

If $s > r$, then the second sum in the right-hand of the above equality is empty and the desired formula follows since $\min(2r,2s) = \min(2r,2(s+1))$.

If $s \leq r$, then this second sum contains exactly two terms ($v \in \{2s-1, 2s\}$). Thus

$$\begin{aligned} T(p^r)T(p^{s+1}) = & \sum_{v=0}^{\min(2r,2s)} p^{v(k_2-1)}\chi_{\underline{L}}(p^v)T(p^{r+s+1-v}) \\ & + p^{(2s+1)(k_2-1)}\chi_{\underline{L}}(p)T(p^{r-s}) + p^{(2s+2)(k_2-1)}T(p^{r-s+1}) \\ = & \sum_{v=0}^{\min(2r,2(s+1))} p^{v(k_2-1)}\chi_{\underline{L}}(p^v)T(p^{r+s+1-v}) \end{aligned}$$

which is what we wanted. □

Proof of Theorem 2.7.4. Let $m = \prod_p p^r$ and $n = \prod_p p^s$ be the expressions as powers of prime numbers. Then by Lemma 2.7.5

$$\sum_{d|m^2,n^2} d^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(d)T\left(\tfrac{mn}{d}\right) = \prod_p \Big(\sum_{v=0}^{2\min(r,s)} p^{v(k-\frac{\mathrm{rk}(\underline{L})}{2}-1)}\chi_{\underline{L}}(p^v)T(p^{r+s-v})\Big).$$

Therefore we have only to prove Theorem 2.7.4 when m and n are powers of a prime p. For this we refer to Lemma 2.7.7. Now, the proof is complete. □

As an easy consequence of this theorem, we obtain an important insight into the arithmetic proprieties of the eigenfunctions.

Proposition 2.7.8. *Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L,\beta)$ which is an eigenfunction of the Hecke operators $T(\ell)$ for all $\ell \in \mathbb{N}_{\underline{L}}$. For each prime number $p \in \mathbb{N}_{\underline{L}}$ we set:*

$$g_p(\phi) := \lambda(p) - p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p).$$

The L-function $L(s,\phi)$ has the product expansion of the form

$$L(s,\phi) = \frac{L_{\mathrm{lev}(\underline{L})}(s-k+\frac{\mathrm{rk}(\underline{L})}{2}+1,\chi_{\underline{L}})}{L_{\mathrm{lev}(\underline{L})}(2s-2k+\mathrm{rk}(\underline{L})+2,\chi_{\underline{L}}^2)} \prod_{\substack{p\in\mathbb{N}_{\underline{L}} \\ p \text{ prime}}} \left(1 - g_p(\phi)p^{-s} + p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s)}\right)^{-1}.$$

Proof. Since $\lambda(\ell_1)\cdot\lambda(\ell_2) = \lambda(\ell_1\ell_2)$ if $(\ell_1,\ell_2) = 1$, we can write

$$L(s,\phi) = \prod_{p\in\mathbb{N}_{\underline{L}},\, p\,\text{prime}} L_p(s,\phi),$$

where for a prime p, the p-local zeta function of ϕ is given by

$$L_p(s,\phi) = \sum_{v=0}^{\infty} \frac{\lambda(p^v)}{p^{vs}}.$$

One has, using Theorem 2.7.4, the following multiplicative relation:

$$\lambda(p)\cdot\lambda(p^v) = \lambda(p^{v+1}) + p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p)\lambda(p^v) + p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1)}\lambda(p^{v-1}).$$

Thus

$$\lambda(p)L_p(s,\phi)$$

$$
\begin{aligned}
&=\lambda(p)+\sum_{v=1}^{\infty}\frac{\lambda(p)\lambda(p^v)}{(p^v)^s}\\
&=\lambda(p)+\sum_{v=1}^{\infty}\frac{\lambda(p^{v+1})}{(p^v)^s}+p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p)\sum_{v=1}^{\infty}\frac{\lambda(p^v)}{(p^v)^s}+p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1)}\sum_{v=1}^{\infty}\frac{\lambda(p^{v-1})}{(p^v)^s}\\
&=\lambda(p)+\big(L_p(s,\phi)-1-\tfrac{\lambda(p)}{p^s}\big)p^s+p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p)\big(L_p(s,\phi)-1\big)\\
&\quad+p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1)}p^{-s}L_p(s,\phi).
\end{aligned}
$$

Hence

$$
\begin{aligned}
L_p(s,\phi)&=\big(1+p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s}\chi_{\underline{L}}(p)\big)\big(1+p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s}\chi_{\underline{L}}(p)-\lambda(p)p^{-s}+p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s)}\big)^{-1}\\
&=\frac{1-p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s)}\chi_{\underline{L}}(p)^2}{1-p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s}\chi_{\underline{L}}(p)}\big(1-\big(\lambda(p)-p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p)\big)p^{-s}+p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s)}\big)^{-1}
\end{aligned}
$$

as stated in the proposition. □

Theorem 2.7.9. *For each $r\in L^{\#}$ and $D\leq 0$ such that $D\equiv\beta(r)\bmod\mathbb{Z}$ and $\mathrm{lev}(r)D$ is a square-free integer. Setting $k_2:=k-\frac{\mathrm{rk}(\underline{L})}{2}$, one has*

$$
\Big(\sum_{\substack{\ell\mid\mathrm{lev}(\underline{L})D\\(\ell,\mathrm{lev}(\underline{L}))=1}}\chi_{\underline{L}}(\ell)\ell^{k_2-1-s}\Big)\sum_{\substack{\ell\geq 1\\(\ell,\mathrm{lev}(\underline{L}))=1}}C_\phi\big(\ell^2D,\ell r\big)\,\ell^{-s}
=C_\phi(D,r)\prod_{p\nmid\mathrm{lev}(\underline{L})}\frac{1+\chi_{\underline{L}}(p)p^{k_2-1-s}}{1-\big(\lambda(p)-p^{k_2-1}\chi_{\underline{L}}(p)\big)p^{-s}+p^{2(k_2-1-s)}}.
$$

The products are over all primes p not dividing $\mathrm{lev}(\underline{L})$.

Proof. According to Theorem 2.6.3 if $\mathrm{lev}(r)D$ is square-free, we have

$$
\sum_{a\mid\ell,\mathrm{lev}(\underline{L})D}a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(a)C_\phi\big(\tfrac{\ell^2}{a^2}D,\ell a'r\big)=\lambda(\ell)C_\phi(D,r). \tag{2.78}
$$

Thus by Definition 2.7.2 of the Lfunction of ϕ , one has

$$
\begin{aligned}
C_\phi(D,r)\sum_{\ell\in\mathbb{N}_{\underline{L}}}\lambda(\ell)\ell^{-s}&=\sum_{\ell\in\mathbb{N}_{\underline{L}}}\sum_{a\mid\ell}a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(a)\boldsymbol{\delta}(a\mid\mathrm{lev}(\underline{L})D)C_\phi\big(\tfrac{\ell^2}{a^2}D,\tfrac{\ell}{a}r\big)\ell^{-s}\\
&=\Big(\sum_{\ell\in\mathbb{N}_{\underline{L}}}\ell^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(\ell)\boldsymbol{\delta}(\ell\mid\mathrm{lev}(\underline{L})D)\ell^{-s}\Big)\Big(\sum_{\ell\in\mathbb{N}_{\underline{L}}}C_\phi\big(\ell^2D,\ell r\big)\ell^{-s}\Big).
\end{aligned}
$$

Now, Proposition 2.7.8,which describes $L(s,\phi)$ as an Euler product, completes the proof. □

Remark 2.7.10. If we think of an elliptic modular form of weight $k_2 = k - \frac{\mathrm{rk}(\underline{L})}{2}$, with nebentypus, say, $\chi_{\underline{L}}\xi$, and with Hecke eigenvalues $\gamma(\ell)$, then $\sum_\ell \overline{\xi(\ell)}\gamma(\ell^2)\ell^{-s}$ (taken over all ℓ coprime to $\mathrm{lev}(\underline{L})$) equals $L(\phi,s)$ if we replace $\lambda(p)$ by $\overline{\xi(p)}\gamma(p^2)$. This suggests, for each ξ and suitable levels m, the existence of maps from $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(m,\chi_{\underline{L}}\xi)$ to $J_{k,\underline{L}}$ such that $T(\ell^2)$ on the left corresponds to $\xi(\ell)T(\ell)$ on the Jacobi form side. We shall construct in this thesis examples for such maps.

2.7.2 Odd Rank Lattices Case

In this subsection we assume that the rank of the lattice $\underline{L} = (L,\beta)$ is odd.

Theorem 2.7.11. *The Hecke operators $T(\ell)$ on $J_{k,\underline{L}}$ satisfy the following multiplicative relation:*

$$T(m)\cdot T(n) = \sum_{d|m,n} d^{2k-\mathrm{rk}(\underline{L})-2}\,T\left(\frac{mn}{d^2}\right) \tag{2.79}$$

for every $m,n \in \mathbb{N}_{\underline{L}}$.

To prove this theorem we need the following lemmas

Lemma 2.7.12. *Let r be a positive integer. For a prime $p \in \mathbb{N}_{\underline{L}}$ one has*

$$T(p^r)T(p) = T(p^{r+1}) + p^{2k-\mathrm{rk}(\underline{L})-2}\,T(p^{r-1}). \tag{2.80}$$

Proof. The action of the left-hand side of Equation (2.80) in terms of Fourier coefficients is given by Theorem 2.6.1 as follows:

$$C_{T(p^r)(T(p)\phi)}(D,s) = \sum_a\sum_b (ab)^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\varrho(D,a)\varrho(\tfrac{p^{2r}}{a^2}D,b)C_\phi\big(\tfrac{p^{2r}p^2}{a^2b^2}D,p^rp(ab)'r\big).$$

In the first sum a runs over all positive divisors of p^{2r} such that $a^2 \mid p^{2r}\,\mathrm{lev}(\underline{L})D$. In the second sum b runs over all positive divisors of p^2 such that $b^2 \mid \frac{p^{2r+2}}{a^2}\,\mathrm{lev}(\underline{L})D$. We set $h := k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1$, and $\Delta := \mathrm{lev}(\underline{L})D$. The previous equation can be written as

$$C_{T(p^r)(T(p)\phi)}(D,s) = \Omega_1+\Omega_2+\Omega_3,$$

where

$$\Omega_1 = \sum_{a|p^{2r}} a^h \varrho(D,a) C_\phi\big(\tfrac{p^{2r+2}}{a^2}D, p^{r+1}a's\big)\boldsymbol{\delta}(a^2|p^{2r+2}D),$$

$$\Omega_2 = \sum_{a|p^{2r}} (ap)^h \varrho(D,a)\varrho(\tfrac{p^{2r}}{a^2}D,p) C_\phi\big(\tfrac{p^{2r}}{a^2}D, p^{r}a's\big)\boldsymbol{\delta}(a^2|p^{2r}\Delta),$$

$$\Omega_3 = \sum_{a|p^{2r}} (ap^2)^h \varrho(D,a)\varrho(\tfrac{p^{2r}}{a^2}D,p^2) C_\phi\big(\tfrac{p^{2r-2}}{a^2}D, p^{r-1}a's\big)\boldsymbol{\delta}(a^2|p^{2r-2}\Delta).$$

The first sum can be written as $\Omega_1 = C_{T(p^{r+1})\phi}(D,s) - A_1 - B_1$, where

$$A_1 = p^{(2r+1)h}\varrho(D,p^{2r+1})C_\phi\big(\tfrac{D}{p^{2r}}, p^{r+1}(p^{2r+1})'s\big)\boldsymbol{\delta}(p^{2r}|\Delta),$$
$$B_1 = p^{(2r+2)h}\varrho(D,p^{2r+2})C_\phi\big(\tfrac{D}{p^{2r+2}}, p^{r+1}(p^{2r+2})'s\big)\boldsymbol{\delta}(p^{2r+2}|\Delta).$$

Also, the third sum Ω_3 can be written as $\Omega_3 = A_3 + B_3 + C_3$, where

$$A_3 = p^{(2r+1)h}\varrho(D,p^{2r-1})\varrho(\tfrac{D}{p^{2r-2}},p^2)C_\phi(\tfrac{D}{p^{2r}}, p^{r+1}(p^{2r+1})'s)\boldsymbol{\delta}(p^{2r}|\Delta),$$
$$B_3 = p^{(2r+2)h}\varrho(D,p^{2r})\varrho(\tfrac{D}{p^{2r}},p^2)C_\phi(\tfrac{D}{p^{2r+2}}, p^{r+1}(p^{2r+2})'s)\boldsymbol{\delta}(p^{2r+2}|\Delta)$$
$$C_3 = \sum_{a|p^{2r-2}} (ap^2)^h \varrho(D,a)\varrho(\tfrac{p^{2r}}{a^2}D,p^2) C_\phi\big(\tfrac{p^{2r-2}}{a^2}D, p^{r-1}a's\big)\boldsymbol{\delta}(a^2|p^{2r-2}\Delta).$$

The condition $p^{2r}|\Delta$ in A_3 implies that $\varrho(D,p^{2r-1}) = 0$, and then $A_3 = 0$. The condition $a^2|p^{2r-2}\Delta$ in C_3 implies that $(\frac{p^{2r}}{a^2}\Delta, p^2) = p^2$, $\varrho(\frac{p^{2r}}{a^2}D,p^2) = p$, and then $C_3 = p^{2k-\mathrm{rk}(\underline{L})-2}C_{T(p^{r-1}\phi)}(D,s)$. Inserting this into the last formula of Ω_3 gives

$$\Omega_3 = p^{2k-\mathrm{rk}(\underline{L})-2}C_{T(p^{r-1}\phi)}(D,s) + B_3.$$

In fact, the condition $p^{2r+2}|\Delta$ in B_3 and B_1 implies that $\varrho(D,p^{2r})\varrho(\frac{D}{p^{2r}},p^2) = p^r p = \varrho(D,p^{2r+2})$, and then $B_1 = B_3$. Thus

$$\begin{aligned} C_{T(p^r)(T(p)\phi)}(D,s) &= \Omega_1 + \Omega_2 + \Omega_3 \\ &= C_{T(p^{r-1}\phi)}(D,s) + p^{2k-\mathrm{rk}(\underline{L})-2}C_{T(p^{r-1}\phi)}(D,s) + \Omega_2 - A_1. \end{aligned}$$

To complete the proof we still need to show that $\Omega_2 = A_1$. It is obvious that both A_1 and Ω_2 are 0 unless $\mathrm{ord}_p(\Delta) = 2r$. If $\mathrm{ord}_p(\Delta) = 2r$, then the sum over a in Ω_2 contains exactly one term ($a = p^{2r}$), i.e.,

$$\Omega_2 = p^{(2r+2)h}\varrho(D,p^{2r})\varrho(\tfrac{D}{p^{2r}},p)C_\phi(\tfrac{D}{p^{2r}}, p^{r+1}(p^{2r+1})'s)\boldsymbol{\delta}(\mathrm{ord}_p(\Delta) = 2r).$$

Since $\mathrm{ord}_p(\Delta) = 2r$, one has $\varrho(D,p^{2r})\varrho(\frac{D}{p^{2r}},p) = p^r = \varrho(D,p^{2r+1})$, and then $A_1 = \Omega_2$. The proof is complete. □

Lemma 2.7.13. *Let $m,n \in \mathbb{N}_{\underline{L}}$ be coprime positive integers. Then one has*

$$T(m)T(n) = T(mn).$$

Proof. We omit the proof, which is similar to the proof of Lemma 2.7.5. □

Proof of Theorem 2.7.11. Let $m = \prod_p p^r$ and $n = \prod_p p^s$ be the expressions as powers of prime numbers. Then by Lemma 2.7.13 one has

$$\sum_{d|m,n} d^{2k-\mathrm{rk}(\underline{L})-2} T\left(\frac{mn}{d^2}\right) = \prod_p \Big(\sum_{v=0}^{\min(r,s)} p^{v(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s-2v}) \Big).$$

Therefore we have only to prove Theorem 2.7.11 when $m = p^r$ and $n = p^s$ are powers of a prime p. We fix r and prove inductively for prime powers p^s. For $s = 1$ we refer to Lemma 2.7.12. Assume that $s \geq 2$ and the equation

$$T(p^r)T(p^s) = \sum_{v=0}^{\min(r,s)} p^{v(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s-2v})$$

has been proven for $1, p, \cdots, p^s$. We want to prove it for p^{s+1}. One has

$$\begin{aligned} T(p^r)T(p^{s+1}) &= T(p^r)\big(T(p^s)T(p) - p^{2k-\mathrm{rk}(\underline{L})-2}T(p^{s-1})\big) \\ &= \sum_{v=0}^{\min(r,s)} p^{v(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s-2v})T(p) + p^{2k-\mathrm{rk}(\underline{L})-2} \sum_{v=0}^{\min(r,s-1)} p^{v(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s-1-2v}), \end{aligned}$$

again, by using Lemma 2.7.12, we obtain

$$\begin{aligned} T(p^r)T(p^{s+1}) = &\sum_{v=0}^{\min(r,s)} p^{v(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s+1-2v}) \\ &+ \sum_{\substack{v>\min(r,s-1) \\ 2v+1 \leq r+s}}^{\min(r,s)} p^{(v+1)(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s-1-2v}), \end{aligned}$$

where the second sum contains exactly one term ($v = s$) if $r > s$, and is empty otherwise. This gives

$$\begin{aligned} T(p^r)T(p^{s+1}) &= \sum_{v=0}^{\min(r,s)} p^{v(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s+1-2v}) + p^{(s+1)(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r-s-1}) \\ &= \sum_{v=0}^{\min(r,s+1)} p^{v(2k-\mathrm{rk}(\underline{L})-2)} T(p^{r+s+1-2v}). \end{aligned}$$

Now, the proof is complete. □

Lemma 2.7.14. *Let* $(D,r) \in \operatorname{supp}(\underline{L})$. *If* $\operatorname{lev}(r)D$ *is a square-free integer, then for all* $\ell \in \mathbb{N}_{\underline{L}}$ *we have the following multiplicative relation:*

$$C_\phi\big(\ell^2 D, \ell r\big) = C_\phi(D,r) \sum_{a \mid \ell} a^{k - \lceil \frac{\operatorname{rk}(\underline{L})}{2} \rceil - 1} \chi_{\underline{L}}(D,a)\mu(a)\lambda(\ell/a). \tag{2.81}$$

Proof. By Remark 2.6.2 we have

$$\sum_{a \mid \ell} a^{k - \lceil \frac{\operatorname{rk}(\underline{L})}{2} \rceil - 1} \chi_{\underline{L}}(D,a) C_\phi\big(\tfrac{\ell^2}{a^2} D, \tfrac{\ell}{a} r\big) = \lambda(\ell) C_\phi(D,r). \tag{2.82}$$

Now, the claimed formula follows from this via Möbius inversion. □

Proposition 2.7.15. *Let* ϕ *be a Jacobi form of weight* k *and index* $\underline{L} = (L,\beta)$ *which is an eigenfunction of the Hecke operators* $T(\ell)$ *for all* $\ell \in \mathbb{N}_{\underline{L}}$. *One has the product expansion of the form*

$$L(s,\phi) = \prod_{\substack{p \in \mathbb{N}_{\underline{L}} \\ p \ \text{prime}}} \big(1 - p^{-s}\lambda(p) + p^{2k - \operatorname{rk}(\underline{L}) - 2 - 2s}\big)^{-1}. \tag{2.83}$$

Proof. Since $\lambda(m) \cdot \lambda(n) = \lambda(mn)$ if $(m,n) = 1$, we can write

$$L(s,\phi) = \prod_{\substack{p \in \mathbb{N}_{\underline{L}} \\ p \ \text{prime}}} L_p(s,\phi),$$

where for a prime p, the p-local L-function of ϕ is given by

$$L_p(s,\phi) = \sum_{v=0}^{\infty} \lambda(p^v) p^{-vs}.$$

By Theorem 2.7.11, we have

$$\begin{aligned}
&\lambda(p) L_p(s,\phi) \\
=&\lambda(p) + \sum_{v=1}^{\infty} \frac{\lambda(p)\lambda(p^v)}{p^{vs}} \\
=&\lambda(p) + p^s \sum_{v=1}^{\infty} \frac{\lambda(p^{v+1})}{p^{(v+1)s}} + p^{(2k-1-\operatorname{rk}(\underline{L}))-1-s} \sum_{v=1}^{\infty} \frac{\lambda(p^{v-1})}{p^{(v-1)s}} \\
=&\lambda(p) + \big(-1 - \tfrac{\lambda(p)}{p^s}\big)p^s + p^s L_p(s,\phi) + p^{(2k-1-\operatorname{rk}(\underline{L}))-1-s} L_p(s,\phi).
\end{aligned}$$

This gives $L_p(s,\phi) = \big(1 - p^{-s}\lambda(p) + p^{2(k - \frac{\operatorname{rk}(\underline{L})}{2} - 1 - s)}\big)^{-1}$, which completes the proof.

□

Remark 2.7.16. If the ϕ in Proposition 2.7.15 lifts to an elliptic modular form f of weight $k_1 = 2k - 1 - \mathrm{rk}(\underline{L})$, then $L(s,\phi)$ should be (up to a finite number of Euler factors) the L-series of f. We observe that the right-hand side of Equation (2.83) has indeed the right shape (compare with Equation (1.8)).

Theorem 2.7.17. *Let* $(D,r) \in \mathrm{Supp}(\underline{L})$ *such that* $\mathrm{lev}(r)D$ *is a square-free integer. One has the following factorization:*

$$L_{\mathrm{lev}(\underline{L})}\big(s-k+\lceil \mathrm{rk}(\underline{L})/2\rceil + 1, \chi_{\underline{L}}(D,\cdot)\big)\, F(s,D,r,\phi) = C_\phi(D,r)\, L(s,\phi). \qquad (2.84)$$

Proof. By Lemma 2.7.14 one has

$$\begin{aligned} F(s,D,r,\phi) &= \sum_{\ell\in\mathbb{N}_{\underline{L}}} C_\phi(\ell^2 D, \ell r)\ell^{-s} \\ &= C_\phi(D,r) \sum_{\ell\in\mathbb{N}_{\underline{L}}} \Big(\sum_{\delta\mid\ell} \delta^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1} \chi_{\underline{L}}(D,\delta)\mu(\delta)\lambda(\ell/\delta)\Big)\ell^{-s}, \end{aligned}$$

which can be written using the Dirichlet convolution as

$$\begin{aligned} F(s,D,r,\phi) &= C_\phi(D,r)\Big(\sum_{\ell\in\mathbb{N}_{\underline{L}}} \ell^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\chi_{\underline{L}}(D,\ell)\mu(\ell)\ell^{-s}\Big)\Big(\sum_{\ell\in\mathbb{N}_{\underline{L}}} \lambda(\ell)\ell^{-s}\Big) \\ &= C_\phi(D,r)\, L(s,\phi)\, L_{\mathrm{lev}(\underline{L})}\big(s-k+\lceil\tfrac{\mathrm{rk}(\underline{L})}{2}\rceil+1, \chi_{\underline{L}}(D,\cdot)\big)^{-1}. \end{aligned}$$

In the last step we applied the Möbius inversion to pull out the reciprocal of the Dirichlet L-function. □

This page is intentionally left blank.

Chapter 3

Basis of Simultaneous Eigenforms

The aim of this chapter is to show that there exist bases of simultaneous Hecke eigenforms (i.e., bases consisting of functions, which are eigenforms to all Hecke operators $T(\ell)$ (ℓ coprime to $\operatorname{lev}(\underline{L})$)) both of the subspace of Jacobi cusp forms $S_{k,\underline{L}}$ and of the subspace of Jacobi-Eisenstein series $E_{k,\underline{L}}$ (to be defined in section 3.3). To that end we will define a scalar product on $S_{k,\underline{L}}$, called the Petersson Inner Product, and show that all Hecke operators are Hermitian with respect to that product. The rest will follow readily via some linear algebra.

3.1 The Action of the Orthogonal Group

Let $\underline{L} = (L, \beta)$ be a positive definite even lattice. Let $D_{\underline{L}} = (L^{\#}/L, \beta)$ be the associated discriminant form. The orthogonal group $O(D_{\underline{L}})$ consists of all automorphisms α of $L^{\#}/L$ such that $\beta \circ \alpha = \beta$.

Proposition 3.1.1. *$O(D_{\underline{L}})$ acts on $J_{k,\underline{L}}$ from left as follows:*

$$(\alpha, \phi) \mapsto W(\alpha)\phi,$$

where, for

$$\phi(\tau, z) = \sum_{x \in L^{\#}/L} h_x(\tau)\vartheta_{\underline{L},x}(\tau, z),$$

we set

$$\big(W(\alpha)\phi\big)(\tau,z) := \sum_{x\in L^\#/L} h_{\alpha(x)}(\tau)\vartheta_{\underline{L},x}(\tau,z).$$

Proof. We prove, first of all, that $W(\alpha)\phi$ is a Jacobi form of weight k and index $\underline{L}=(L,\beta)$. The action of $W(\alpha)$ maps the collection of tuples $(h_x)_{x\in L^\#/L}$ into itself by the permutation $\big(h_x\big)_{x\in L^\#/L} \mapsto \big(h_{\alpha(x)}\big)_{x\in L^\#/L}$. Since $\beta\circ\alpha=\beta$, it is clear that the matrix representations in Proposition 2.4.9 are preserved under this permutation, i.e., for each $\alpha\in O(D_{\underline{L}})$

$$\big(W(\alpha)\phi\big)(\tau,z) = \sum_{x\in L^\#/L} h_{\alpha(x)}(\tau)\vartheta_{\underline{L},x}(\tau,z)$$

is a Jacobi form of weight k and index $\underline{L}=(L,\beta)$. Next, we prove the group axioms. Let $\mathbf{1}\in O(D_{\underline{L}})$ denote the trivial automorphism. It is clear that

$$\big(W(\mathbf{1})\phi\big)(\tau,z) = \phi(\tau,z).$$

For $\alpha,\gamma\in O(D_{\underline{L}})$ we have

$$\begin{aligned}
&\big(W(\alpha)\big(W(\gamma)\phi\big)\big)(\tau,z)\\
=&\sum_{x\in L^\#/L} h_{\alpha(\gamma(x))}(\tau)\vartheta_{\underline{L},x}(\tau,z)\\
=&\sum_{x\in L^\#/L} h_{(\alpha\gamma)(x)}(\tau)\vartheta_{\underline{L},x}(\tau,z)\\
=&\big(W(\alpha\gamma)\phi\big)(\tau,z).
\end{aligned}$$

Now, the proof is complete. □

Theorem 3.1.2. *The action of the orthogonal group on the vector space of Jacobi forms commutes with the action of the Hecke operators $T(\ell)$ $(\ell\in\mathbb{N}_{\underline{L}})$, i.e.,*

$$T(\ell)\big(W(\alpha)\phi\big) = W(\alpha)\big(T(\ell)\phi\big) \qquad (\phi\in J_{k,\underline{L}})$$

for all $\ell\in\mathbb{N}_{\underline{L}}$ and $\alpha\in O(D_{\underline{L}})$.

Proof. For each pair $(D,x)\in\operatorname{supp}(\underline{L})$ we have the identity $C_{W(\alpha)\phi}(D,x) = C_\phi(D,\alpha(x))$. If $\operatorname{rk}(\underline{L})$ is even, one has

$$C_{W(\alpha)(T(\ell)\phi)}(D,x) = C_{T(\ell)\phi}(D,\alpha(x))$$

$$= \sum_{a|\gcd(\ell^2, \mathrm{lev}(\underline{L})D)} a^{k-1-\frac{\mathrm{rk}(\underline{L})}{2}} \chi_{\underline{L}}(a) C_\phi\big(\tfrac{\ell^2}{a^2}D, \ell a' \alpha(x)\big)$$
$$= \sum_{a|\gcd(\ell^2, \mathrm{lev}(\underline{L})D)} a^{k-1-\frac{\mathrm{rk}(\underline{L})}{2}} \chi_{\underline{L}}(a) C_{W(\alpha)\phi}\big(\tfrac{\ell^2}{a^2}D, \ell a' x\big)$$
$$= C_{T(\ell)(W(\alpha)\phi)}(D, x).$$

The case of odd rank lattices can be proved exactly in the same way, so we will omit its proof. □

3.2 Jacobi Cusp Forms and Hecke Operators

For $\tau \in \mathfrak{H}$, $z \in L \otimes_{\mathbb{Z}} \mathbb{C}$, let $\tau = u + iv$ and $z = x + iy$ be the decompositions into real and imaginary parts. We define a volume element on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ by

$$dV_{\underline{L},(\tau,z)} := v^{-\mathrm{rk}(\underline{L})-2} du\, dv\, dx\, dy.$$

The volume element $dV_{\underline{L},(\tau,z)}$ is invariant under the action of $J_{\underline{L}}(\mathbb{Q})$ on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$ (see e.g. [Zie89, p. 202]).

Definition 3.2.1. Let ϕ and ψ be $|_{k,\underline{L}}$-invariant under a subgroup Λ of $J_{\underline{L}}(\mathbb{Z})$ of finite index. We set

$$\omega_{\phi,\psi}(\tau, z) := \phi(\tau, z)\overline{\psi(\tau, z)} v^k e^{-4\pi\beta(y).v^{-1}}. \tag{3.1}$$

Lemma 3.2.2. *One has* $\omega_{\phi|_{k,\underline{L}} g, \psi|_{k,\underline{L}} g}(\tau, z) = \omega_{\phi,\psi}(g(\tau, z))$ *for all* $g \in J_{\underline{L}}(\mathbb{Q})$*. In particular the function* $\omega_{\phi,\psi}(\tau, z)$ *is* Λ*-invariant (i.e.,* $\omega_{\phi,\psi}(M(\tau, z)) = \omega_{\phi,\psi}(\tau, z)$ *for all* $M \in \Lambda$*).*

Proof. See e.g. [Bri04, Lemma 2.23]. □

Definition 3.2.3. We set

$$\mathscr{F}_{J_{\underline{L}}(\mathbb{Z})} = \big\{(\tau, z) \in \mathfrak{H} \times (L \otimes \mathbb{C}) \mid \tau \in F_\Gamma, z \text{ in a fundamental mesh for } L \otimes \mathbb{C}/\tau L + L\big\} \\ /\big\{(\tau, z) \mapsto (\tau, -z)\big\},$$

where

$$F_\Gamma := \{\tau \in \mathfrak{H} \mid -1/2 \le \mathrm{Re}\,\tau \le 1/2, \text{ and } |\tau| \ge 1\}$$

denotes the classical fundamental domain for the operation of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Lemma 3.2.4. *$\mathscr{F}_{J_{\underline{L}}(\mathbb{Z})}$ is a fundamental domain of the action of $J_{\underline{L}}(\mathbb{Z})$ on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$*

Proof. See e.g. [Bri04, Remark 2.25]. □

Definition 3.2.5. Let ϕ and ψ be $|_{k,\underline{L}}$-invariant under a subgroup Λ of finite index in $J_{\underline{L}}(\mathbb{Z})$. Suppose that ϕ or ψ is a cusp form. We define the Petersson scalar product of ϕ and ψ with respect to Λ by

$$<\phi,\psi>_\Lambda := \frac{1}{[J_{\underline{L}}(\mathbb{Z}):\Lambda]} \int_{\mathscr{F}_\Lambda} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)}, \tag{3.2}$$

where $\mathscr{F}_\Lambda$ denotes a fundamental domain of the action of Λ on $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$.

Remark 3.2.6. Writing $J_{\underline{L}}(\mathbb{Z}) = \cup_M \Lambda M$ as decompositions into classes, then

$$<\phi,\psi>_\Lambda = \frac{1}{[J_{\underline{L}}(\mathbb{Z}):\Lambda]} \sum_{M \in \Lambda \backslash J_{\underline{L}}(\mathbb{Z})} \int_{M \mathscr{F}_{J_{\underline{L}}(\mathbb{Z})}} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)}. \tag{3.3}$$

Proposition 3.2.7. *The integral in (3.2) is absolutely convergent. The scalar product (3.2) does not depend on the choice of the fundamental domain $\mathscr{F}_\Lambda$. Moreover, the scalar product (3.2) is positive definite.*

Proof. See e.g. [Zie89, p 202-203]. □

Proposition 3.2.8. *The scalar product (3.2) does not depend on the choice of Λ, i.e., if Λ' is another subgroup of finite index in $J_{\underline{L}}(\mathbb{Z})$ such that ϕ and ψ are $|_{k,\underline{L}}$-invariant under Λ', then*

$$<\phi,\psi>_\Lambda = <\phi,\psi>_{\Lambda'} .$$

Proof. Let $\mathscr{F}_{\Lambda\cup\Lambda'}$ be a fundamental domain of the action of $\Lambda \cup \Lambda'$. Write $\Lambda \cup \Lambda' = \cup_v \Lambda M_v = \cup_{v'} \Lambda' M_{v'}$ as decompositions into classes. Then

$$\mathscr{F}_\Lambda := \bigcup_v M_v \mathscr{F}_{\Lambda\cup\Lambda'}, \quad \mathscr{F}_{\Lambda'} := \bigcup_{v'} M_{v'} \mathscr{F}_{\Lambda\cup\Lambda'}$$

are fundamental domains of the action of Λ and Λ' respectively. This gives

$$<\phi,\psi>_\Lambda = \frac{1}{[J_{\underline{L}}(\mathbb{Z}):\Lambda]} \int_{\mathscr{F}_\Lambda} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)}$$

$$
\begin{aligned}
&= \frac{1}{[J_{\underline{L}}(\mathbb{Z}):\Lambda]} \sum_v \int_{M_v \mathscr{F}_{\Lambda\cup\Lambda'}} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)} \\
&= \frac{[\Lambda\cup\Lambda':\Lambda]}{[J_{\underline{L}}(\mathbb{Z}):\Lambda]} \frac{[J_{\underline{L}}(\mathbb{Z}):\Lambda\cup\Lambda']}{[J_{\underline{L}}(\mathbb{Z}):\Lambda\cup\Lambda']} \int_{\mathscr{F}_{\Lambda\cup\Lambda'}} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)} \\
&= \frac{1}{[J_{\underline{L}}(\mathbb{Z}):\Lambda\cup\Lambda'][\Lambda\cup\Lambda':\Lambda']} \sum_{v'} \int_{M_{v'} \mathscr{F}_{\Lambda\cup\Lambda'}} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)} \\
&= \frac{1}{[J_{\underline{L}}(\mathbb{Z}):\Lambda']} \int_{\mathscr{F}_{\Lambda'}} \omega_{\phi,\psi}(\tau,z) dV_{\underline{L},(\tau,z)} \\
&= <\phi,\psi>_{\Lambda'}
\end{aligned}
$$

which completes the proof. □

Definition 3.2.9 (Notation)**.** Since $<\phi,\psi>_\Lambda$ does not depend on the choice of Λ (see Proposition 3.2.8), we often write it simply $<\phi,\psi>$.

Proposition 3.2.10. *Let*

$$
\phi = \sum_{x\in L^\#/L} h_x \vartheta_{\underline{L},x} \quad , \quad \psi = \sum_{x\in L^\#/L} g_x \vartheta_{\underline{L},x}
$$

be two Jacobi forms in $J_{k,\underline{L}}$ and at least one of ϕ and ψ is a cusp form, then

$$
<\phi(\tau,z),\psi(\tau,z)>= 2^{-\frac{\mathrm{rk}(\underline{L})}{2}} \left(\det(\underline{L})\right)^{-\frac{1}{2}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \sum_{x\in L^\#/L} h_x(\tau)\overline{g_x(\tau)} v^{k-\frac{\mathrm{rk}(\underline{L})}{2}-2} du dv. \tag{3.4}
$$

In other words, the Petersson scalar product of ϕ and ψ is equal (up to constant) to the Petersson product in the usual sense of the vector-valued modular forms $\vec{h}=(h_x)_{x\in L^\#/L}$ and $\vec{g}=(g_x)_{x\in L^\#/L}$ of weight $k-\frac{\mathrm{rk}(\underline{L})}{2}$.

Remark 3.2.11. Note that $\sum_{x\in L^\#/L} h_x(\tau)\overline{g_x(\tau)} v^{k-\frac{\mathrm{rk}(\underline{L})}{2}-2}$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$, which follows from the fact the Weil representation $\rho_{\underline{L}}$ is unitary.

Proof of Proposition 3.2.10. In the scalar index case, a proof can be found in [EZ85, P.61]. For the higher-rank case, one can prove (using the computations in [BK93, P.504]) the following statement: in a fixed fiber ($\tau\in\mathfrak{H}$ fixed), the scalar product of $\vartheta_{\underline{L},\mu}, \vartheta_{\underline{L},\lambda}$ $(\mu,\lambda\in L^\#/L)$ is equal to

$$
\int_{(L\otimes_{\mathbb{Z}}\mathbb{C})/(L+\tau L)} \vartheta_{\underline{L},\mu}(\tau,z)\vartheta_{\underline{L},\lambda}(\tau,z) e^{-4\pi\beta(y).v^{-1}} dx dy
$$

$$= 2^{-\frac{\mathrm{rk}(\underline{L})}{2}} \left(\det(\underline{L})\right)^{-\frac{1}{2}} v^{\frac{\mathrm{rk}(\underline{L})}{2}} \boldsymbol{\delta}(\mu - \lambda \in L). \quad (3.5)$$

Hence, the proposition immediately follows from the above statement. □

Now, we state the main results of this section.

Theorem 3.2.12. *Let ϕ and ψ be Jacobi forms of weight k and index $\underline{L} = (L,\beta)$ such that one of them at least is a Jacobi cusp form. Then for each $\ell \in \mathbb{N}_{\underline{L}}$ we have*

$$< T(\ell)\phi, \psi >_{J_{\underline{L}}(\mathbb{Z})} = < \phi, T(\ell)\psi >_{J_{\underline{L}}(\mathbb{Z})} . \quad (3.6)$$

Theorem 3.2.13. *The space of Jacobi cusp forms $S_{k,\underline{L}}$ has a basis of simultaneous eigenforms for all operators $T(\ell)$ and all operators $W(\alpha)$ $(\ell \in \mathbb{N}_{\underline{L}}, \alpha \in O(D_{\underline{L}}))$.*

The rest of this section deals with the proof of the theorems. For that, we need the following lemmas:

Lemma 3.2.14. *Let $g \in J_{\underline{L}}(\mathbb{Q})$. Assuming that $\phi, \psi, \phi|_{k,\underline{L}} g, \psi|_{k,\underline{L}} g^{-1}$ are $|_{k,\underline{L}}$-invariant under a subgroup Λ_g of finite index in $J_{\underline{L}}(\mathbb{Z})$. Then*

$$< \phi|_{k,\underline{L}} g, \psi >_{\Lambda_g} = < \phi, \psi|_{k,\underline{L}} g^{-1} >_{\Lambda_g} .$$

Proof. Recall that the Petersson scalar product of ϕ, ψ is independent of the choice of the fundamental domain (see Proposition 3.2.7). Thus

$$\begin{aligned} < \phi|_{k,\underline{L}} g, \psi >_{\Lambda_g} &= \frac{1}{[J_{\underline{L}}(\mathbb{Z}) : \Lambda_g]} \int_{\mathscr{F}_{\Lambda_g}} \omega_{\phi|_{k,\underline{L}} g, \psi}(\tau, z)\, dV_{\underline{L},(\tau,z)} \\ &= \frac{1}{[J_{\underline{L}}(\mathbb{Z}) : \Lambda_g]} \int_{g^{-1}\mathscr{F}_{\Lambda_g}} \omega_{\phi|_{k,\underline{L}} g, \psi}(g^{-1}(\tau, z))\, dV_{\underline{L}, g^{-1}(\tau,z)}. \end{aligned}$$

The volume element $dV_{\underline{L},(\tau,z)}$ is invariant under the action of $J_{\underline{L}}(\mathbb{Q})$ (see e.g. [Zie89, p. 202]). Using this and Lemma 3.2.2 we obtain

$$< \phi|_{k,\underline{L}} g, \psi >_{\Lambda_g} = \frac{1}{[J_{\underline{L}}(\mathbb{Z}) : \Lambda_g]} \int_{\mathscr{F}_{\Lambda_g}} \omega_{\phi, \psi|_{k,\underline{L}} g^{-1}}(\tau, z)\, dV_{\underline{L},(\tau,z)} = < \phi, \psi|_{k,\underline{L}} g^{-1} >_{\Lambda_g} .$$

The proof is complete. □

Lemma 3.2.15. *Let $g \in J_{\underline{L}}(\mathbb{Q})$. Then $J_{\underline{L}}(\mathbb{Z})$ and $J_{\underline{L}}(\mathbb{Z}) \cap g^{-1}J_{\underline{L}}(\mathbb{Z})g$ are commensurable.*

Proof. One has

$$J_{\underline{L}}(\mathbb{Z})gJ_{\underline{L}}(\mathbb{Z}) = \bigcup_{s \in (J_{\underline{L}}(\mathbb{Z}) \cap g^{-1}J_{\underline{L}}(\mathbb{Z})g)\backslash J_{\underline{L}}(\mathbb{Z})} J_{\underline{L}}(\mathbb{Z})gs$$

as a disjoint union, and

$$[J_{\underline{L}}(\mathbb{Z}) : J_{\underline{L}}(\mathbb{Z}) \cap g^{-1}J_{\underline{L}}(\mathbb{Z})g] = \#\big(J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})gJ_{\underline{L}}(\mathbb{Z})\big).$$

Write $g = (A,x)$. Let $\gamma : J_{\underline{L}}(\mathbb{Z})gJ_{\underline{L}}(\mathbb{Z}) \to \Gamma A \Gamma$ be the canonical map $(M,h) \mapsto M$, and let $\gamma_* : J_{\underline{L}}(\mathbb{Z})\backslash J_{\underline{L}}(\mathbb{Z})gJ_{\underline{L}}(\mathbb{Z}) \to \Gamma\backslash\Gamma A \Gamma$ be the induced map. Each class in the fibre $\gamma_*^{-1}(\mathrm{SL}_2(\mathbb{Z})M)$ has a representative of the form $(M, x^{B_1}h)$, where $h \in H_{\underline{L}}(\mathbb{Z})$ and $B_1 \in A^{-1}\Gamma M \cap \Gamma$. Write $x = (\lambda, \mu, \xi)$. Let N be a positive integer such that $(N\lambda, N\mu, 1) \in H_{\underline{L}}(\mathbb{Z})$, $\ell \in \mathbb{N}$ such that $\ell M \in \mathrm{GL}_2(\mathbb{Z})$, let $B_2 \in A^{-1}\Gamma M \cap \Gamma$ with $B_1 \equiv B_2 \bmod \ell N$, then

$$\big(M, x^{B_1}h\big)\big(M, x^{B_2}h\big)^{-1} = \big(1, (x^{B_1}(x^{B_2})^{-1})^{M^{-1}}\big) = \big(1, (\lambda, \mu, 1)^{(B_1 - B_2)M^{-1}}\big) \in J_{\underline{L}}(\mathbb{Z}).$$

Thus $(M, x^{B_2}h)$ is also a representative of the same coset. The number of such possibilities $\bmod\ \ell N$ is finite since $A^{-1}\Gamma M \cap \Gamma \bmod \ell N \subseteq \mathrm{SL}_2(\mathbb{Z}/\ell N\mathbb{Z})$. Let $M \in \Gamma\backslash\Gamma A \Gamma$, $B \in A^{-1}\Gamma M \cap \Gamma$, and $h_1, h_2 \in H_{\underline{L}}(\mathbb{Z})$. The two classes $J_{\underline{L}}(\mathbb{Z})(M, x^B h_1)$ and $J_{\underline{L}}(\mathbb{Z})(M, x^B h_2)$ are equal if and only if

$$h_1 \equiv h_2 \bmod \Big((M, x^B)^{-1} H_{\underline{L}}(\mathbb{Z})(M, x^B)\Big) \cap H_{\underline{L}}(\mathbb{Z}).$$

Hence in the view of the well-known fact that $\Gamma\backslash\Gamma A \Gamma$ is finite, and $A^{-1}\Gamma M \cap \Gamma \bmod \ell N$ is finite, it is enough to show that

$$(H_{\underline{L}}(\mathbb{Z}) \cap (M, x^B)^{-1} H_{\underline{L}}(\mathbb{Z})(M, x^B))\backslash H_{\underline{L}}(\mathbb{Z})$$

is finite. But this is immediate, since

$$\Big\{(\ell N\lambda, \ell N\mu, 1) \mid \lambda, \mu \in L\Big\} \subseteq H_{\underline{L}}(\mathbb{Z}) \cap \big((M, x^B)^{-1} H_{\underline{L}}(\mathbb{Z})(M, x^B)\big).$$

Now, the proof is complete. □

Proof of Theorem 3.2.12. By Definition 2.5.7, the operator $T(\ell)$ can be written as a linear combination of the double coset Hecke operator $T_0(\ell)$, so it suffices to prove that

$$< T_0(\ell)\phi, \psi >_{J_{\underline{L}}(\mathbb{Z})} = < \phi, T_0(\ell)\psi >_{J_{\underline{L}}(\mathbb{Z})} . \tag{3.7}$$

For each $g \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})$ we set $\Lambda_g := (g^{-1} J_{\underline{L}}(\mathbb{Z}) g) \cap (g J_{\underline{L}}(\mathbb{Z}) g^{-1}) \cap J_{\underline{L}}(\mathbb{Z})$. By Lemma 3.2.15 we conclude that Λ_g is a subgroup of finite index in $J_{\underline{L}}(\mathbb{Z})$. Moreover, by [Zie89, Lemma 1.4], we see that ϕ, ψ, $\phi|_{k,\underline{L}} g$, $\psi|_{k,\underline{L}} g^{-1}$ are $|_{k,\underline{L}}$-invariant under Λ_g. The intersection

$$\Lambda = \bigcap_{g \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})} \Lambda_g$$

is a subgroup of finite index in $J_{\underline{L}}(\mathbb{Z})$. This gives

$$\begin{aligned}
< T_0(\ell)\phi, \psi >_{J_{\underline{L}}(\mathbb{Z})} &= < T_0(\ell)\phi, \psi >_{\Lambda} \\
&= \ell^{k-2-\mathrm{rk}(\underline{L})} \sum_{g \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})} < \phi|_{k,\underline{L}} g, \psi >_{\Lambda} \\
&= \ell^{k-2-\mathrm{rk}(\underline{L})} \sum_{g \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})} < \phi, \psi|_{k,\underline{L}} g^{-1} >_{\Lambda} \\
&= < \phi, \ell^{k-2-\mathrm{rk}(\underline{L})} \sum_{g \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})} \psi|_{k,\underline{L}} g^{-1} >_{\Lambda} \\
&= < \phi, T_0(\ell)\psi >_{J_{\underline{L}}(\mathbb{Z})} .
\end{aligned}$$

The first identity follows from the fact that $<,>$ is independent of the choice of Λ (see Proposition 3.2.8), the second from the bilinearity of $<,>$, the third from Lemma 3.2.14, the fourth from the anti-linearity of $<,>$ in the second argument, the last from the fact that $J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z}) = J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})$. Thus for each $\ell \in \mathbb{N}_{\underline{L}}$, the operators $T_0(\ell)$ $T(\ell)$ is a self-adjoint operator with respect to the Petersson scalar product. □

We recall a lemma from basic linear algebra.

Lemma 3.2.16. *Let V be a finite-dimensional complex vector space equipped with a positive definite Hermitian form $<,>$.*

1. *Let* $f : V \to V$ *be a linear map which is Hermitian; in other words, if* $v, w \in V$ *then* $<f(v),w> = <v, f(w)>$*. Then V has a basis consisting of eigenvectors for* f*.*

2. *Let* $f_1, f_2, \cdots$ *be a sequence of Hermitian operators sending V to V which commute with each other. Then V has a basis consisting of vectors that are eigenvectors for all of the* f_i*.*

Proof of Theorem 3.2.13. It is known from [BS14] that the space $S_{k,\underline{L}}$ is a finite-dimensional Hilbert space under the Petersson scalar product. If $\alpha \in O(D_{\underline{L}})$, then the action of $W(\alpha)$ permutes the functions h_x $(x \in L^\#/L)$. It follows from Proposition 3.2.10 that $W(\alpha)$ is Hermitian. Also by Theorem 3.2.12, we see if $\ell \in \mathbb{N}_{\underline{L}}$ then $T(\ell)$ is Hermitian with respect to the Petersson scalar product. Therefore, we can apply Lemma 3.2.16 to the set of $T(\ell)$ $(\ell \in \mathbb{N}_{\underline{L}})$ and to the set of $W(\alpha)$ $(\alpha \in O(D_{\underline{L}}))$. The first part of Lemma 3.2.16 says that there exist eigenforms which form an orthonormal basis for $S_{k,\underline{L}}$. These need not be simultaneous eigenforms for all $T(\ell), W(\alpha)$. However, since the operators $T(\ell)$ (resp. $W(\alpha)$) commute with each other, the second part of Lemma 3.2.16 shows that $S_{k,\underline{L}}$ has an orthonormal basis consisting of simultaneous eigenforms. Each of these can be multiplied by a constant factor to get a new basis of simultaneous normalized eigenforms. □

3.3 Jacobi-Eisenstein Series and Hecke Operators

As in the usual theory of modular forms, we will obtain an example of the Jacobi form of lattice index by constructing Jacobi-Eisenstein series.

Let $\underline{L} = (L, \beta)$ be an even positive definite lattice over $\mathbb{Z}$. Recall that each $\phi \in J_{k,\underline{L}}$ has Fourier expansion

$$\phi(\tau, z) = \sum_{(D,r) \in \mathrm{supp}(\underline{L})} C_\phi(D, r) \mathfrak{e}\big(n\tau + \beta(r, z)\big) \qquad (\tau, z) \in \mathfrak{h} \times (L \otimes_{\mathbb{Z}} \mathbb{C}).$$

The Jacobi form ϕ is called a cusp form if $C_\phi(0,r)=0$ for all r such that $\beta(r)\in\mathbb{Z}$. By $S_{k,\underline{L}}$ as before, we mean the subspace of Jacobi forms in $J_{k,\underline{L}}$ consisting of cusp forms.

Definition 3.3.1. Let $D_{\underline{L}}$ be the associated discriminant form with $\underline{L}$. We set

$$\operatorname{Iso}\left(D_{\underline{L}}\right) := \left\{x\in L^\#/L \;\middle|\; \beta(x)\in\mathbb{Z}\right\}.$$

Definition 3.3.2. For $r\in L^\#$, we set

$$g_{\underline{L},r}(\tau,z) := \mathfrak{e}\left(\tau\beta(r)+\beta(r,z)\right). \tag{3.8}$$

Definition 3.3.3. We set

$$J_{\underline{L}}(\mathbb{Z})_\infty := \left\{\left(\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right),(0,\mu,1)\right); n\in\mathbb{Z},\mu\in L\right\}. \tag{3.9}$$

Proposition 3.3.4. *Let* $\gamma_\infty = \left(\left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right),(0,\mu,1)\right)\in J_{\underline{L}}(\mathbb{Z})_\infty$. *For each* $r\in\operatorname{Iso}\left(D_{\underline{L}}\right)$ *and* $k\in\mathbb{N}$, *the function* $g_{\underline{L},r}$ *satisfies the following:*

$$g_{\underline{L},r}\big|_{k,\underline{L}}\gamma_\infty = g_{\underline{L},r} \tag{3.10}$$

$$g_{\underline{L},r}\big|_{k,\underline{L}} - I_2 = (-1)^k g_{\underline{L},-r} \tag{3.11}$$

Proof. The following identity proves Equation (3.10):

$$\begin{aligned}\left(g_{\underline{L},r}\big|_{k,\underline{L}}\gamma_\infty\right)(\tau,z) &= (g_{\underline{L},r}\big|_{k,\underline{L}}A)(\tau,z+\mu) = g_{\underline{L},r}\left(A\tau,\tfrac{z+\mu}{w(\tau)^2}\right)\\ &= g_{\underline{L},r}(\tau,z+\mu) = g_{\underline{L},r}(\tau,z).\end{aligned}$$

One also has

$$\left(g_{\underline{L},r}\big|_{k,\underline{L}}\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right)(\tau,z) = (-1)^k\,\mathfrak{e}\left(\tau\beta(r)+\beta(r,-z)\right) = (-1)^k g_{\underline{L},-r}(\tau,z)$$

which proves Equation (3.11). □

Definition 3.3.5 (Jacobi-Eisenstein series)**.** Let $k\in\mathbb{N}$ with $k>\frac{\operatorname{rk}(L)}{2}+2$. For $r\in\operatorname{Iso}\left(D_{\underline{L}}\right)$ we define the Jacobi-Eisenstein series of weight k and index $\underline{L}$ as follows:

$$E_{k,\underline{L},r} := \frac{1}{2}\sum_{\gamma\in J_{\underline{L}}(\mathbb{Z})_\infty\backslash J_{\underline{L}}(\mathbb{Z})} g_{\underline{L},r}\big|_{k,\underline{L}}\gamma. \tag{3.12}$$

Proposition 3.3.6. *For each $r \in \mathrm{Iso}(D_{\underline{L}})$, $k \in \mathbb{N}$ with $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$, the series $E_{k,\underline{L},r}$ is absolutely uniformly convergent on compact sets.*

Proof. See e.g. [BK93, P.503]. □

Lemma 3.3.7. *The sum in Equation (3.12) is independent of the choice of coset representatives for $J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})$.*

Proof. Let $\gamma_\infty \in J_{\underline{L}}(\mathbb{Z})_\infty, \gamma \in J_{\underline{L}}(\mathbb{Z})$. One has

$$g_{\underline{L},r}|_{k,\underline{L}}\gamma_\infty\gamma = g_{\underline{L},r}|_{k,\underline{L}}\gamma_\infty|_{k,\underline{L}}\gamma = g_{\underline{L},r}|_{k,\underline{L}}\gamma.$$

Hence, each term in Equation (3.12) does not depend on the choice of the representative, but only on the $J_{\underline{L}}(\mathbb{Z})_\infty$-orbit of γ. □

Proposition 3.3.8. *The series $E_{k,\underline{L},r}$ given by Definition 3.3.5 transforms like a Jacobi form of weight k and index $\underline{L}$. Moreover, one has $E_{k,\underline{L},r} = (-1)^k E_{k,\underline{L},-r}$.*

Proof. Any $A \in J_{\underline{L}}(\mathbb{Z})$ permutes $J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})$ by right multiplication, i.e. the map

$$\psi : J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z}) \to J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z}), \quad J_{\underline{L}}(\mathbb{Z})_\infty\gamma \mapsto J_{\underline{L}}(\mathbb{Z})_\infty\gamma A$$

is bijective. Thus one has

$$\begin{aligned} E_{k,\underline{L},r}|_{k,\underline{L}}A &= \frac{1}{2}\sum_{\gamma \in J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})} g_{\underline{L},r}|_{k,\underline{L}}\gamma A \\ &= \frac{1}{2}\sum_{\gamma A \in J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})} g_{\underline{L},r}|_{k,\underline{L}}\gamma A \\ &= E_{k,\underline{L},r}. \end{aligned}$$

Hence, $E_{k,\underline{L},r}$ transforms like a Jacobi form of weight k and index $\underline{L}$. The equality $E_{k,\underline{L},r} = (-1)^k E_{k,\underline{L},-r}$ follows from Equation (3.11). □

Proposition 3.3.9. *One has*

$$< \phi, E_{k,\underline{L},r} >= 0 \tag{3.13}$$

for all $\phi \in S_{k,\underline{L}}$.

Proof. according to [BK93, Lemma 1], there is $\lambda_{k,r} \in \mathbb{C}$ such that $< \phi, E_{k,\underline{L},r} >= \lambda_{k,r} C_\phi(0,r)$. Since ϕ is a cusp form, we have $C_\phi(0,r) = 0$. Thus $< \phi, E_{k,\underline{L},r} >= 0$. □

To write the Jacobi-Eisenstein series in more explicit form, we need to find a set of coset representatives for $J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})$. One can take the set

$$\left\{\left(A, (\lambda, 0, 1)^A\right) \mid A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z}), \lambda \in L\right\} \tag{3.14}$$

as a complete set of coset representatives for $J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})$ (see [BK93, P.504]).

Proposition 3.3.10. *The series $E_{k,\underline{L},r}$ given by Definition 3.3.5 can be written in terms of theta series as follows:*

$$E_{k,\underline{L},r} = \frac{1}{2} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \vartheta_{\underline{L},r}\big|_{k,\underline{L}} A. \tag{3.15}$$

Proof. As a set of coset representatives for $J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})$ we take the set which is given by Equation (3.14). One has

$$E_{k,\underline{L},r} = \frac{1}{2} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\lambda \in L} g_{\underline{L},r}\big|_{k,\underline{L}}(\lambda, 0, 1)\big|_{k,\underline{L}} A.$$

This can be written, using the definition of the slash operator and Definition 3.3.1 of the function $g_{\underline{L},r}$, as

$$\begin{aligned} E_{k,\underline{L},r}(\tau, z) =& \frac{1}{2} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\lambda \in L} \mathfrak{e}\left(A\tau \beta(r+\lambda) + \beta(r+\lambda, \tfrac{z}{w_A(\tau)^2})\right) \mathfrak{e}\left(\tfrac{-c\beta(z)}{w_A(\tau)^2}\right) w_A(\tau)^{-2k} \\ =& \frac{1}{2} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \Big(\sum_{\lambda \in L} g_{k,L,r+\lambda}\Big)\Big|_{k,\underline{L}} A(\tau, z). \end{aligned}$$

Comparing the inner sum with Definition 2.3.2 of the theta series, we see that $\sum_{\lambda \in L} g_{k,L,r+\lambda} = \vartheta_{\underline{L},r}$. Now, the proof is complete. □

Definition 3.3.11. Let ϕ be a Jacobi form of weight k and index $\underline{L} = (L, \beta)$. The singular term of ϕ is defined as

$$\big(\text{singular-term}(\phi)\big)(\tau, z) := \sum_{\substack{s \in L^\# \\ \beta(s) \equiv 0 \bmod \mathbb{Z}}} C_\phi(0, s)\mathfrak{e}\left(\tau\beta(s) + \beta(s, z)\right). \tag{3.16}$$

Proposition 3.3.12. *The singular term of the series $E_{k,\underline{L},r}$ is given by*

$$\begin{aligned}\big(\text{singular-term}(E_{k,\underline{L},r})\big)(\tau,z) &= \tfrac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z)+(-1)^k\vartheta_{\underline{L},-r}(\tau,z)\right)\\ &= \sum_{\substack{s\in L^{\#}\\ \beta(s)\equiv 0 \bmod \mathbb{Z}}} C_{E_{k,\underline{L},r}}(0,s)\mathfrak{e}\left(\tau\beta(s)+\beta(s,z)\right), \end{aligned} \tag{3.17}$$

where

$$C_{E_{k,\underline{L},r}}(0,s) = \tfrac{1}{2}\left(\boldsymbol{\delta}(s\equiv r \bmod L)+(-1)^k\boldsymbol{\delta}(s\equiv -r \bmod L)\right). \tag{3.18}$$

Remark 3.3.13 (and Definition)**.** If $\phi\in J_{k,\underline{L}}$, then

$$\phi-\sum_{r\in \mathrm{Iso}(D_{\underline{L}})} C_\phi(0,r)E_{k,\underline{L},r}\in S_{k,\underline{L}}.$$

Thus, in the view of Proposition 3.3.9, we may write

$$J_{k,\underline{L}} = S_{k,\underline{L}}\oplus J^{\mathrm{Eis}}_{k,\underline{L}},$$

where $J^{\mathrm{Eis}}_{k,\underline{L}}$ is the Jacobi-Eisenstein subspace of the space of the Jacobi forms of weight k and index $\underline{L}=(L,\beta)$ which consists of all functions $E_{k,\underline{L},r}$ ($r\in\mathrm{Iso}(D_{\underline{L}})$).

Proof of Proposition 3.3.12. As a complete system of coset representatives for $\mathrm{SL}_2(\mathbb{Z})_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$ one takes the set

$$\left\{A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})\mid (c,d)=1\right\}.$$

Thus we can split the sum (3.15) into two parts, according to whether $c=0$ or not. If $c=0$, then these terms give a contribution of the singular term, which is

$$\text{singular-term}(E_{k,\underline{L},r}) = \tfrac{1}{2}\left(\vartheta_{\underline{L},r}+(-1)^k\vartheta_{\underline{L},-r}\right).$$

Now, Definition 2.3.2 of the theta function completes the proof. □

Lemma 3.3.14. *The dimension of the vector space of the Jacobi-Eisenstein series is given by*

$$\dim J^{\mathrm{Eis}}_{k,\underline{L}} = \tfrac{1}{2}\left(|\mathrm{Iso}(D_{\underline{L}})|+(-1)^k\#\left\{r\in\mathrm{Iso}(D_{\underline{L}})\;\middle|\;2r\in L\right\}\right). \tag{3.19}$$

Proof. If k is even, then $E_{k,\underline{L},r} = E_{k,\underline{L},-r}$. In this case, the space of Jacobi Eisenstein series is spanned by $E_{k,\underline{L},s}$ where s runs through a set Δ of representatives for $\mathrm{Iso}(D_{\underline{L}})$ modulo the action of $\{\pm 1\}$. One has $|\Delta| = \frac{1}{2}\#\{r \in \mathrm{Iso}(D_{\underline{L}}) \mid 2r \notin L\} + \#\{r \in \mathrm{Iso}(D_{\underline{L}}) \mid 2r \in L\}$. If k is odd, then $E_{k,\underline{L},r} = -E_{k,\underline{L},-r}$, and $E_{k,\underline{L},r} = 0$ if $2r \in L$. In this case, $J_{k,\underline{L}}^{\mathrm{Eis}}$ is spanned by $E_{k,\underline{L},s}$ where s runs through $\Delta \backslash \{r \in \Delta \mid 2r \in L\}$. The linearity independence of the Jacobi theta series implies that singular-term$(E_{k,\underline{L},s}) = \frac{1}{2}\left(\vartheta_{\underline{L},s} + (-1)^k \vartheta_{\underline{L},-s}\right)$, where s is as above, are linearly independent. Thus $E_{k,\underline{L},s}$ are linearly independent. □

Remark 3.3.15. For even k, the dimension formula (3.19) can also be found in [Bru02, p.26].

Definition 3.3.16 (Notation)**.** For each positive integer F, we use $\mathrm{Pr}(F)$ to denote the set of all primitive Dirichlet characters mod F.

We now state our main results of this section

Definition 3.3.17. Let $k \in \mathbb{N}$ such that $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$. We set

$$E_{k,\underline{L},x,\xi} := \sum_{d \in \mathbb{Z}_{N_x}^\times} \xi(d) E_{k,\underline{L},dx}.$$

Here $x \in \mathscr{R}_{\mathrm{Iso}}$ where $\mathscr{R}_{\mathrm{Iso}}$ is a set of representatives for the orbits in the orbit space $\mathrm{Iso}(D_{\underline{L}})/\mathbb{Z}_{\mathrm{lev}(\underline{L})}^\times$, N_x is the smallest positive integer such that $N_x x \in L$, and ξ is a primitive Dirichlet character modulo F with $F \mid N_x$ and $\xi(-1) = (-1)^k$.

Theorem 3.3.18. *In the same notations as in Definition 3.3.17, the series $E_{k,\underline{L},x,\xi}$, where x runs through the set $\mathscr{R}_{\mathrm{Iso}}$ and ξ runs through all primitive Dirichlet characters* mod *F with $F \mid N_x$ such that $\xi(-1) = (-1)^k$, form a basis of Hecke eigenforms of $J_{k,\underline{L}}^{\mathrm{Eis}}$. More precisely, for all $\ell \in \mathbb{N}$ with* $\gcd(\ell, \mathrm{lev}(\underline{L})) = 1$, *one has*

$$T(\ell) E_{k,\underline{L},x,\xi} = \lambda(\ell, E_{k,\underline{L},x,\xi})\, E_{k,\underline{L},x,\xi}, \tag{3.20}$$

where

$$\lambda(\ell, E_{k,\underline{L},x,\xi}) := \begin{cases} \sigma_{2k-\mathrm{rk}(\underline{L})-2}^{\xi,\overline{\xi}}(\ell) & \text{if } \mathrm{rk}(\underline{L}) \text{ is odd,} \\ \overline{\xi(\ell)}\, \sigma_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}^{\xi,\chi_{\underline{L}}}(\ell^2) & \text{if } \mathrm{rk}(\underline{L}) \text{ is even.} \end{cases}$$

Here, for any two Dirichlet characters ξ and χ we set $\sigma_k^{\xi,\chi}(\ell) := \sum_{d \mid \ell} \xi\left(\frac{\ell}{d}\right) \chi(d) d^k$.

Remark 3.3.19. Note that Theorem 3.3.18 is in complete accordance with Remark 2.7.16 and Remark 2.7.10. More precisely, let N,h be two positive integers where $h \geq 3$. For any two Dirichlet characters ξ modulo u and χ modulo v such that $uv = N$ and $\xi\chi(-1) = (-1)^h$ and ξ is primitive, we consider the Eisenstein series

$$E_h^{\xi,\chi}(\tau) = \sum_{n=0}^{\infty} \sigma_{h-1}^{\xi,\chi}(n) q^n,$$

where $\sigma_{h-1}^{\xi,\chi}(0) = \frac{1}{2}L(1-h,\chi)$ or $= 0$ according to whether $u = 1$ (ξ is the trivial character modulo 1) or not. It is known that $E_h^{\xi,\chi} \in M_h(\Gamma_0(N), \xi\chi)$ (see e.g. [DS06, Theorem 4.5.1]).

Let k, x, ξ as in Theorem 3.3.18. We set

$$\mathscr{S}(E_{k,\underline{L},x,\xi})(\tau) := \sum_{\ell \in \mathbb{N}_{\underline{L}}} \lambda(\ell, E_{k,\underline{L},x,\xi}) \mathfrak{e}(\ell\tau).$$

Then, for odd $\mathrm{rk}(\underline{L})$, one has

$$\mathscr{S}(E_{k,\underline{L},x,\xi}) = E_{2k-\mathrm{rk}(\underline{L})-1}^{\xi,\bar{\xi}} \otimes \varphi,$$

where $\varphi(\cdot) := \left(\frac{\mathrm{lev}(\underline{L})^2}{\cdot}\right)$. Also, for even $\mathrm{rk}(\underline{L})$, it is obvious that

$$\xi(\ell)\lambda(\ell, E_{k,\underline{L},x,\xi}) = \text{ the } \ell^2\text{-th coefficient of } E_{k-\frac{\mathrm{rk}(\underline{L})}{2}}^{\xi,\chi_{\underline{L}}}.$$

To prove Theorem 3.3.18 we need the following lemma:

Lemma 3.3.20. *In the same notations as in Definition 3.3.17, one has*

$$\sum_{x \in \mathscr{R}_{\mathrm{Iso}}} \sum_{F | N_x} \#\left\{\xi \in \mathrm{Pr}(F) \;\middle|\; \xi(-1) = (-1)^k\right\} = \dim J_{k,\underline{L}}^{\mathrm{Eis}}. \tag{3.21}$$

Proof. First, we assume that k is even. If $-1 \in \mathrm{Stab}_{\mathbb{Z}_{\mathrm{lev}(\underline{L})}^\times}(x)$, i.e. if $\mathrm{Orb}(x)$ consists of elements s such that $2s \in L$, then

$$\sum_{F|N_x} \#\left\{\xi \in \mathrm{Pr}(F) \;\middle|\; \xi(-1) = (-1)^k\right\} = \varphi(N_x) = |\,\mathrm{Orb}(x)|.$$

If $-1 \notin \mathrm{Stab}_{\mathbb{Z}_{\mathrm{lev}(\underline{L})}^\times}(x)$ ($N_x \neq 1,2$), then half of the characters are even and half of them are odd. Thus

$$\sum_{F|N_x} \#\left\{\xi \in \mathrm{Pr}(F) \;\middle|\; \xi(-1) = (-1)^k\right\} = \frac{1}{2}\varphi(N_x) = \frac{1}{2}|\,\mathrm{Orb}(x)|.$$

We therefore conclude by the Orbit-Stabilizer theorem that

$$\sum_{x\in\mathscr{R}_{\mathrm{Iso}}}\sum_{F|N_x}\#\left\{\xi\in\Pr(F)\;\middle|\;\xi(-1)=(-1)^k\right\}=\frac{1}{2}\sum_{\substack{x\in\mathscr{R}_{\mathrm{Iso}}\\2x\notin L}}|\operatorname{Orb}(x)|+\sum_{\substack{x\in\mathscr{R}_{\mathrm{Iso}}\\2x\in L}}|\operatorname{Orb}(x)|$$
$$=\frac{1}{2}\#\left\{x\in\operatorname{Iso}\left(D_{\underline{L}}\right)\;\middle|\;2x\notin L\right\}+\#\left\{x\in\operatorname{Iso}\left(D_{\underline{L}}\right)\;\middle|\;2x\;\in L\right\}$$
$$=\frac{1}{2}\left(|\operatorname{Iso}\left(D_{\underline{L}}\right)|+(-1)^k\#\left\{x\in\operatorname{Iso}\left(D_{\underline{L}}\right)\;\middle|\;2x\in L\right\}\right)$$
$$=\dim J^{\mathrm{Eis}}_{k,\underline{L}}.$$

Now, we assume that k is odd. One has $\xi(-1)=-1$. Since there is no odd characters modulo 1 or 2, and $N_x\notin\{1,2\}$ $(2x\notin L)$. This gives

$$\sum_{F|N_x}\#\left\{\xi\in\Pr(F)\;\middle|\;\xi(-1)=(-1)^k\right\}=\frac{1}{2}\varphi(N_x)=\frac{1}{2}|\operatorname{Orb}(x)|.$$

We therefore conclude

$$\sum_{x\in\mathscr{R}_{\mathrm{Iso}}}\sum_{F|N_x}\#\left\{\xi\in\Pr(F)\;\middle|\;\xi(-1)=(-1)^k\right\}=\frac{1}{2}\sum_{\substack{x\in\mathscr{R}_{\mathrm{Iso}}\\2x\notin L}}|\operatorname{Orb}(x)|$$
$$=\frac{1}{2}\#\left\{x\in\operatorname{Iso}\left(D_{\underline{L}}\right)\;\middle|\;2x\notin L\right\}$$
$$=\frac{1}{2}\left(|\operatorname{Iso}\left(D_{\underline{L}}\right)|+(-1)^k\#\left\{x\in\operatorname{Iso}\left(D_{\underline{L}}\right)\;\middle|\;2x\in L\right\}\right)$$
$$=\dim J^{\mathrm{Eis}}_{k,\underline{L}}.$$

Now, the proof is complete. □

Proof of Theorem 3.3.18. From Lemma 3.3.20 we have

$$\sum_{x\in\mathscr{R}_{\mathrm{Iso}}}\sum_{F|N_x}\#\left\{\xi\in\Pr(F)\;\middle|\;\xi(-1)=(-1)^k\right\}=\dim J^{\mathrm{Eis}}_{k,\underline{L}}.$$

To prove that the set of series $\{E_{k,\underline{L},x,\xi}\}_{x,\xi}$ is a basis for $J^{\mathrm{Eis}}_{k,\underline{L}}$ we still need to show that they are linearly independent. Recall that each Jacobi-Eisenstein series $E_{k,\underline{L},x,\xi}$ is determined by its singular-term, which is given by

$$\text{singular-term}\left(E_{k,\underline{L},x,\xi}\right)=\sum_{d\in\mathbb{Z}^\times_{N_x}}\xi(d)\vartheta_{\underline{L},dx},$$

or equivalently by its Fourier coefficients $C_{E_{k,\underline{L},x,\xi}}(0,s)$ $(\beta(s)\equiv 0 \bmod \mathbb{Z})$, which are given by

$$C_{E_{k,\underline{L},x,\xi}}(0,s)=\sum_{d\in\mathbb{Z}^\times_{N_x}}\xi(d)C_{E_{k,\underline{L},dx}}(0,s)=\Xi_{x,\xi}(s),\tag{3.22}$$

where

$$\Xi_{x,\xi}(s) = \begin{cases} \xi(d) & \text{if } s \equiv dx \text{ for some } d \bmod N_x \text{ with } (d,N_x)=1, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to check the linear independence at the singular-parts. If a linear combination

$$\sum_{\xi,x} g_{\xi,x} \, \text{singular-term}\left(E_{k,\underline{L},x,\xi}\right) = 0 \qquad (g_{\xi,x} \in \mathbb{C}),$$

then the linearity independence of the Jacobi theta series gives

$$\sum_{F|N_x} \sum_{\xi \in \mathrm{Pr}(F)} g_{\xi,x} \xi(s) = 0 \qquad (sx \in \mathrm{Orb}(x)). \tag{3.23}$$

Now, let $L^2(\mathbb{Z}_{N_x}^\times, \mathbb{C})$ the Hilbert space of functions $\mathbb{Z}_{N_x}^\times \to \mathbb{C}$, with inner product $\langle g,h\rangle = \frac{1}{|\mathbb{Z}_{N_x}^\times|}\sum_{g\in\mathbb{Z}_{N_x}^\times} f(g)\overline{h(g)}$. We can view $\mathrm{Hom}(\mathbb{Z}_{N_x}^\times, \mathbb{C}^\times)$ as a subset of $L^2(\mathbb{Z}_{N_x}^\times, \mathbb{C})$. The elements of $\mathrm{Hom}(\mathbb{Z}_{N_x}^\times, \mathbb{C}^\times)$ are pairwise orthogonal under this inner product. It follows that $\mathrm{Hom}(\mathbb{Z}_{N_x}^\times, \mathbb{C}^\times)$ is a linearly independent subset of $L^2(\mathbb{Z}_{N_x}^\times, \mathbb{C})$. Thus by the orthogonality relations for such characters we see that each coefficient $g_{\xi,x}$ in Equation (3.23) is 0, which proves the linearly independent part.

To describe the action of our Hecke operators on this series, recall that the space of the Jacobi-Eisenstein series is invariant under all operators $T(\ell)$ ($\ell \in \mathbb{N}_{\underline{L}}$), since it is the orthogonal complement of $S_{k,\underline{L}}$ in $J_{k,\underline{L}}$ with respect to the Petersson scalar product (see Proposition 3.3.6) and the operators $T(\ell)$ for $\ell \in \mathbb{N}_{\underline{L}}$ are Hermitian (see Theorem 3.2.12). So to verify Equation (3.20) it suffices to check the action of the operators on the Fourier coefficient $C_{E_{k,\underline{L},x,\xi}}(0,s)$. Thus, if $\mathrm{rk}(\underline{L})$ is even, one has

$$\begin{aligned} C_{T(\ell)E_{k,\underline{L},x,\xi}}(0,s) &= \sum_{a|\ell^2} a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \chi_{\underline{L}}(a) C_{E_{k,\underline{L},x,\xi}}\left(0, \ell a' s\right) \\ &= \sum_{a|\ell^2} a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \chi_{\underline{L}}(a) \xi(\ell a') C_{E_{k,\underline{L},x,\xi}}(0,s) \\ &= \Big(\sum_{a|\ell^2} \chi_{\underline{L}}(a) \overline{\xi(\ell)} \xi\left(\tfrac{\ell^2}{a}\right) a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \Big) C_{E_{k,\underline{L},x,\xi}}(0,s), \end{aligned}$$

where the first identity follows from Theorem 2.6.3, and the second by Equation (3.22). Similarly, if $\mathrm{rk}(\underline{L})$ is odd one has

$$C_{T(\ell)E_{k,\underline{L},x,\xi}}(0,s) = \sum_{d\mid\ell} d^{2k-\mathrm{rk}(\underline{L})-2}\, C_{E_{k,\underline{L},x,\xi}}(0,\ell g s),$$

where g here is an integer such that $gd^2 \equiv 1 \bmod \mathrm{lev}(\underline{L})$. Since $C_{E_{k,\underline{L},x,\xi}}(0,\ell g s) = \xi(\ell g) C_{E_{k,\underline{L},x,\xi}}(0,s)$, one obtains

$$C_{T(\ell)E_{k,\underline{L},x,\xi}}(0,s) = \Big(\sum_{d\mid\ell} \overline{\xi(d)}\xi\left(\tfrac{\ell}{d}\right) d^{2k-\mathrm{rk}(\underline{L})-2}\Big) C_{E_{k,\underline{L},x,\xi}}(0,s),$$

as stated in the theorem. □

Chapter 4

Lifting to Elliptic Modular Forms

A natural question in view chapter 2 , in particular, Proposition 2.7.15, is whether we can construct in general a relation between Jacobi forms of lattice index of odd rank and elliptic modular forms which extends the work of Skoruppa and Zagier [SZ88] for the case of scalar index. In this chapter we shall prove certain results which suggest that there should be indeed such relations.

4.1 Lifting via Shimura Correspondence for Half Integral Weight

For positive integers N, λ and an even Dirichlet character $\chi \bmod 4N$ let $S_{\lambda+\frac{1}{2}}(4N,\chi)$ be the space of cusp forms of weight $\lambda+\frac{1}{2}$ on $\Gamma_0(4N)$ and character χ (see Definition 1.1.8). For $f(\tau)=\sum_{n=0}^{\infty} a_f(n)\mathfrak{e}(n\tau) \in S_{\lambda+\frac{1}{2}}(N,\chi)$ and a square-free positive integer t, we set

$$\mathscr{S}_t(f)(\tau) = \sum_{n=1}^{\infty} A_t(n)\mathfrak{e}(n\tau), \tag{4.1}$$

where $A_t(n)$ is determined by the relation

$$\sum_{n=1}^{\infty} A_t(n) n^{-s} = L(s-\lambda+1, \chi_t^{(\lambda)}) \sum_{m=1}^{\infty} a_f(tm^2) m^{-s}, \tag{4.2}$$

and $\chi_t^{(\lambda)}(m) = \chi(m)\left(\frac{-1}{m}\right)^\lambda\left(\frac{t}{m}\right)$. Note that $\chi_t^{(\lambda)}$ is a character mod $4Nt$.

Theorem 4.1.1 ([Shi73, p. 458],[Niw75, 3]). *Suppose $f \in S_{\lambda+\frac{1}{2}}(4N,\chi)$. Then*

$$\mathscr{S}_t(f) \in \begin{cases} M_{2\lambda}(\Gamma_0(2N),\chi^2) & \text{if } \lambda = 1, \\ S_{2\lambda}(\Gamma_0(2N),\chi^2) & \text{if } \lambda > 1. \end{cases}$$

Moreover, $\mathscr{S}_t$ commutes with Hecke operators, i.e.,

$$\mathscr{S}_t(T(p^2)f) = T(p)\mathscr{S}_t(f).$$

Remark 4.1.2. In [Shi73] Shimura proved the cited theorem apart from the fact that $\mathscr{S}_t(f)$ has a level $2N$, which only conjectured. The latter was proven by Niwa in [Niw75, 3] and later by Cipra in [Cip83].

We know that every Jacobi form of lattice index has a theta expansion which implies, for odd rank index, a connection to half integral weight modular forms. We can try to use the Shimura correspondence for half integral weight forms to map Jacobi forms of lattice index to modular forms of integral weight.

Definition 4.1.3. Let $\underline{L}$ be of odd rank, and k be a positive integer. For $\phi \in S_{k,\underline{L}}$, $x \in L^\#$, and $D_0 \in \mathbb{Q}$ such that $D_0 \equiv \beta(x) \bmod \mathbb{Z}$, set

$$\mathscr{S}_{D_0,x}(\phi) := \sum_{\ell=1}^{\infty}\Big(\sum_{a|\ell} a^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\chi_{\underline{L}}(D_0,a)C_\phi\big(\tfrac{\ell^2}{a^2}D_0,\tfrac{\ell}{a}x\big)\Big)\mathfrak{e}(\ell\tau).$$

$$\mathscr{S}^{\xi}_{D_0,x}(\phi) := \sum_{\substack{s \bmod N_x \\ D_0\equiv\beta(sx) \bmod \mathbb{Z}}} \xi(s)\big(\mathscr{S}_{D_0,sx}(\phi)\otimes\xi\big),$$

with $\xi(\cdot) = \left(\frac{(-1)^k {N_x}^2}{\cdot}\right)$ where N_x is the order of $x + L \in L^\#/L$ ($N_x x \in L$), and $\mathscr{S}_{D_0,sx}(\phi)\otimes\xi$ denotes the function obtained from $\mathscr{S}_{D_0,sx}(\phi)$ by multiplying its n-th Fourier coefficient by $\xi(n)$.

Theorem 4.1.4. *Let the notations be as in Definition 4.1.3. Assume that $2k - \mathrm{rk}(\underline{L}) - 1 \geq 2$, and ${N_x}^2 D_0$ is a square free negative integer. Then $\mathscr{S}^{\xi}_{D_0,x}(\phi)$*

is an elliptic modular form of weight $2k-1-\mathrm{rk}(\underline{L})$ *on* $\Gamma_0(\mathrm{lev}(\underline{L})N_x{}^2/2)$*, and, in fact, a cusp form if* $2k-1-\mathrm{rk}(\underline{L})>2$*. Moreover, one has*

$$T(p)\mathscr{S}^{\xi}_{D_0,x}(\phi)=\xi(p)\mathscr{S}^{\xi}_{D_0,x}(T(p)\phi) \tag{4.3}$$

for all primes p *with* $\gcd(p,\mathrm{lev}(\underline{L}))=1$.

Proof. By Proposition 2.4.7, ϕ can be written as a sum

$$\phi(\tau,z)=\sum_{r\in L^{\#}/L} h_r(\tau)\vartheta_{\underline{L},r}(\tau,z),$$

where

$$h_r(\tau)=\sum_{\substack{D\in\mathbb{Q}\\(D,r)\in\mathrm{supp}(\underline{L})}} C_\phi(D,r)\mathfrak{e}(-D\tau).$$

For $x\in L^{\#}$, we set

$$F_\phi(\tau):=\sum_{s \bmod N_x}\xi(s)h_{sx}(N_x{}^2\tau). \tag{4.4}$$

According to Proposition 2.4.7, which gives a closed formula for the Fourier expansion of h_{sx}, we see that

$$F_\phi(\tau)=\sum_{D\in\mathbb{Q}}\Bigg(\sum_{\substack{s \bmod N_x\\(D,sx)\in\mathrm{supp}(\underline{L})}}\xi(s)C_\phi(D,sx)\Bigg)\mathfrak{e}\left(-DN_x{}^2\tau\right). \tag{4.5}$$

Let $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(\mathrm{lev}(\underline{L})N_x{}^2)$. One has

$$\begin{aligned}j(A,\tau)^{-2(k-\frac{\mathrm{rk}(\underline{L})}{2})}F_\phi(A\tau)&=j(A,\tau)^{-2(k-\frac{\mathrm{rk}(\underline{L})}{2})}\sum_{s \bmod N_x}\xi(s)h_{sx}\left(N_x{}^2\tfrac{a\tau+b}{c\tau+d}\right)\\&=j(A,\tau)^{-2(k-\frac{\mathrm{rk}(\underline{L})}{2})}\sum_{s \bmod N_x}\xi(s)h_{sx}\left(BN_x{}^2\tau\right),\end{aligned}$$

where $B:=\begin{pmatrix}a&N_x{}^2b\\c/N_x{}^2&d\end{pmatrix}\in\Gamma_0(\mathrm{lev}(\underline{L}))$. It is obvious that $j(A,\tau)=j(B,N_x{}^2\tau)$. Thus

$$\begin{aligned}j(A,\tau)^{-2(k-\frac{\mathrm{rk}(\underline{L})}{2})}F_\phi(A\tau)&=\sum_{s \bmod N_x}\xi(s)\left(\frac{(-1)^k2\det(\underline{L})}{d}\right)h_{asx}(N_x{}^2\tau)\\&=\left(\frac{N_x{}^22\det(\underline{L})}{d}\right)\sum_{s \bmod N_x}\xi(as)h_{asx}(N_x{}^2\tau)=\left(\frac{2\det(\underline{L})}{d}\right)F_\phi(\tau),\end{aligned}$$

where the first identity follows from Theorem 2.3.4 and Proposition 2.3.1, and the second identity using that $ad\equiv1 \bmod 4$. Thus F_ϕ defines an element in $S_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\big(\mathrm{lev}(\underline{L})N_x{}^2,\psi\big)$, where $\psi(\cdot)=\left(\frac{2\det(\underline{L})}{\cdot}\right)$(a quadratic character). Set $\lambda = k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil$, $t=-N_x{}^2D_0$, and $\psi_t^{(\lambda)}(m)=\psi(m)\left(\frac{-1}{m}\right)^{\lambda}\left(\frac{t}{m}\right)=\xi(m)\chi_{\underline{L}}(D_0,m)$. One has

$$\begin{aligned}
\mathscr{S}^{\xi}_{D_0,x}(\phi)(\tau) &= \sum_{\substack{s \bmod N_x\\ (D_0,sx)\in\mathrm{supp}(\underline{\mathrm{L}})}} \xi(s)\big(\mathscr{S}_{D_0,sx}(\phi)\otimes\xi\big)(\tau)\\
&= \sum_{\substack{s \bmod N_x\\ (D_0,sx)\in\mathrm{supp}(\underline{\mathrm{L}})}} \xi(s)\sum_{\ell=1}^{\infty}\xi(\ell)\Big(\sum_{a\mid\ell}a^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\chi_{\underline{L}}(D_0,a)C_{\phi}\big(\tfrac{\ell^2}{a^2}D_0,\tfrac{\ell}{a}sx\big)\Big)\mathfrak{e}(\ell\tau)\\
&= \sum_{\ell=1}^{\infty}\Big(\sum_{a\mid\ell}a^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\psi_t^{(\lambda)}(a)\sum_{\substack{s \bmod N_x\\ (D_0,sx)\in\mathrm{supp}(\underline{\mathrm{L}})}}\xi(\tfrac{\ell}{a}s)C_{\phi}\big(\tfrac{\ell^2}{a^2}D_0,\tfrac{\ell}{a}sx\big)\Big)\mathfrak{e}(\ell\tau).
\end{aligned}$$

Now, inserting the Fourier expansion of the F_ϕ as given in Equation (4.5) gives

$$\mathscr{S}^{\xi}_{D_0,x}(\phi)(\tau)=\sum_{\ell=1}^{\infty}\Big(\sum_{a\mid\ell}a^{\lambda-1}\psi_t^{(\lambda)}(a)c_{F_\phi}(t\tfrac{\ell^2}{a^2})\Big)\mathfrak{e}(\ell\tau)=\mathscr{S}_t(F_\phi)(\tau).$$

Thus by Shimura correspondence (see Theorem 4.1.1), we obtain

$$\mathscr{S}^{\xi}_{D_0,x}(\phi)\in\begin{cases} M_{2k-\mathrm{rk}(\underline{L})-1}(\Gamma_0(\mathrm{lev}(\underline{L})N_x{}^2/2)) & \text{if } k=\frac{\mathrm{rk}(\underline{L})+3}{2},\\ S_{2k-\mathrm{rk}(\underline{L})-1}(\Gamma_0(\mathrm{lev}(\underline{L})N_x{}^2/2)) & \text{if } k>\frac{\mathrm{rk}(\underline{L})+3}{2}.\end{cases}$$

Moreover, for each prime integer $p\in\mathbb{N}_{\underline{L}}$

$$T(p)\mathscr{S}^{\xi}_{D_0,x}(\phi)=T(p)\mathscr{S}_t(F_\phi)=\mathscr{S}_t(T(p^2)F_\phi)=\mathscr{S}_t(F_{\xi(p)(T(p)\phi)})=\xi(p)\mathscr{S}^{\xi}_{D_0,x}(T(p)\phi)$$

as stated in the theorem. □

Remark 4.1.5. In fact, Theorem 4.1.4 supports our expectations. Namely, if $\mathscr{S}_{D_0,x}$ takes $J_{k,\underline{L}}$ to elliptic modular forms on $\Gamma_0(\mathrm{lev}(\underline{L})/2)$, then the twisted version $\mathscr{S}^{\xi}_{D_0,x}$ takes $J_{k,\underline{L}}$ to elliptic modular forms on $\Gamma_0(N_x{}^2\,\mathrm{lev}(\underline{L})/2)$, which we proved indeed. If in addition, $\mathscr{S}_{D_0,x}$ commutes with Hecke operators, we deduce for $\mathscr{S}^{\xi}_{D_0,x}$ the relation (4.3), which is again what we proved.

4.2 Lifting via Stable Isomorphisms between Lattices

For lattices $\underline{L_1} = (L_1, \beta_1), \underline{L_2} = (L_2, \beta_2)$ over $\mathbb{Z}$ we define their orthogonal sum by $\underline{L_1} \oplus \underline{L_2} := (L_1 \oplus L_2, f)$, where $f(x_1 \oplus x_2, y_1 \oplus y_2) = \beta_1(x_1, y_1) + \beta_2(x_2, y_2)$. If G_1 and G_2 are Gram matrices associated to $\underline{L_1}$ and $\underline{L_2}$ respectively, then the block sum $G_1 \oplus G_2$ is a Gram matrix associated to $\underline{L_1} \oplus \underline{L_2}$. Two even lattices $\underline{L_1}$ and $\underline{L_2}$ are said to be *stably isomorphic* if and only if there exists even unimodular lattices $\underline{U_1}$ and $\underline{U_2}$ such that $\underline{L_1} \oplus \underline{U_1} \cong \underline{L_2} \oplus \underline{U_2}$.

We shall show in this section that Jacobi form whose index is stably isomorphic to a rank one even lattice lift to elliptic modular forms. For this we need the following theorems:

Theorem 4.2.1 ([Nik80, Theorem 1.3.1]). *Two even integral lattices $\underline{L_1}$ and $\underline{L_2}$ are* stably isomorphic *if and only if their discriminant modules $D_{\underline{L_1}}$ and $D_{\underline{L_2}}$ are isomorphic.*

Theorem 4.2.2 ([BS14, Theorem 2.3]). *Let $\underline{L_1}$ and $\underline{L_2}$ be two even positive definite lattices over $\mathbb{Z}$. Assume that $j : D_{\underline{L_2}} \xrightarrow{\cong} D_{\underline{L_1}}$ is an isomorphism of finite quadratic modules. Then the map*

$$I_j : J_{k+\lceil \frac{\mathrm{rk}(\underline{L_2})}{2} \rceil, \underline{L_2}} \longrightarrow J_{k+\lceil \frac{\mathrm{rk}(\underline{L_1})}{2} \rceil, \underline{L_1}}$$

given by

$$\phi(\tau, z) = \sum_{r \in L_2{}^{\#}/L_2} h_r(\tau) \vartheta_{\underline{L_2}, r}(\tau, z) \mapsto I_j(\phi) = \sum_{r \in L_2{}^{\#}/L_2} h_r(\tau) \vartheta_{\underline{L_1}, j^{-1}(r)}(\tau, z)$$

is an isomorphism.

Remark 4.2.3. In terms of Fourier coefficients we have

$$C_{I_j(\phi)}(D, s) = C_\phi(D, j(s)). \tag{4.6}$$

Here by abuse of language, we use $C_\phi(D, j(s))$ for $C_\phi(D, s')$, where $s' + L = j(s + L)$. Recall that $C_\phi(D, s)$ depends only D and on $s \bmod L$.

It is an interesting question whether the isomorphism of Theorem 4.2.2 commutes with Hecke operators. We shall show that is the case.

Theorem 4.2.4. *The isomorphism*

$$I_j : J_{k+\lceil \frac{\mathrm{rk}(L_2)}{2} \rceil, \underline{L_2}} \xrightarrow{\cong} J_{k+\lceil \frac{\mathrm{rk}(L_1)}{2} \rceil, \underline{L_1}},$$

which is defined in Theorem 4.2.2, commutes with the Hecke operators $T(\ell)$, i.e.,

$$T(\ell) I_j = I_j T(\ell) \tag{4.7}$$

for all $\ell \in \mathbb{N}_{\underline{L}}$.

Proof. Let ϕ be a Jacobi form of index (L_2, β_2) and of weight $k + \lceil \frac{\mathrm{rk}(L_2)}{2} \rceil$. According to Remark 4.2.3 we have the identity

$$C_{I_j(\phi)}(D,s) = C_\phi(D, j(s)) \tag{4.8}$$

for all $(D,s) \in \mathrm{supp}(\underline{\mathrm{L}})$. We will prove Equation (4.7) for Jacobi forms of even rank lattices; the case of odd rank lattices can be verified similarly. One has

$$\begin{aligned}
C_{I_j(T(\ell)\phi)}(D,s) =& C_{T(\ell)\phi}(D, j(s)) \\
=& \sum_a a^{k-1} \chi_{\underline{L}}(a) C_\phi\big(\tfrac{\ell^2}{a^2} D, \ell a' j(s)\big) \\
=& \sum_a a^{k-1} \chi_{\underline{L}}(a) C_{I_j(\phi)}\big(\tfrac{\ell^2}{a^2} D, \ell a' s\big) \\
=& C_{T(\ell)(I_j(\phi))}(D,s).
\end{aligned}$$

Here the third identity follows from Equation (4.8). □

The main result of this section is the following:

Theorem 4.2.5. *If the lattice $\underline{L} = (L, \beta)$ is stably isomorphic to the lattice $\big(\mathbb{Z}, (x,y) \mapsto \det(\underline{L})xy\big)$, then there is a Hecke-equivariant isomorphism*

$$J_{k,\underline{L}} \xrightarrow{\cong} \mathfrak{M}_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)^-, \tag{4.9}$$

where $\mathfrak{M}_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)$ is the Certain Space *inside $M_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)$ which was introduced in [SZ88, 3], and where the "−" sign denotes the subspace of all $f \in \mathfrak{M}_{2k-1-\mathrm{rk}(\underline{L})}\big(\mathrm{lev}(\underline{L})/4\big)$ such that $f \mid W_{\mathrm{lev}(\underline{L})/4} = -(-1)^{k/2} f$.*

Proof. Since $\underline{L} = (L, \beta)$ is stably isomorphic to the lattice $\underline{B} = \big(\mathbb{Z}, (x, y) \mapsto \det(\underline{L})xy\big)$, then according to Theorem 4.2.1 and Theorem 4.2.4, there is an isomorphism $J_{k,\underline{L}} \xrightarrow{\cong} J_{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil+1,\underline{B}}$ which commutes with the action of the Hecke operators. Now, the result follows by applying the main theorem of [SZ88] on the vector space $J_{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil+1,\underline{B}}$ and the fact that in this case $\mathrm{lev}(\underline{L}) = 2\det(\underline{L})$. □

This page is intentionally left blank.

Chapter 5

Lifting from Elliptic Modular Forms

Recall that we use the notation $\underline{L} = (L, \beta)$ to denote a positive definite even lattice over $\mathbb{Z}$. We shall use $\mathrm{lev}(\underline{L})$, $\mathrm{rk}(\underline{L})$, and $\det(\underline{L})$ to denote the level, the rank, and the determinant of the lattice $\underline{L}$, as defined at section 1.2 respectively.

We observed in the previous chapters that the multiplicative properties of Jacobi forms of weight k and index $\underline{L}$ are similar to the multiplicative properties of elliptic modular forms of weight $k - \frac{\mathrm{rk}(\underline{L})}{2}$ if the rank $\mathrm{rk}(\underline{L})$ is even. In this chapter we will provide some examples of this relation between the two vector spaces.

The first obvious example is the case of Jacobi forms whose indexes are unimodular lattices. Let $\underline{L}$ be unimodular, and $\phi \in J_{k,\underline{L}}$. The function ϕ has an expansion $\phi(\tau, z) = h_0(\tau)\vartheta_{\underline{L},0}(\tau, z)$, where $h_0(\tau) \in M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\mathrm{SL}_2(\mathbb{Z}))$. The Fourier expansion of h_0 is given by $h_0(\tau) = \sum_{n\geq 0} a_{h_0}(n)\mathfrak{e}(n\tau)$, with $a_{h_0}(n) := C_\phi(-n, 0)$. For every positive integer ℓ one has

$$C_{T(\ell)\phi}(-n, 0) = \sum_{a|\ell^2,n} a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} C_\phi\left(\frac{-\ell^2}{a^2}n, 0\right) = a_{T(\ell^2)h_0}(n).$$

Namely

$$(T(\ell)\phi)(\tau, z) = (T(\ell^2)h_0)(\tau)\vartheta_{\underline{L},0}(\tau, z)$$

which support the mentioned expectation.

5.1 Jacobi Forms of Prime Discriminant and Hecke Operators

In this section we assume that the determinant of the lattice $\underline{L} = (L, \beta)$ is an odd prime p. It is known from Lemma 1.2.16 that $\mathrm{rk}(\underline{L})$ is even and $(-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} p \equiv 1 \bmod 4$. The application $a \mapsto \chi(a) := \chi_{\underline{L}}(a) = \left(\frac{a}{p}\right)$ defines a Dirichlet character modulo p. According to Theorem 1.2.3 there is an isomorphism $\varphi : D_{\underline{L}} \xrightarrow{\cong} (\mathbb{Z}/p\mathbb{Z}, x \mapsto \frac{\alpha x^2}{p})$, where $\alpha \in \mathbb{Z}$ with $\gcd(\alpha, p) = 1$. We know from Milgram's formula that

$$\sum_{x \in L^\#/L} \mathfrak{e}\left(\beta(x)\right) = \sqrt{p}\ \mathfrak{e}\left(\mathrm{rk}(\underline{L})/8\right). \tag{5.1}$$

By Lemma 2.1.7, which gives the well-known formula for Gauss sum, the left-hand side of Equation (5.1) is equal to

$$\sum_{x \in L^\#/L} \mathfrak{e}\left(\beta(x)\right) = \chi(\alpha)\epsilon(p)\sqrt{p},$$

where, as usual, $\epsilon(p) = \sqrt{\left(\frac{-4}{p}\right)}$. Thus we find that $\mathfrak{e}\left(\mathrm{rk}(\underline{L})/8\right) = \chi(\alpha)\epsilon(p)$. In fact, we obtain the following table for $\mathrm{rk}(\underline{L})$ modulo 8:

	$p \equiv 1 \bmod 4$	$p \equiv 3 \bmod 4$
$\chi(\alpha) = 1$	0	2
$\chi(\alpha) = -1$	4	6

Table 5.1: $\mathrm{rk}(\underline{L})$ modulo 8

For a positive even integer k we set $k_2 := k - \frac{\mathrm{rk}(\underline{L})}{2}$. We shall explicitly construct a map

$$\mathscr{S} : M_{k_2}\left(p, \chi\right) \to J_{k, \underline{L}}.$$

Note that $M_{k_2}\left(p, \chi\right) = 0$ unless k is even. The map $\mathscr{S}$ will turn out to be surjective. However, in general, it will not be injective, but we shall see that we can restrict $\mathscr{S}$ to a natural subspace of $M_{k_2}\left(p, \chi\right)$, invariant under all $T(\ell^2)$, so that we obtain an isomorphism.

Definition 5.1.1. For $t \in \{\pm 1\}$ and $k \in \mathbb{N}$ we shall use $M_k^t\left(p, \chi\right)$ for the subspace of elliptic modular forms f of integral weight k on $\Gamma_0(p)$ with nebentypus χ

whose Fourier expansion is of the form

$$f(\tau) = \sum_{\substack{n\geq 0 \\ \chi(-n)\neq -t}} a_f(n)q^n. \tag{5.2}$$

It is known that $M_k(p,\chi) = M_k^{+1}(p,\chi) \bigoplus M_k^{-1}(p,\chi)$ (see e.g. [BB03, p.3]). Moreover, it is easy to see that the spaces $M_k^{\pm 1}(p,\chi)$ are invariant under the action of the Hecke operators $T(\ell^2)$ for all positive integers ℓ such that $(\ell, p) = 1$.

Theorem 5.1.2. *Let k be an even positive integer, and $k_2 := k - \frac{\mathrm{rk}(\underline{L})}{2}$. The applications*

$$f \mapsto \tfrac{1}{2} \sum_{A\in\Gamma_0(p)\backslash \mathrm{SL}_2(\mathbb{Z})} \Theta\big|_{k,\underline{L}} A, \tag{5.3}$$

where

$$\Theta(\tau,z) := (f\big|_{k_2} W_p)(\tau)\vartheta_{\underline{L},0}(\tau,z),$$

and

$$\phi \mapsto (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} h_0|_{k_2} W_p, \tag{5.4}$$

define maps $\mathscr{S} : M_{k_2}^{\chi(\alpha)}(p,\chi) \to J_{k,\underline{L}}$ and $\Omega : J_{k,\underline{L}} \to M_{k_2}^{\chi(\alpha)}(p,\chi)$ respectively. The maps $\mathscr{S}$ and Ω are mutually inverse isomorphisms.

To prove this theorem, we will need the next lemmas.

Lemma 5.1.3. *The application Ω defined by Equation (5.4) maps $J_{k,\underline{L}}$ to $M_{k_2}^{\chi(\alpha)}(p,\chi)$.*

Proof. Given $\phi = \sum_{x\in L^\#/L} h_x \vartheta_{\underline{L},x} \in J_{k,\underline{L}}$ then we deduce by Theorem 2.3.4 that $h_0 \in M_{k_2}(p,\chi)$. Since the Fricke involution is an automorphism of $M_{k_2}(p,\chi)$, then $h_0|_{k_2} W_p$ also belongs to the same vector space. For showing that $\Omega(\phi)$ is indeed in the right subspace, we write $W_p = S\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)$. So that

$$\Omega(\phi)(\tau) = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \Big(h_0|_{k_2} S|_{k_2} \left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)\Big)(\tau).$$

Applying the transformation laws from Proposition 2.4.9 and Proposition 2.2.8, we obtain

$$\Omega(\phi)(\tau) = i^{\frac{-\mathrm{rk}(\underline{L})}{2}} p^{(k_2-1)/2} \sum_{x\in L^\#/L} h_x(p\tau).$$

Now, inserting the Fourier expansion of the h_x as given in Proposition 2.4.7 gives

$$\Omega(\phi)(\tau) = i^{\frac{-\mathrm{rk}(\underline{L})}{2}} p^{(k_2-1)/2} \sum_{n\geq 0} \sum_{\substack{x\in L^\#/L \\ -n/p\equiv\beta(x) \bmod \mathbb{Z}}} C_\phi(-n/p,x)\mathfrak{e}(n\tau).$$

Then the n-th Fourier expansion of $\Omega(\phi)$ is 0 unless there exists an $x \in L^\#$ such that $-n/p \equiv \beta(x) \bmod \mathbb{Z}$. But this condition equivalent, in view of the isomorphism $D_{\underline{L}} \cong (\mathbb{Z}/p\mathbb{Z}, t \mapsto \frac{\alpha t^2}{p})$, to the exists of $t \in \mathbb{Z}$ such that $-n \equiv \alpha t^2 \bmod p$, i.e., it is equivalent to $\left(\frac{-n}{p}\right) = \left(\frac{\alpha}{p}\right)$ or $p \mid n$, i.e. equivalent to $\left(\frac{-n}{p}\right) \neq -\left(\frac{\alpha}{p}\right)$. Thus $\Omega(\phi) \in M_{k_2}^{\chi(\alpha)}(p,\chi)$. □

Lemma 5.1.4. *The application $\mathscr{S}$ defined by Equation (5.3) is well-defined. (i.e. the sum on the right does not depend on the choice of the representative A.)*

Proof. For $G = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, one has

$$\Theta\big|_{k,\underline{L}} G = (f\big|_{k_2} W_p G)\ \vartheta_{\underline{L},0}\big|_{\frac{\mathrm{rk}(\underline{L})}{2}} G.$$

Now By Theorem 2.3.4 and Proposition 2.3.1 we have

$$\vartheta_{\underline{L},0}\big|_{\frac{\mathrm{rk}(\underline{L})}{2}} G = \chi(d)\vartheta_{\underline{L},0}.$$

Moreover, $f\big|_{k_2} W_p G = \chi(d) f\big|_{k_2} W_p$ since $f\big|_{k_2} W_p$ is in $M_{k_2}(p,\chi)$ (see preceding proof.). Now, the identity $\chi(d)^2 = 1$ completes the proof. □

Proposition 5.1.5. *Let k be an even positive integer, $k_2 := k - \frac{\mathrm{rk}(\underline{L})}{2}$, and $f = \sum_{n\in\mathbb{N}} a_f(n)\mathfrak{e}(n\tau) \in M_{k_2}(p,\chi)$. Then $\mathscr{S}(f) \in J_{k,\underline{L}}$. Moreover, if $f \in M_{k_2}^{\chi(\alpha)}(p,\chi)$ then*

$$\mathscr{S}(f)(\tau,z) = \sum_{x\in L^\#/L} h_{x,f}(\tau)\vartheta_{\underline{L},x}(\tau,z),$$

with

$$h_{x,f}(\tau) = \sum_{\substack{D\in\mathbb{Q}, D\leq 0 \\ D\equiv\beta(x) \bmod \mathbb{Z}}} C_{\mathscr{S}(f)}(D,x)\mathfrak{e}(-D\tau), \tag{5.5}$$

where

$$C_{\mathscr{S}(f)}(D,x) = \frac{i^{\frac{\mathrm{rk}(\underline{L})}{2}} p^{\frac{1-k_2}{2}}}{1+\delta(x \notin L)} a_f(-pD).$$

Proof. First, we want to show that $\mathscr{S}(f)$ is a Jacobi form of weight k and index $\underline{L}$. Any $B \in \mathrm{SL}_2(\mathbb{Z})$ permutes $\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})$ by right multiplication. That is, the map

$$\varphi : \Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z}) \to \Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z}) \text{ given by } \Gamma_0(p)A \mapsto \Gamma_0(p)AB$$

is well defined and bijective. So if $\{A_i\}$ is a set of representatives for $\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})$ then $\{A_iB\}$ is a set of representatives for $\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})$ as well. Thus

$$\mathscr{S}(f)\big|_{k,\underline{L}}B = \frac{1}{2}\sum_{AB\in\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})}\Theta\big|_{k,\underline{L}}AB = \mathscr{S}(f).$$

Let $(\lambda,\mu,1)\in H_{\underline{L}}(\mathbb{Z})$, the identity

$$\vartheta_{\underline{L},0}(\tau,z+\lambda\tau+\mu)\mathfrak{e}\left(\tau\beta(\lambda)+\beta(\lambda,z)\right) = \vartheta_{\underline{L},0}(\tau,z)$$

implies that $\mathscr{S}(f)\big|_{k,\underline{L}}(\lambda,\mu,1) = \mathscr{S}(f)$. Therefore $\mathscr{S}(f)$ transforms like a Jacobi form of weight k and index $\underline{L} = (L,\beta)$. Now, we want to prove the claimed expansion, which also implies the holomorphicity of $\mathscr{S}(f)$ at infinity. As a set of representatives for $\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})$ we take the elements

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad ST^j = \begin{pmatrix}0&-1\\1&j\end{pmatrix} \text{ for } 0\le j<p.$$

Therefore

$$2\mathscr{S}(f) = \Theta + \sum_{j \bmod p}\left(f\big|_{k_2}W_pST^j\right)\left(\vartheta_{\underline{L},0}|_{\frac{\mathrm{rk}(\underline{L})}{2}}ST^j\right).$$

By applying the transformation laws of the Jacobi theta series (see Theorem 2.3.3) and $W_PST^j = \left(\begin{smallmatrix}-1&-j\\0&-p\end{smallmatrix}\right)$, we obtain

$$\begin{aligned}2\mathscr{S}(f)(\tau,z) &= \Theta(\tau,z) + i^{\frac{\mathrm{rk}(\underline{L})}{2}}p^{-\frac{k_2}{2}-\frac{1}{2}}\sum_{j \bmod p} f\left(\tfrac{\tau+j}{p}\right)\sum_{x\in L^\#/L}\vartheta_{\underline{L},x}(\tau,z)\mathfrak{e}\left(j\beta(x)\right)\\ &= \Theta(\tau,z) + i^{\frac{\mathrm{rk}(\underline{L})}{2}}p^{\frac{1-k_2}{2}}\sum_{x\in L^\#/L}\vartheta_{\underline{L},x}(\tau,z)\sum_{\substack{n\ge 0\\ -n/p\equiv\beta(x) \bmod \mathbb{Z}}} a_f(n)\mathfrak{e}\left(\tfrac{n\tau}{p}\right),\end{aligned}$$

where, in the second identity, we used

$$\sum_{j \bmod p}\mathfrak{e}\left(\left(\tfrac{n}{p}+\beta(x)\right)j\right) = p\ \boldsymbol{\delta}\left(-\tfrac{n}{p}\equiv\beta(x) \bmod \mathbb{Z}\right).$$

If $f \in M_{k_2}^{\chi(\alpha)}(p,\chi)$ then by [Kri91, P.672] one has

$$\big(f\,|_{k_2} W_p\big)(\tau) = i^{\frac{\mathrm{rk}(\underline{L})}{2}} p^{\frac{1-k_2}{2}} \sum_{\substack{n\in\mathbb{N}\\ n\equiv 0 \bmod p}} a_f(n)\mathfrak{e}\left(\tfrac{n\tau}{p}\right). \tag{5.6}$$

Inserting this into the last formula for $\mathscr{S}(f)$ gives

$$\mathscr{S}(f)(\tau,z) = \sum_{x\in L^\#/L} \vartheta_{\underline{L},x}(\tau,z) \sum_{\substack{n\geq 0\\ -n/p\equiv\beta(x) \bmod \mathbb{Z}}} \frac{i^{\frac{\mathrm{rk}(\underline{L})}{2}} p^{\frac{1-k_2}{2}}}{1+\boldsymbol{\delta}(x\notin L)} a_f(n)\mathfrak{e}\left(\tfrac{n\tau}{p}\right).$$

Now, the proof is complete. □

Proof of Theorem 5.1.2. Let $f \in M_{k_2}^{\chi(\alpha)}(p,\chi)$, and let $\phi \in J_{k,\underline{L}}$ with an expansion

$$\sum_{x\in L^\#/L} h_x(\tau)\vartheta_{\underline{L},x}(\tau,z).$$

By virtue of Lemma 5.1.3 and Proposition 5.1.5 we have only to prove that

$$\mathscr{S}(\Omega(\phi)) = \phi \quad \text{and} \quad \Omega(\mathscr{S}(f)) = f.$$

By Equation (5.3) and Equation (5.4) we have

$$\mathscr{S}(\Omega(\phi)) = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \mathscr{S}(h_0|_{k_2}W_p) = \sum_{x\in L^\#/L} h_{x,h_0|_{k_2}W_p}(\tau)\vartheta_{\underline{L},x}(\tau,z),$$

where

$$\begin{aligned} h_{x,h_0|_{k_2}W_p}(\tau) &= (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} p^{\frac{1-k_2}{2}} \frac{i^{\frac{\mathrm{rk}(\underline{L})}{2}}}{1+\boldsymbol{\delta}(x\notin L)} \sum_{\substack{n\geq 0\\ -n/p\equiv\beta(x) \bmod \mathbb{Z}}} a_{h_0|_{k_2}W_p}(n)\mathfrak{e}(n\tau/p) \\ &= \frac{1}{1+\boldsymbol{\delta}(x\notin L)} \sum_{\substack{n\geq 0\\ -n/p\equiv\beta(x) \bmod \mathbb{Z}}} \sum_{\substack{y\in L^\#/L\\ -n/p\equiv\beta(y) \bmod \mathbb{Z}}} C_\phi(-n/p,y)\mathfrak{e}(n\tau/p). \end{aligned}$$

Since $\det(\underline{L}) = \mathrm{lev}(\underline{L}) = p$ is an odd prime and k is even, the condition $\beta(x) \equiv \beta(y) \bmod \mathbb{Z}$ is equivalent to $C_\phi(-n/p,y) = C_\phi(-n/p,x)$. Thus

$$h_{x,h_0|_{k_2}W_p}(\tau) = \frac{\#\{y\in L^\#/L \mid \beta(y)\equiv\beta(x) \bmod \mathbb{Z}\}}{1+\boldsymbol{\delta}(x\notin L)} h_x = h_x.$$

Thus the claimed identity $\mathscr{S}(\Omega(\phi)) = \phi$ holds true. Also the second claimed identity $\Omega(\mathscr{S}(f)) = f$ holds true since

$$\Omega(\mathscr{S}(f)) = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} h_{0,f}|_{k_2}W_p = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} f|_{k_2}{W_p}^2 = f.$$

Now, the proof is complete. □

Theorem 5.1.6. *The isomorphism* $\mathscr{S} : M_{k_2}^{\chi(\alpha)}(p,\chi) \to J_{k,\underline{L}}$, *which is defined in Theorem 5.1.2, commutes with the action of the Hecke operators. More precisely, for each* $f \in M_{k_2}^{\chi(\alpha)}(p,\chi)$ *and* $\ell \in \mathbb{N}_{\underline{L}}$, *one has*

$$T(\ell)(\mathscr{S}(f)) = \mathscr{S}\left(T(\ell^2)f\right).$$

Proof. Using the explicit action of the operator $T(\ell)$ on Fourier coefficients (see Theorem 2.6.3, one has

$$C_{T(\ell)\mathscr{S}(f)}(D,x) = \sum_{a|\ell^2,pD} a^{k_2-1}\,\chi_{\underline{L}}(a) C_{\mathscr{S}(f)}\big(\tfrac{\ell^2}{a^2}D, \ell a' x\big),$$

where $a' \in \mathbb{Z}$ with $aa' \equiv 1 \bmod p$, and

$$C_{\mathscr{S}(f)}\big(\tfrac{\ell^2}{a^2}D, \ell a' x\big) = \frac{i^{\frac{\mathrm{rk}(\underline{L})}{2}} p^{\frac{1-k_2}{2}}}{1+\boldsymbol{\delta}(\ell a' x \notin L)} a_f(-\tfrac{\ell^2}{a^2}pD) \qquad \text{(see Proposition 5.1.5).}$$

Since $(\ell,p)=1$, $\boldsymbol{\delta}(\ell a' x \notin L) = \boldsymbol{\delta}(x \notin L)$. This gives

$$C_{T(\ell)\mathscr{S}(f)}(D,x) = \frac{i^{\frac{\mathrm{rk}(\underline{L})}{2}} p^{\frac{1-k_2}{2}}}{1+\boldsymbol{\delta}(x \notin L)} \sum_{a|\ell^2,pD} a^{k_2-1}\,\chi_{\underline{L}}(a) a_f(-\tfrac{\ell^2}{a^2}pD).$$

Now by using Equation (1.7)), which gives a closed formula for the action of Hecke operators on modular forms of integral weights in terms of Fourier coefficients, we obtain

$$C_{T(\ell)\mathscr{S}(f)}(D,x) = \frac{i^{\frac{\mathrm{rk}(\underline{L})}{2}} p^{\frac{1-k_2}{2}}}{1+\boldsymbol{\delta}(x \notin L)} a_{T(\ell^2)f}(-pD) = C_{\mathscr{S}(T(\ell^2)f)}(D,x)$$

as claimed in the theorem. □

Example 5.1.7. Consider the Gram matrix $F = \left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$. One has $\det F = 3$. The application $a \mapsto \chi(a) := \left(\frac{3}{a}\right)$ defines a Dirichlet Character mod 3. Let $\underline{L} = (\mathbb{Z}^2, (x,y) \mapsto x^t F y)$. One has $D_{\underline{L}} \cong (\mathbb{Z}_3, x \mapsto \alpha x^2/3)$, where $\alpha \in \mathbb{Z}$ with $(\alpha,3)=1$. By Table 5.1 we have $\chi(\alpha) = +1$ (since $\mathrm{rk}(\underline{L}) = 2$). The vector space $M_3(3,\chi)$ is two dimensional as can be easily seen by running the following Sage session

Listing 5.1: Sage input

```
e = DirichletGroup(3, RationalField()).gen()
M=ModularForms(e,3,prec=20);print M
```

Listing 5.2: Sage output

```
Modular Forms space of dimension 2, character [-1] and weight 3
    over Rational Field
```

The theta function $\theta(\tau) := \vartheta_{\underline{L},0}(\tau,0) = \sum_{x,y\in\mathbb{Z}} q^{x^2+xy+y^2}$ is an element of $M_1(3,\chi)$ (see Proposition 2.3.1 and Theorem 2.3.4). Clearly $A := \theta^3 \in M_3(3,\chi)$. Also, by using modular derivative (see e.g. [Lan76, Sec. 10.5]) we see that $B := \theta E_2 - 12q\frac{d}{dq}\theta \in M_3(3,\chi)$ $(E_2(\tau) = 1-24\sum_{n\geq 1}\sigma_1(n)q^n)$. The functions A and B are linearly independent. Since $M_3(3,\chi)$ is two-dimensional (see Listing 5.1 and Listing 5.2), A and B form a basis. The first Fourier coefficients of A and B are

$$\begin{aligned} A =& 1+18q+108q^2+234q^3+234q^4+864q^5+756q^6+900q^7+ \\ &+1836q^8+2178q^9+O(q^{10}), \\ B =& 1-90q-216q^2-738q^3-1170q^4-1728q^5-2160q^6-4500q^7 \\ &-3672q^8-6570q^9+O(q^{10}), \end{aligned}$$

which can be computed using the following Sage (a computer algebra system [S⁺11]) code

Listing 5.3: Sage input

```
ec = lambda n : sum([d for d in divisors(n) if 1 == gcd(3,d)])
R.<q> = PowerSeriesRing(ZZ)
E = 1 + 12 * sum( [ec(n)*q^n for n in range( 1, 11)]) + O(q^10)
th = sum( q^(x^2+x*y+y^2) for x in range(-19,20) for y in range
    (-19,20)) + O(q^10)
E2 = 1 - 24*sum( sigma(n)*q^n for n in range( 1, 11)) + O(q^10)
# Note E*th == th^3
A = th^3
B = -12*(q*th.derivative() - 1/12 * E2*th)
print "A= ",A
print "B= ",B
```

Listing 5.4: Sage output

```
A=  1 + 18*q + 108*q^2 + 234*q^3 + 234*q^4 + 864*q^5 + 756*q^6 +
    900*q^7 + 1836*q^8 + 2178*q^9 + O(q^10)
```

```
B=  1 - 90*q - 216*q^2 - 738*q^3 - 1170*q^4 - 1728*q^5 - 2160*q^6
    -4500*q^7 - 3672*q^8 - 6570*q^9 + O(q^10)
```

Next, we would like to find the Hecke eigenfunctions with respect to all Hecke operators $T(\ell)$ ($\ell \in \mathbb{N}$ with $\gcd(\ell,3)=1$). Let $\ell=2$. For all $f=\sum_{n\geq 1} a_f(n)q^n \in M_3(3,\chi)$ one has

$$a_{T(2)f}(n) = a_f(2n) - 4a_f(n/2)\boldsymbol{\delta}(n \text{ is even}). \tag{5.7}$$

Thus we find $T(2)A = -3+108q+O(q^2)$, and $T(2)B = -3-216q+O(q^2)$. The matrix $M(2)$ of the action of $T(2)$ satisfies $(T(2)A, T(2)B) = (A,B)M(2)$, i.e., $M(2) = \begin{pmatrix} -\frac{3}{2} & -\frac{9}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{pmatrix}$. The eigenvalues of $M(2)$ are $\{3,-3\}$. The normalized eigenfunctions are

$$\mathbf{e}_3 := \tfrac{A-B}{108} = q+3q^2+9q^3+13q^4+24q^5+27q^6+50q^7+51q^8+O(q^9),$$
$$\mathbf{e}_{-3} := \tfrac{3A+B}{2} = 1-9q+27q^2-9q^3-117q^4+216q^5+27q^6-450q^7+459q^8O(q^9).$$

Of course, by Equation (5.7) for the Hecke action we deduce that

$$\mathbf{e}_{-3} = 1-9\sum_{n\geq 1}\Big(\sum_{d|n} d^2\chi(d)\Big)q^n, \qquad \mathbf{e}_3 = \sum_{n\geq 1}\Big(\sum_{d|n} d^2\chi(n/d)\Big)q^n.$$

It is obvious that

$$\begin{aligned} f^+ :=& 9\mathbf{e}_3+\mathbf{e}_{-3} = 1+9\sum_{n\geq 1}\Big(\sum_{d|n} d^2(\chi(n/d)-\chi(d))\Big)q^n \\ =& 1+54q^2+72q^3+432q^5+270q^6+918q^8+720q^9+O(q^{10}) \end{aligned}$$

is in the subspace $M_3^{+1}(3,\chi)$, and

$$\begin{aligned} f^- :=& 9\mathbf{e}_3-\mathbf{e}_{-3} = -1+18\sum_{n\geq 1}\Big(\sum_{d|n} d^2(\chi(n/d)+\chi(d))\Big)q^n \\ =& -1+18q+90q^3+234q^4+216q^6+900q^7+738q^9+O(q^{10}) \end{aligned}$$

is in the subspace $M_3^{-1}(3,\chi)$. Since $M_3(3,\chi) = M_3^{+1}(3,\chi) \oplus M_3^{-1}(3,\chi)$ is two dimensional, we observe that $f^\pm$ are Hecke eigenforms for all Hecke operators $T(\ell)$ ($\gcd(\ell,3)=1$). The corresponding eigenvalue $\lambda(\ell, f^+)$ of f^+ equals $\sum_{a|\ell}\chi(a)a^2$ (using Proposition 1.1.11). Now we construct the Jacobi form

$\mathscr{S}(f^+) \in J_{4,\underline{L}}$ using Proposition 5.1.5. Note that $\dim J_{4,\underline{L}} = 1$. Thus $\mathscr{S}(f^+)$ is a Hecke eigenfunction for all Hecke operators $T(\ell)$ ($\gcd(\ell,3) = 1$) with Hecke eigenvalues $\lambda(\ell, \mathscr{S}(f^+))$ given by the equation

$$\lambda(\ell, \mathscr{S}(f^+)) C_{\mathscr{S}(f^+)}(D,0) = \sum_{a \mid \ell^2} a^2 \chi_{\underline{L}}(a) C_{\mathscr{S}(f^+)}\left(\tfrac{\ell^2}{a^2} D, 0\right).$$

If we choose $D = 0$, then $C_{\mathscr{S}(f^+)}(0,0) = \text{constant} \cdot a_{f^+}(0) \neq 0$ and

$$\lambda(\ell, \mathscr{S}(f^+)) = \sum_{a \mid \ell^2} a^2 \chi_{\underline{L}}(a) = \lambda(\ell^2, f^+)$$

as claimed.

5.2 Operators on the Vector-Valued Components

In this section we recall the definition of the vector-valued modular forms associated with Weil representation. Then we define Hecke operators $T(\ell)$ on these forms. Next, we relate these operators to the ones which developed in [BS07]. Again, we shall use $\underline{L} = (L, \beta)$ to denote an even positive definite lattice over $\mathbb{Z}$ (see section 1.2), and $\rho_{\underline{L}}$ for the Weil representation $\rho_{\underline{L}} : \widetilde{\mathrm{SL}_2(\mathbb{Z})} \to GL(\mathbb{C}[L^\#/L])$ associated with $\underline{L}$ (see section 2.3).

Definition 5.2.1. A vector-valued function $h : \mathfrak{H} \to \mathbb{C}[L^\#/L]$ is a vector-valued modular form on $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ of weight $k \in \frac{1}{2}\mathbb{Z}$ and type $\rho_{\underline{L}}$ if

1. $h \mid_k \widetilde{A} = \rho_{\underline{L}}(\widetilde{A}) h$ for all $\widetilde{A} \in \widetilde{\mathrm{SL}_2(\mathbb{Z})}$.
2. f is holomorphic on $\mathfrak{H}$ and at the cusp ∞.

We shall use $M_k(\rho_{\underline{L}})$ to denote the vector space of all such vector-valued modular forms.

Let $h \in M_k(\rho_{\underline{L}})$. We denote the components of h by h_x, so that

$$h = \sum_{x \in L^\#/L} h_x \delta_x.$$

The invariance of h under $|_k\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),1\right)$ implies that $\mathfrak{e}\left(\beta(x)\right)h_x$ are periodic with period 1. Thus each h_x has a Fourier expansion

$$h_x = \sum_{\substack{D\in\mathbb{Q}\\(D,x)\in\operatorname{supp}(\underline{\mathrm{L}})}} C_h(D,x)\,\mathfrak{e}(-D\tau). \tag{5.8}$$

Recall that the vector-valued Jacobi theta series

$$\Theta = \sum_{x\in L^{\#}/L} \vartheta_{\underline{L},x}\delta_x$$

is a vector-valued modular form of weight $\frac{\operatorname{rk}(\underline{L})}{2}$ and type $\rho_{\underline{L}}$ (see Theorem 2.3.3).

For a positive integer k it is well-known from [BS14] that $M_{k-\frac{\operatorname{rk}(\underline{L})}{2}}(\rho_{\underline{L}})$ is isomorphic to $J_{k,\underline{L}}$ as $\mathbb{C}$-linear spaces via the correspondence

$$[\![\,\cdot \mid \overline{\Theta}]\!] : M_{k-\frac{\operatorname{rk}(\underline{L})}{2}}(\rho_{\underline{L}}) \to J_{k,\underline{L}} \quad \text{given by} \quad h \mapsto [\![h \mid \overline{\Theta}]\!]. \tag{5.9}$$

Definition 5.2.2. Let $\ell \in \mathbb{N}$ such that $\gcd(\ell,\operatorname{lev}(\underline{L})) = 1$. We define a Hecke operator $T(\ell)$ acting on the vector space $M_{k-\frac{\operatorname{rk}(\underline{L})}{2}}(\rho_{\underline{L}})$ via

$$[\![T(\ell)h \mid \overline{\Theta}]\!] := T(\ell)[\![h \mid \overline{\Theta}]\!] \qquad \left(h \in M_{k-\frac{\operatorname{rk}(\underline{L})}{2}}(\rho_{\underline{L}})\right). \tag{5.10}$$

It is obvious from the definition that for all pairs $(D,x) \in \operatorname{supp}(\underline{\mathrm{L}})$ one has

$$C_{T(\ell)h}(D,x) = C_{T(\ell)[\![h \mid \overline{\Theta}]\!]}(D,x), \tag{5.11}$$

where the Fourier coefficients $C_{T(\ell)[\![h \mid \overline{\Theta}]\!]}(D,x)$ of the Jacobi form $T(\ell)[\![h \mid \overline{\Theta}]\!]$ are given by Theorem 2.6.3 for even rank lattice, and by Theorem 2.6.1 for odd rank lattice. Moreover, the operator $T(\ell)$ on $M_{k-\frac{\operatorname{rk}(\underline{L})}{2}}(\rho_{\underline{L}})$ has a multiplicative property as in Theorem 2.7.11 for odd $\operatorname{rk}(\underline{L})$ and as in Theorem 2.7.4 for even $\operatorname{rk}(\underline{L})$.

The main result of [BS07] is a well-defined double coset operator acting on the space of vector-valued modular forms of weight k and type $\rho_{\underline{L}}$. In the next lines we will recall their construction briefly and then we shall show the relation between our Hecke operators $T(\ell)$ and those operators.

In the rest of this section we assume that $\mathrm{rk}(\underline{L})$ is even. It is well known that $\rho_{\underline{L}}$ factor through $\mathrm{SL}_2(\mathbb{Z})$. Moreover, it is trivial on the subgroup $\Gamma(\mathrm{lev}(\underline{L}))$. Recall that the map

$$\varphi : \mathrm{SL}_2(\mathbb{Z})/\Gamma(\mathrm{lev}(\underline{L})) \to \mathrm{SL}_2(\mathbb{Z}_{\mathrm{lev}(\underline{L})}) \text{ given by } A\Gamma(\mathrm{lev}(\underline{L})) \mapsto A \bmod \mathrm{lev}(\underline{L})$$

is an isomorphism of groups. Therefore, $\rho_{\underline{L}}$ factor through the finite group $\mathrm{SL}_2(\mathbb{Z}_{\mathrm{lev}(\underline{L})})$. Let N be a positive integer. We set

$$Q(N) = \left\{(M,r) \in \mathrm{GL}_2(\mathbb{Z}_N) \times \mathbb{Z}_N{}^{\times}; \det(M) \equiv r^2 \bmod N\right\}.$$

with a product defined component-wise. For $(M,r) \in Q(N)$ the assignment $(M,r) \mapsto (M\left(\begin{smallmatrix} r & 0 \\ 0 & r\end{smallmatrix}\right)^{-1}, r)$ defines an isomorphism $Q(N) \cong \mathrm{SL}_2(\mathbb{Z}_N) \times \mathbb{Z}_N{}^{\times}$.

Proposition 5.2.3. *Let $(M,r) \in Q(\mathrm{lev}(\underline{L}))$. Then (M,r) acts on $\mathbb{C}[D_{\underline{L}}]$ by Weil representation $\rho_{\underline{L}}$ as follows:*

$$\rho_{\underline{L}}((M,r)) \cdot \delta_\gamma = \chi_{\underline{L}}(r) \rho_{\underline{L}}\left(\varphi^{-1}(M\left(\begin{smallmatrix} r & 0 \\ 0 & r\end{smallmatrix}\right)^{-1})\right) \cdot \delta_\gamma$$

for all $\gamma \in D_{\underline{L}}$.

Proof. See [BS07, (3.5)] and [BS07, Proposition 3.3]. □

Definition 5.2.4 ([BS07, Definition 4.1])**.** Consider the Hecke pair $(Q(\mathrm{lev}(\underline{L})), \Gamma)$ in sense [Shi71]. For each $(M,r) \in Q(\mathrm{lev}(\underline{L}))$, the corresponding double cosets decompose into a finite union of left cosets

$$\Gamma(M,r)\Gamma = \bigcup_{\gamma \in \Gamma \backslash \Gamma M \Gamma} \Gamma(\gamma, r).$$

We define the corresponding double coset Hecke operator $\mathcal{H}(M,r)$ that is acting on the vector space of vector-valued modular forms $\mathcal{M}_k(\rho_{\underline{L}})$ by

$$h \big| \mathcal{H}(M,r) = \sum_{\gamma \in \Gamma \backslash \Gamma M \Gamma} \rho_L(\gamma, r)^{-1} h \big|_{k,\underline{L}} \det(\gamma)^{-\frac{1}{2}} \gamma. \tag{5.12}$$

Remark 5.2.5. The operator $\mathcal{H}(M,r)$ is exactly the operator $T(M,r)$ in [BS07, Definition 4.1] but without the normalization term.

Now we are ready to introduce our result:

Theorem 5.2.6. *Let* $\underline{L} = (L, \beta)$ *be an even positive definite lattice of even rank. Let* $\ell \in \mathbb{N}$ *such that* $(\ell, \mathrm{lev}(\underline{L})) = 1$. *For* $h \in M_k(\rho_{\underline{L}})$ *one has*

$$T(\ell)h = \ell^{k-2} \sum_{\substack{\ell' \mid \ell \\ \ell/\ell' = \square}} \sum_{\substack{s \mid \ell' \\ s \text{ is square-free}}} \chi_{\underline{L}}(s)\, h \,\Big|\, \mathscr{H}\Big(\Big(\begin{smallmatrix} (\ell'/s)^2 & 0 \\ 0 & 1 \end{smallmatrix}\Big), \ell'/s\Big). \tag{5.13}$$

Proof. We omit the proof, which is similar to the proof of Theorem 2.6.3. □

Remark 5.2.7. Here, in the last part of this section, we restricted ourselves to lattices of even rank. A similar treatment for lattices of odd rank can be done by replacing $\mathrm{SL}_2(\mathbb{Z})$ with its metaphoric cover $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$, the double coset Hecke operator $\mathscr{H}$ with the one which is defined in [BS07, 4.21], and $\chi_{\underline{L}}$ with the character which is defined in [McG03, Lemma 4.5].

This page is intentionally left blank.

Chapter 6

Appendix

6.1 Examples Using the Method of Theta Blocks

The Fourier expansion of theta blocks can be computed from the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\in\mathbb{N}} (1-q^n) = \sum_{n\in\mathbb{Z}} \left(\frac{24}{n}\right) q^{n^2}, \quad q = e^{2\pi i\tau} \tag{6.1}$$

which is a modular form of weight $\frac{1}{2}$ with a multiplier system on $\mathrm{SL}_2(\mathbb{Z})$, and the Jacobi theta function (a Jacobi form of weight and index $\frac{1}{2}$)

$$\vartheta(\tau,z) = \sum_{r\in\mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{r}{2}}. \tag{6.2}$$

Theorem 6.1.1 (The construction method [GSZ])**.** *Let $\alpha : L \to \mathbb{Z}^m$ be an isometric embedding into the m-fold orthogonal sum of $\underline{\mathbb{Z}}$. Then the function*

$$\vartheta(\tau,\alpha_1(z))\vartheta(\tau,\alpha_2(z))\dots\vartheta(\tau,\alpha_m(z))\eta(\tau)^t \tag{6.3}$$

defines an element of $J_{k,\underline{L}}$, where α_i is the i-th coordinate function of $\alpha, t \in \mathbb{N}$ such that $t + 3m \equiv 0 \pmod{24}$, and $k = \frac{t+m}{2}$.

Gram Matrix	Jacobi form	weight
$\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$	$\vartheta(\tau,z_2)\vartheta(\tau,z_1)\vartheta(\tau,z_1+z_2)\eta(\tau)^{15}$	9

$\left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)$	$\vartheta(\tau,z_2)\vartheta(\tau,z_2)\vartheta(\tau,z_2)\vartheta(\tau,z_1)\vartheta(\tau,z_1+z_2)\eta(\tau)^9$	7
$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\vartheta(\tau,z_3)\vartheta(\tau,z_3)\vartheta(\tau,z_1+z_2)\vartheta(\tau,z_1-z_2)\eta(\tau)^{12}$	8

Table 6.1: Examples of Jacobi forms using the method of Theta Blocks

Note that in the first row of Table 6.1 we used the embedding $(z_1,z_2)\mapsto(z_1,z_2,z_1+z_2)$, in the second row we used the embedding $(z_1,z_2)\mapsto(z_1,z_2,z_2,z_2,z_1+z_2)$, in the third we used the embedding $(z_1,z_2,z_3)\mapsto(z_3,z_3,z_1+z_2,z_1-z_2)$.

Lemma 6.1.2. *The maximal even sub-lattice $\mathbb{Z}^m_{\text{ev}}$ in $\mathbb{Z}^m$ is*

$$\mathbb{Z}^m_{\text{ev}} = \left\{(x_1,x_2,\ldots,x_m) \mid \sum_i x_i \in 2\mathbb{Z}\right\} \quad (m \geq 1). \tag{6.4}$$

Proof. We observe that $\mathbb{Z}^m_{\text{ev}}$ is even. Assuming that $\mathbb{Z}^m_{\text{ev}}$ is not the maximal even sub-lattice in $\mathbb{Z}^m$,i.e., there exists an even lattice such that $\mathbb{Z}^m_{\text{ev}} \subset L \subseteq \mathbb{Z}^m$. Let $z=(z_1,\cdots,z_m)$ be an element in L which is not in $\mathbb{Z}^m_{\text{ev}}$, i.e., $\sum_i z_i$ is odd. Then one has $z^t z = z_1^2+\cdots+z_m^2$ is odd. This contradicts the assumption that L is even. Hence, the assumption is false and the lemma is true. □

One has$|\mathbb{Z}^{m\,\#}_{\text{ev}}/\mathbb{Z}^m_{\text{ev}}| = 4$ (since $\mathbb{Z}^m_{\text{ev}} \subseteq_{\text{index}2} \mathbb{Z}^m = \mathbb{Z}^{m\,\#} \subseteq_{\text{index}2} \mathbb{Z}^{m\,\#}_{\text{ev}}$) . As a set of coset representatives of $\mathbb{Z}^{m\,\#}_{\text{ev}}/\mathbb{Z}^m_{\text{ev}}$ we take the set

$$\begin{aligned}\Delta = \Big\{&[0] = (0,0,\cdots,0),\\ &[1] = (\tfrac{1}{2},\tfrac{1}{2},\cdots,-\tfrac{1}{2}),\\ &[2] = (0,0,\cdots,1),\\ &[3] = (\tfrac{1}{2},\tfrac{1}{2},\cdots,\tfrac{1}{2})\Big\}.\end{aligned}$$

We have $D_{\underline{\mathbb{Z}^m_{\text{ev}}}} = (\mathbb{Z}^{m\,\#}_{\text{ev}}/\mathbb{Z}^m_{\text{ev}}, Q : x+\mathbb{Z}^m_{\text{ev}} \mapsto \frac{x^t x}{2} \bmod \mathbb{Z})$. Let $t\in\mathbb{N}$ such that $t+3m \equiv 0 \pmod{24}$. Set $k := \frac{t+m}{2}$. According to Theorem 6.1.1, the product

$$\Theta_{k,\mathbb{Z}^m_{\text{ev}}}(\tau,(z_1,\cdots,z_m)) := \vartheta(\tau,z_1)\vartheta(\tau,z_2)\ldots\vartheta(\tau,z_m)\eta(\tau)^t \tag{6.5}$$

belongs to $J_{k,\mathbb{Z}^m_{\text{ev}}}$. It is easy to see that the function $\Theta_{k,\mathbb{Z}^m_{\text{ev}}}$ given by Equation (6.5) has an expansion of the form

$$\Theta_{k,\mathbb{Z}^m_{\text{ev}}}(\tau,z) = \sum_{x=(x_1,\cdots,x_m)\in\frac{1}{2}\mathbb{Z}^m} h_x(\tau)\mathfrak{e}\left(\beta(x,z)+\beta(x)\tau\right),$$

where

$$h_x(\tau) = \left(\frac{-4}{(2x_1)\cdots(2x_m)}\right)\eta(\tau)^t.$$

Note that $\Theta_{k,\mathbb{Z}_{\mathrm{ev}}^m}$ is a cusp form (since h_x equal 0 or $\pm\eta^t$, which are all cusp forms).

For odd m we observe that

$$\begin{aligned}
[0] &\equiv 0\cdot[1] \bmod \mathbb{Z}_{\mathrm{ev}}^m,\\
[1] &\equiv 1\cdot[1] \bmod \mathbb{Z}_{\mathrm{ev}}^m,\\
[2] &\equiv 2\cdot[1] \bmod \mathbb{Z}_{\mathrm{ev}}^m,\\
[3] &\equiv 3\cdot[1] \bmod \mathbb{Z}_{\mathrm{ev}}^m.
\end{aligned}$$

Thus $\mathbb{Z}_{\mathrm{ev}}^{m\,\#}/\mathbb{Z}_{\mathrm{ev}}^m = <[1]> \cong \mathbb{Z}_4$ (By $\varphi : [x] + \mathbb{Z}_{\mathrm{ev}}^m \mapsto x + 4\mathbb{Z}$). Moreover, if we endow $\mathbb{Z}_4$ with the quadratic form $\underline{Q} : x + 4\mathbb{Z} \mapsto \frac{mx^2}{8} \bmod \mathbb{Z}$, then $\underline{Q}\circ\varphi = Q$, i.e.,

$$D_{\underline{\mathbb{Z}_{\mathrm{ev}}^m}} \cong (\mathbb{Z}_4, x + 4\mathbb{Z} \mapsto \frac{mx^2}{8} \bmod \mathbb{Z}) \tag{6.6}$$

as finite quadratic modules. We consider for each of these odd m ($m = 1,3,5,7$) the smallest $t \in \mathbb{N}$ such that $t \equiv -3m \bmod 24$

m	1	3	5	7
t	21	15	9	3
$k = \frac{t+m}{2}$	11	9	7	5
$2k - m - 1$	20	14	8	2
$\dim M_{2k-m-1}(\Gamma_0(2))$	6	4	3	1
$\dim S_{2k-m-1}(\Gamma_0(2))$	4	2	1	0
$\dim S^{\mathrm{new}}_{2k-m-1}(\Gamma_0(2))$	2	2	1	0
$\dim J_{k,\mathbb{Z}_{\mathrm{ev}}^m}$	1	1	1	1

Table 6.2

Note that the last line in Table 6.2 is obtained by using $J_{k,\mathbb{Z}_{\mathrm{ev}}^m} \cong M_{k-\frac{m}{2}}\big((\mathbb{Z}_4, x + 4\mathbb{Z} \mapsto -\frac{mx^2}{8} \bmod \mathbb{Z})\big)$ (using also Equation (6.6)) and the dimension formula for the vector-valued modular form as in [ES95, p.12]. (For $m = 7$ we need an additional argument, e.g. $\dim J_{9,\mathbb{Z}_{\mathrm{ev}}^7} = 1$ and $\dim J_{5,\mathbb{Z}_{\mathrm{ev}}^7} \cdot E_4 \subseteq J_{9,\mathbb{Z}_{\mathrm{ev}}^7}$). Note also that $\Theta_{5,\mathbb{Z}_{\text{ev}}^7}$ is a cusp form, but its h_x equal 0 or

$$h_x(8\tau) = \pm\eta(8\tau)^3 = \pm\sum_{n\geq 1}^{\infty}\left(\frac{-4}{n}\right)nq^{n^2}$$

which are "trivial" cusp forms of weight 3/2 in the sense of [Shi73, Proposition 2.2] and therefore, in the view of [Shi73, (c) p.478] and Remark 2.7.16, $\Theta_{5,\mathbb{Z}_{\text{ev}}^7}$ should lift to an Eisenstein series. This is in complete accordance with the fact that $M_2(\Gamma_0(2)) = \mathbb{C}(E_2(\tau) - 2E_2(2\tau))$.

From Table 6.2 we see that all $\Theta_{k,\mathbb{Z}_{\text{ev}}^m}$ ($m = 1,3,5,7$) are Hecke eigenfunctions. We calculate the first eigenvalues $\lambda(\ell,\Theta_{k,\mathbb{Z}_{\text{ev}}^m})$ of $T(\ell)$:

Table 6.3: The first eigenvalues $\lambda(\ell,\Theta_{k,\mathbb{Z}_{\text{ev}}^m})$

ℓ	$\lambda(\ell,\Theta_{11,\mathbb{Z}_{\text{ev}}^1})$	$\lambda(\ell,\Theta_{9,\mathbb{Z}_{\text{ev}}^3})$	$\lambda(\ell,\Theta_{7,\mathbb{Z}_{\text{ev}}^5})$	$\lambda(\ell,\Theta_{5,\mathbb{Z}_{\text{ev}}^7})$
1	1	1	1	1
3	−53028	−1836	12	4
5	−5556930	3990	−210	6
7	−44496424	−433432	1016	8
9	1649707317	1776573	−2043	13
11	6320674932	1619772	1092	12
13	−33124973098	−10878466	1382	14
15	294672884040	−7325640	−2520	24
17	−722355252174	60569298	14706	18
19	−1312620671860	−243131740	−39940	20
21	2359556371872	795781152	12192	32
23	3379752742152	−606096456	68712	24
25	11805984696775	−1204783025	−34025	31
27	−25848278533800	−334611000	−50760	40
29	−29378097714810	5258639310	−102570	30
31	131976476089952	−1824312928	227552	32
33	−335172750294096	−2973901392	13104	48
35	247263513418320	−1729393680	−213360	48

Continued on next page

Table 6.3 – continued from previous page

ℓ	$\lambda(\ell,\Theta_{11,\mathbb{Z}_{ev}^1})$	$\lambda(\ell,\Theta_{9,\mathbb{Z}_{ev}^3})$	$\lambda(\ell,\Theta_{7,\mathbb{Z}_{ev}^5})$	$\lambda(\ell,\Theta_{5,\mathbb{Z}_{ev}^7})$
37	−466464103652194	−3005875402	160526	38
39	1756551073440744	19972863576	16584	56
41	1889447681239482	−49704880758	10842	42
43	−4323507451065388	58766693084	−630748	44
45	−9167308081056810	7088526270	429030	78
47	12103384387771536	−42095878032	472656	48
49	−9418963436585367	90974288217	208713	57
51	38305054312282872	−111205231128	176472	72
53	−30593935900444338	−181140755706	−1494018	54
55	−35123548149878760	6462890280	−229320	72
57	69605648987392080	446389874640	−479280	80
59	9908742512283780	206730587820	2640660	60
61	−91638145794467098	−124479015058	827702	62
63	−73406076253134408	−770023588536	−2075688	104
65	184073156757469140	−43405079340	−290220	84
67	−103349440678278244	95665133588	−126004	68
69	−179221528410836256	1112793093216	824544	96
71	285448322456957592	−371436487128	−1414728	72
73	875008267167254042	−1800576064726	980282	74
75	−626047756500584700	2211981633900	−408300	124
77	−281247431740443168	−702061017504	1109472	96
79	−1081394522969090320	1557932091920	−3566800	80
81	−546708732286707639	−2218085399079	3858921	121
83	−665085275193888948	2492790917604	5672892	84
85	4014077571463265820	241671499020	−3088260	108
87	1557861765620944680	−9654861773160	−1230840	120
89	−2020985164277790390	2994235754490	−11951190	90
91	1473942847957201552	4715075275312	1404112	112
93	−6998448574097974656	3349438535808	2730624	128

Continued on next page

Table 6.3 – continued from previous page

ℓ	$\lambda(\ell,\Theta_{11,\mathbb{Z}^1_{\mathrm{ev}}})$	$\lambda(\ell,\Theta_{9,\mathbb{Z}^3_{\mathrm{ev}}})$	$\lambda(\ell,\Theta_{7,\mathbb{Z}^5_{\mathrm{ev}}})$	$\lambda(\ell,\Theta_{5,\mathbb{Z}^7_{\mathrm{ev}}})$
95	7294141190078989800	−970095642600	8387400	120
97	−12825578365118067934	4382492665058	8682146	98
99	10427263683698877444	2877643201356	−2230956	156

The first column in Table 6.3 ($m=1$) is covered by Theorem 4.2.5, i.e., $\Theta_{11,\mathbb{Z}^1_{\mathrm{ev}}}$ lift to elliptic modular form f of weight 20 on $\Gamma_0(2)$ with $f \mid W_2 = -f$. To determine this f, we use Sage to find the generators of $S_{20}^{\mathrm{new}}(\Gamma_0(2))$. Recall that the $\dim S_{20}^{\mathrm{new}}(\Gamma_0(2)) = 2$ (see Table 6.2)

Listing 6.1: Sage input

```
f,g=Newforms(2,20)
print "f=",f.qexp(12)
print "g=",g.qexp(12)
```

Listing 6.2: Sage input

```
f= q - 512*q^2 - 13092*q^3 + 262144*q^4 + 6546750*q^5 + 6703104*q
   ^6 +96674264*q^7 - 134217728*q^8 - 990861003*q^9 - 3351936000*
   q^10 +11799694452*q^11 + O(q^12)
g= q + 512*q^2 - 53028*q^3 + 262144*q^4 - 5556930*q^5 - 27150336*
   q^6 -44496424*q^7 + 134217728*q^8 + 1649707317*q^9 -
   2845148160*q^10 +6320674932*q^11 + O(q^12)
```

According to Theorem 1.1.15, the newform $f \in S_{20}^{\mathrm{new}}(\Gamma_0(2))$ given by (see Listing 6.2)

$$\begin{aligned} f(\tau) =& q + 512q^2 - 53028q^3 + 262144q^4 - 5556930q^5 - 27150336q^6 - 44496424q^7 \\ &+ 134217728q^8 + 1649707317q^9 - 2845148160q^{10} + O(q^{11}) \end{aligned}$$

is a normalized Hecke eigenfunction. We compute its Atkin-Lehner eigenvalue. Again, using Theorem 1.1.15 one has

$$f \mid W_2 = -2^{1-\frac{20}{2}} a_f(2) f = -f.$$

In fact, one has $\lambda(\ell,f)=\lambda(\ell,\Theta_{11,\mathbb{Z}^1_{\rm ev}})$ for all odd ℓ as follows from Theorem 4.2.5 and as can be verified numerically by comparing the eigenvalues $\lambda(\ell,f)=a_f(\ell)$ of f with the eigenvalues $\lambda(\ell,\Theta_{11,\mathbb{Z}^5_{\rm ev}})$ from Table 6.3.

Recall that if the $\Theta_{k,\mathbb{Z}^m_{\rm ev}}$ lifts to an elliptic modular form f of weight $k_1 = 2k-1-m$ then $L(s,\Theta_{k,\mathbb{Z}^m_{\rm ev}}) = \sum_{\ell \text{ is odd}} \lambda(\ell,\Theta_{k,\mathbb{Z}^m_{\rm ev}})\ell^{-s}$ should be (up to a finite number of Euler factors) the L-series of f (see Remark 2.7.16).

- $m=3$: The vector space $S_{14}(\Gamma_0(2))$ is of dimension 2. The generators of this space are

$$\begin{aligned} g_1 =& q-300q^3+4096q^4-26730q^5+98304q^6-184600q^7\\ &+854973q^9-1966080q^{10}+2042172q^{11}+O(q^{12}),\\ g_2 =& q^2+24q^3-480q^5-300q^6+3888q^7+4096q^8\\ &-14400q^9-26730q^{10}+6600q^{11}+O(q^{12}), \end{aligned}$$

which can be easily seen by running the following Sage session

Listing 6.3: Sage input

```
S=CuspForms(2,14,prec=12)
print S
g1,g2=S.gens()
print "g1=",g1
print "g2=",g2
```

Next, we would like to find the Hecke eigenfunctions with respect to all Hecke operators $T(\ell)$ ($\ell \in \mathbb{N}$ with $\gcd(\ell,2)=1$). Let $\ell=3$. Using Sage, we compute the matrix $M(3)$ of the action of $T(3)$, i.e. the matrix $M(3)$ which satisfies $(T(3)g_1, T(3)g_2) = (g_1,g_3)M(3)$. One has

$$M(3)=\begin{pmatrix} -300 & 98304\\ 24 & -300\end{pmatrix}.$$

The eigenvalues of $M(3)$ are $\{1236,-1836\}$, and the corresponding eigenfunctions are

$$f^{\pm} := g_1 \pm 64 g_2.$$

The first Fourier coefficients of f^+ and f^- are:

$$f^-(\tau) := q-64q^2-1836q^3+4096q^4+3990q^5+117504q^6-433432q^7$$

$$-262144q^8 + 1776573q^9 - 255360q^{10} + 1619772q^{11} + O(q^{12})$$
$$f^+(\tau) := q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + 79104q^6 + 64232q^7$$
$$+ 262144q^8 - 66627q^9 - 3676800q^{10} + 2464572q^{11} + O(q^{12}).$$

In fact, f^+ , $f^- \in S_{14}^{\text{new}}(\Gamma_0(2))$(since $S_{14}^{\text{old}}(\Gamma_0(2)) = 0$). Also we can verify this by running the following Sage session

Listing 6.4: Sage input

```
f_minus,f_plus=Newforms(2,14)
print "f-=",f_minus.qexp(12)
print "f+=",f_plus.qexp(12)
```

Listing 6.5: Sage input

```
f-= q - 64*q^2 - 1836*q^3 + 4096*q^4 + 3990*q^5 + 117504*q^6
   -433432*q^7 - 262144*q^8 + 1776573*q^9 - 255360*q^10 +
   1619772*q^11 +O(q^12)
f+= q + 64*q^2 + 1236*q^3 + 4096*q^4 - 57450*q^5 + 79104*q^6 +
   64232*q^7 + 262144*q^8 - 66627*q^9 - 3676800*q^10 + 2464572*q
   ^11 + O(q^12)
```

By the implementation of Sage, it is guarantied that $f^{\pm} = \sum_n a_{f^{\pm}}(n)q^n$ are simultaneous normalized eigenfunctions for all Hecke operators $T(\ell)$ with eigenvalues $\lambda(\ell, f^{\pm}) = a_{f^{\pm}}(\ell)$.

The function f^- has Atkin-Lehner eigenvalue equals 1, and for each odd ℓ we observe that the eigenvalue $\lambda(\ell, f^-)$ of f^- equals $\lambda(\ell, \Theta_{9,\mathbb{Z}_{\text{ev}}^3})$ from Table 6.3.

• $m = 5$: The cuspidal subspace $S_8(\Gamma_0(2))$ is of dimension 1. The generator of this subspace is $f(\tau) = \eta(\tau)^8\eta(2\tau)^8 = q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8$. The first coefficients of f are

$$f(\tau) = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 - 512q^8 - 2043q^9$$
$$+ 1680q^{10} + 1092q^{11} + O(q^{12}).$$

Since $\dim S_8(\Gamma_0(2)) = 1$, the function f is a normalized Hecke eigenfunction with respect to the operator $T(\ell)$. In fact, f has Atkin-Lehner eigenvalue equals 1, and for odd ℓ the Hecke eigenvalue $\lambda(\ell, f)$ equals $\lambda(\ell, \Theta_{7,\mathbb{Z}_{\text{ev}}^5})$. This can be verified by comparing the eigenvalues $\lambda(\ell, f)$ with $\lambda(\ell, \Theta_{7,\mathbb{Z}_{\text{ev}}^5})$ from Table 6.3.

- $m = 7$: The vector space $M_2(\Gamma_0(2))$ is of dimension 1. The generator of this space is

$$\begin{aligned} f(\tau) =& 24(E_2(\tau) - 2E_2(2\tau)) = 1 + 24 \sum_{\ell \geq 1} \Big(\sum_{\substack{d \mid \ell \\ d \text{ is odd}}} d \Big) q^\ell \\ =& 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + 96q^6 + 192q^7 + 24q^8 + 312q^9 \\ & + 144q^{10} + 288q^{11} + O(q^{12}). \end{aligned}$$

The function f has Atkin-Lehner eigenvalue equals $-\operatorname{sign}(a_f(2)) = -1$. Since $\dim M_2(\Gamma_0(2)) = 1$, the function $f(\tau)$ is a Hecke eigenfunction with eigenvalues $\lambda(\ell, f) = \sum_{d \mid \ell} \left(\frac{d}{4}\right) d$. In fact, one has $\lambda(\ell, f) = \lambda(\ell, \Theta_{5,\mathbb{Z}_{\text{ev}}^7})$ for all odd positive integers ℓ which can be verified by comparing the eigenvalues $\lambda(\ell, f)$ with $\lambda(\ell, \Theta_{5,\mathbb{Z}_{\text{ev}}^7})$ from Table 6.3.

6.1.1 Final remarks

It is obvious that the previous examples support our expectations in section 4.1. Namely, that the function $\mathscr{S}_{D_0,x}$ given by Definition 4.1.3 take Jacobi cusp forms of weight k and index $\underline{L}$ ($\operatorname{rk}(\underline{L})$ is odd) to elliptic modular forms of weight $2k - 1 - \operatorname{rk}(\underline{L})$ on $\Gamma_0(\operatorname{lev}(\underline{L})/2)$. In fact, the level is determined by our examples to be $\operatorname{lev}(\underline{L})/4$ instead of $\operatorname{lev}(\underline{L})/2$. Moreover, we observe that the elliptic modular forms that are lifts of Jacobi forms have Atkin-Lehner eigenvalue equal to $-\left(\frac{\operatorname{rk}(\underline{L})}{2}\right)$. Thus we conjecture the following:

Conjecture 6.1.3. *Let $\underline{L}$ be a positive definite even odd rank lattice over $\mathbb{Z}$. Setting $\varepsilon_L = -1$ if $\operatorname{rk}(\underline{L}) \equiv 1$ or $3 \bmod 8$, and 1 otherwise. There is a Hecke-equivariant isomorphism*

$$J_{k,\underline{L}} \xrightarrow{\cong} \mathfrak{M}_{2k-1-\operatorname{rk}(\underline{L})}\left(\operatorname{lev}(\underline{L})/4\right)^{\varepsilon_L},$$

where $\mathfrak{M}_{2k-1-\operatorname{rk}(\underline{L})}\left(\operatorname{lev}(\underline{L})/4\right)$ is the Certain Space *inside $M_{2k-1-\operatorname{rk}(\underline{L})}\left(\operatorname{lev}(\underline{L})/4\right)$ which was introduced in [SZ88, 3], and where the "ε_L" denotes the subspace of all $f \in \mathfrak{M}_{2k-1-\operatorname{rk}(\underline{L})}\left(\operatorname{lev}(\underline{L})/4\right)$ such that $f \mid W_{\operatorname{lev}(\underline{L})/4} = \varepsilon_L(-1)^{k/2} f$.*

This page is intentionally left blank.

List of Notations

$J_{\underline{L}}(\mathbb{Z})$ $\mathrm{SL}_2(\mathbb{Z}) \ltimes H_{\underline{L}}(\mathbb{Z})$ 29

$\underline{L} = (L, \beta)$ a Lattice over $\mathbb{Z}$, where L is free $\mathbb{Z}$-module of finite rank, and β is a non-degenerate symmetric bilinear form 12

$\lceil x \rceil$ $= \min\{n \in \mathbb{Z} \mid n \geq x\}$ 2

$\left(\frac{a}{n}\right)$ the Kronecker symbol 2

$\lfloor x \rfloor$ $= \max\{n \in \mathbb{Z} \mid n \leq x\}$ 2

$\mathbb{N}_{\underline{L}}$ the set of all positive integer $\ell \in \mathbb{N}$ with $(\ell, \mathrm{lev}(\underline{L})) = 1$ 14

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_p, \mathbf{Z}_p, \mathbf{Q}_p$, and $\mathbb{C}$ 1

$\mathfrak{H}$ the upper half plane (the Poincaré half plane) 3

$\mathfrak{S}_{\underline{L}}(s, \ell)$ 20

$|_k$ Petersson slash operator of weight k 5

$\mathrm{lev}(\underline{L}), \mathrm{lev}(x)$ 14

$\mathrm{ord}_p(n)$ the p-adic order or p-adic valuation 2

$\Pr(F)$ 78

$\mathscr{W}(c, a)$ 16

$\big|_{k,\underline{L}}$ Jacobi slash operator of weight k and index $\underline{L}$ 31

$\mathrm{supp}(\underline{L})$ 36

$T(\ell)$ a Hecke operator 41

$T_0(\ell)$ a double coset Hecke operator 40

$\vartheta_{\underline{L},x}$ the Jacobi theta series 33

a' Let a be a positive integer such that a coprime to the level of the lattice $\underline{L}$. We shall use a' to denote an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L})$45

$D_{\underline{L}}$ the discriminant form associated to the lattice $\underline{L} = (L, \beta)$ 13

$E_{k,\underline{L},r}$ a Jacobi-Eisenstein series 74

This page is intentionally left blank.

Bibliography

[AL70] A.O.L. Atkin and J. Lehner. Hecke operators on g0(m). *Mathematische Annalen*, 185(2):134–160, 1970. URL: http://dx.doi.org/10.1007/BF01359701, doi:10.1007/BF01359701.

[And87] A.N. Andrianov. *Quadratic Forms and Hecke Operators*. Grundlehren Der Mathematischen Wissenschaften. Springer London, Limited, 1987.

[BB03] JanHendrik Bruinier and Michael Bundschuh. On borcherds products associated with lattices of prime discriminant. In Bruce Berndt and Ken Ono, editors, *Number Theory and Modular Forms*, volume 10 of *Developments in Mathematics*, pages 49–61. Springer US, 2003. URL: http://dx.doi.org/10.1007/978-1-4757-6044-6_5, doi:10.1007/978-1-4757-6044-6_5.

[BEW98] Bruce C. Berndt, R.J. Evans, and K.S. Williams. *Gauss and Jacobi sums*. Canadian Mathematical Society series of monographs and advanced texts. Wiley, 1998. URL: http://books.google.de/books?id=UvruAAAAMAAJ.

[BK93] Siegfried Boecherer and Winfried Kohnen. Estimates for fourier coefficients of siegel cusp forms. *Mathematische Annalen*, 297(1):499–517, 1993. URL: http://dx.doi.org/10.1007/BF01459512, doi:10.1007/BF01459512.

[Bor99] R. E. Borcherds. Reflection groups of Lorentzian lattices, September 1999. URL: http://arxiv.org/abs/math/9909123, arXiv:math/9909123.

[Boy11] Hatice Boylan. *Jacobi forms, quadratic modules and Weil representations over number Fields*. PhD thesis, 2011. URL: http://dokumentix.ub.uni-siegen.de/opus/volltexte/2012/597/.

[Bri04] Kathrin Bringmann. *Applications of Poincare series on Jacobi groups*. PhD thesis, 2004. URL: http://archiv.ub.uni-heidelberg.de/volltextserver/4939/.

[Bru02] JanHendrik Bruinier. 1. vector valued modular forms for the metaplectic group. In *Borcherds Products on O(2, l) and Chern Classes of Heegner Divisors*, volume 1780 of *Lecture Notes in Mathematics*, pages 15–38. Springer Berlin Heidelberg, 2002. URL: http://dx.doi.org/10.1007/978-3-540-45872-2_2, doi:10.1007/978-3-540-45872-2_2.

[BS07] Jan H. Bruinier and Oliver Stein. The Weil representation and Hecke operators for vector valued modular forms, April 2007. URL: http://arxiv.org/abs/0704.1868, arXiv:0704.1868.

[BS14] Hatice. Boylan and N.P. Skoruppa. Jacobi Forms of Lattice Index I. Basic Theory, In preparation, 2014.

[Cas08] J. W. S. Cassels. *Rational Quadratic Forms*. Dover books on mathematics. Dover Publications, 2008. URL: http://books.google.de/books?id=aookJwAACAAJ.

[CG11] Fabien Clery and Valery Gritsenko. Modular forms of orthogonal type and Jacobi theta-series, 2011. URL: http://arxiv.org/abs/1106.4733.

[Cip83] Barry A. Cipra. On the niwa-shintani theta-kernel lifting of modular forms. *Nagoya Mathematical Journal*, 91:49–117, 1983. URL: http://projecteuclid.org/euclid.nmj/1118787266.

[DS06] F. Diamond and J. Shurman. *A First Course in Modular Forms*. Graduate Texts in Mathematics. Springer, 2006. URL: http://books.google.de/books?id=EXZCAAAAQBAJ.

[Ebe12] W. Ebeling. *Lattices and Codes: A Course Partially Based on Lectures by Friedrich Hirzebruch*. Advanced Lectures in Mathematics. Springer Fachmedien Wiesbaden, 2012. URL: http://books.google.de/books?id=xKbEFE6d184C.

[ES95] Wolfgang Eholzer and Nils-Peter Skoruppa. Modular invariance and uniqueness of conformal characters. *Communications in Mathematical Physics*, 174(1):117–136, 1995. URL: http://dx.doi.org/10.1007/BF02099466, doi:10.1007/BF02099466.

[EZ85] M. Eichler and D. Zagier. *The theory of Jacobi forms*. Progress in mathematics. Birkhäuser, 1985. URL: http://books.google.de/books?id=dAXvAAAAMAAJ.

[GSZ] V. Gritsenko, N. Skoruppa, and D. Zagier. Theta blocks, In preparation.

[Kob93] N. Koblitz. *Introduction to Elliptic Curves and Modular Forms*. Graduate Texts in Mathematics. Springer New York, 1993.

[Kri] A. Krieg. *Hecke Algebras*. Number no. 435 in American Mathematical Society: Memoirs of the American Mathematical Society. American Mathematical Society. URL: http://books.google.de/books?id=iSux8tEzN0sC.

[Kri91] Aloys Krieg. The maaß spaces on the hermitian half-space of degree 2. *Mathematische Annalen*, 289(1):663–681, 1991. URL: http://dx.doi.org/10.1007/BF01446595, doi:10.1007/BF01446595.

[Lan76] S. Lang. *Introduction to Modular Forms*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Springer, 1976. URL: http://books.google.de/books?id=GtEwY53Nx7wC.

[McG03] William J. McGraw. The rationality of vector valued modular forms associated with the weil representation. *Mathematische Annalen*, 326(1):105–122, 2003. URL: http://dx.doi.org/10.1007/s00208-003-0413-1, doi:10.1007/s00208-003-0413-1.

[Miy06] T. Miyake. *Modular Forms*. Springer monographs in mathematics. Physica-Verlag, 2006.

[Nik80] V V Nikulin. Integral symmetric bilinear forms and some of their applications. *Mathematics of the USSR-Izvestiya*, 14(1):103, 1980. URL: http://stacks.iop.org/0025-5726/14/i=1/a=A06.

[Niw75] Shinji Niwa. Modular forms of half integral weight and the integral of certain theta-functions. *Nagoya Mathematical Journal*, 56:147–161, 1975. URL: http://projecteuclid.org/euclid.nmj/1118795278.

[S+11] W. A. Stein et al. *Sage Mathematics Software*. The Sage Development Team, 2011. http://www.sagemath.org.

[Ser93] Jean-Pierre Serre. *A course in arithmetic*. Springer, 1993.

[Shi71] G. Shimura. *Introduction to the Arithmetic Theory of Automorphic Functions*. Kanô memorial lectures. Princeton University Press, 1971.

[Shi73] Goro Shimura. Modular forms of half integral weight. 320:57–74, 1973. URL: http://dx.doi.org/10.1007/978-3-540-38509-7_3.

[Sko07] Nils-Peter Skoruppa. Jacobi Forms of Degree One and Weil Representations, November 2007. URL: `http://arxiv.org/abs/0711.0525`, `arXiv:0711.0525`.

[Sko14] Nils-Peter Skoruppa. Finite Quadratic Modules, Weil Representations and Vector Valued Modular Forms, 2014.

[Str11] Fredrik Strömberg. Weil representations associated to finite quadratic modules, July 2011. URL: `http://arxiv.org/abs/1108.0202`, `arXiv:1108.0202`.

[SZ87] N.P. Skoruppa and D. Zagier. *A trace formula for Jacobi forms*. Max-Planck-Inst. f. Mathematik, 1987.

[SZ88] Nils-Peter Skoruppa and Don Zagier. *Jacobi forms and a certain space of modular forms*, volume 94. Springer-Verlag, 1988. URL: `http://dx.doi.org/10.1007/BF01394347`, `doi:10.1007/BF01394347`.

[Wal63] C.T.C. Wall. Quadratic forms on finite groups, and related topics. *Topology*, 2(4):281 – 298, 1963. URL: `http://www.sciencedirect.com/science/article/pii/0040938363900120`.

[Zie89] C. Ziegler. Jacobi forms of higher degree. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 59(1):191–224, 1989. URL: `http://dx.doi.org/10.1007/BF02942329`, `doi:10.1007/BF02942329`.

This page is intentionally left blank.